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ON THE CUBIC L-FUNCTION

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Abstract. The cubic L-function is related to the cubic Kubota–Patterson theta function via the Mellin transformation. The cubic L-function obeys a functional equation of the Riemann type (with two gamma factors), but admits no expansion in an Euler product. In the paper, the cubic L-function is studied, and the distribution problem for the real parts of its zeros is considered. Some conjectures based on calculations are stated.

Introduction

The cubic theta function $\Theta_{K\text{-}P}$ of Kubota–Patterson is defined on the hyperbolic space $\mathbb{H} = \{(z,v) \in \mathbb{C} \times \mathbb{R} \mid v > 0\}$ and takes values in the complex field $\mathbb{C}$. We have the Fourier expansion

$$\Theta_{K\text{-}P}(z,v) = v^{2/3} + (6\pi v)^{2/3} \sum_{\nu} \widetilde{\tau}(\nu) Ai\big((6\pi|\nu|v)^{2/3}\big) e(\nu z), \tag{1}$$

where Ai is the Airy function and $e(q) = \exp\big(2\pi i(q+\bar{q})\big)$ for $q \in \mathbb{C}$. For the coefficients, we have the Patterson formulas (15) expressing $\widetilde{\tau}(\nu)$ in terms of the cubic Gauss sums. Put $\omega = \exp(2\pi i/3) = (-1+\sqrt{-3})/2$. Summation in (1) extends over all ν of the form $(\sqrt{-3})^{-3} l$ with $l \in \mathbb{Z}[\omega]$, $l \neq 0$. Here $\mathbb{Z}[\omega] = \{a + b\omega \mid a, b \in \mathbb{Z}\}$ is the ring of integers of the field $\mathbb{Q}(\sqrt{-3})$. The function $\Theta_{K\text{-}P}$ was discovered by Kubota [1], see also [2, 3]. It is an automorphic function of a fairly special type, namely, it is a metapectic form. In particular, this means that $\Theta_{K\text{-}P}$ admits some transformation formulas. One of them looks like this:

$$\Theta_{K\text{-}P}(0,v) = \Theta_{K\text{-}P}(0,v^{-1}) \quad \text{for all} \quad v \in \mathbb{R},\ v > 0. \tag{2}$$

We put

$$L(\tau; s) = \sum_{\nu} \frac{\tau(\nu)}{\|\nu\|^s}, \quad s \in \mathbb{C},\ \operatorname{Re} s > 1, \tag{3}$$

where summation is over the same ν as in (1),

$$\tau(\nu) = \widetilde{\tau}(\nu)\|\nu\|^{1/6} \tag{4}$$

and $\|\cdot\|\colon \mathbb{Q}(\sqrt{-3}) \to \mathbb{Q}$ is the norm map, $\|z\| = z\bar{z}$ for all $z \in \mathbb{Q}(\sqrt{-3})$. The series (3) converges absolutely and yields an analytic function in the domain $\operatorname{Re} s > 1$. The function $L(\tau;\cdot)$ is related to $\Theta_{K\text{-}P}$ via the Mellin transformation. We shall see that this function extends meromorphically to the entire complex plane $\mathbb{C}$ and admits a functional equation of the Riemann type, which relates its values at the points s and $1-s$. Namely, it will be shown that the function

$$s \mapsto (2\pi)^{-2s}\, \Gamma(s - 1/6)\, \Gamma(s + 1/6)\, L(\tau; s) \tag{5}$$

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is invariant with respect to the change of s by $1-s$. We call $L(\tau;\cdot)$ the cubic L-function. We are interested in the distribution of zeros of $L(\tau;\cdot)$, and especially, in the distribution of their real parts on the real line $\mathbb{R}$. The function τ is not multiplicative, so that $L(\tau;\cdot)$ admits no expansion in an Euler product, and it would be too optimistic to expect that all the nontrivial zeros[1] lie on the critical line $\operatorname{Re} s = 1/2$. Observe that the nontrivial zeros of $L(\tau;\cdot)$ are located on the complex plane $\mathbb{C}$ symmetrically relative to the real line $\mathbb{R}$ and to the critical line $\operatorname{Re} s = 1/2$. Therefore, we may restrict ourselves to considering the zeros that lie on the half-line

$$\operatorname{Re} s = 1/2, \quad \operatorname{Im} s \geq 0 \tag{6}$$

and those in the quadrant

$$\operatorname{Re} s > 1/2, \quad \operatorname{Im} s \geq 0. \tag{7}$$

Our computations show that within the limits $\operatorname{Im} s \leq 9002$ there are 27914 zeros on the half-line (6) and 8724 zeros in the quadrant (7). The lists of zeros can be found on the site `http://www.pdmi.ras.ru/~np`. All these zeros are simple. The zeros $\rho_1, \rho_2, \rho_3, \dots$ of $L(\tau;\cdot)$ lying in the quadrant (7) will be enumerated in the order of increasing of their imaginary parts. In the figure at the end of the paper, we depict the histogram of the distribution of the points $\sigma_n = \operatorname{Re} \rho_n$. In §1, we state several conjectures based on computational data. The main properties of the cubic L-function are listed and proved in §3. All facts we can prove about its zeros are presented in §4. Some preliminaries are collected in §2. We do not touch upon computational methods for calculating values and zeros of L-functions; see [4, 5, 6, 7] on this issue.

There is an extensive literature devoted to the distribution of zeros of L-functions with Euler products. In agreement with the general Riemann conjecture, the zeros are only found on the critical lines. In connection with the Montgomery conjectures [8] and their generalizations, much attention is paid to statistical aspects of the zeros distribution problem on the critical lines, i.e., the distribution of the imaginary parts of zeros. On the other hand, the problem of zero distribution for L-functions without Euler products remains open. Besides the cubic L-function, it would be of interest to study also other L-functions, in order to know to what extent the discovered phenomena are typical or unique[2].

§1. On the distribution of zeros

In this section we collect our conjectures about the zeros of the function $L(\tau;\cdot)$. The statements are printed in italic and supplied with brief comments.

(I) *All nontrivial zeros of $L(\tau;\cdot)$ are simple and lie in the critical strip, i.e., in the strip $\{s \in \mathbb{C} \mid 0 \leq \operatorname{Re} s \leq 1\}$.*

In §4, we shall prove that every nontrivial zero ρ of the function $L(\tau;\cdot)$ satisfies $-0.2 < \operatorname{Re} \rho < 1.2$ and present some additional arguments in favor of our conjecture. Except for a single zero, within our calculations the real parts of all zeros are less than 0.98. The real part of the exceptional zero is equal to $0.9948596\dots$, and its imaginary part is $147.1889196\dots$.

For real σ and T, we define $N(\sigma, T)$ as the number of zeros ρ of $L(\tau;\cdot)$ satisfying $\operatorname{Re} \rho \geq \sigma$, $0 < \operatorname{Im} \rho < T$.

[1] i.e., all zeros excluding the trivial ones, see Theorem 1 in §3.

[2] For example, this concerns the Epstein zeta-function of quadratic forms. The papers [9] by Siegel and [10] by Fomenko deliver the theoretic basicals. From the viewpoint of computations, the Epstein zeta-functions are substantially simpler than the cubic L-function.

(II) *Let q be the smallest real number such that*

$$N(\sigma,T) \ll T \quad \text{as} \quad T \to \infty \tag{8}$$

for each real $\sigma > q$ (the constant meant in $\ll$ depends on σ only). Then $q = 7/12$.

This conjecture is in good agreement with the calculations of $N(\sigma,T)$ and the mean values of the functions $t \mapsto |L(\tau;\sigma+it)|^2$. However, we are only able to prove that $q \le 3/4$, see Theorems 10 and 11 in §4. Apparently, the number q is also the exact lower bound of the set of all reals σ such that

$$\int_1^T |L(\tau;\sigma+it)|^2\,dt \ll T \quad \text{as} \quad T \to \infty.$$

Let $J = (1/2, 7/12]$ be the half-open interval in $\mathbb{R}$ formed by all x with $1/2 < x \le 7/12$, and let $J' = (7/12, \infty)$ be the open interval in $\mathbb{R}$ consisting of $x > 7/12$. For a real $T > 0$, consider the zeros ρ of the function $L(\tau;\cdot)$ such that $0 < \operatorname{Im}\rho < T$. Let $N(T)$ be the number of all such zeros ρ. We put

$P(T)$ to be the number of zeros ρ with $\operatorname{Re}\rho = 1/2,\ 0 < \operatorname{Im}\rho < T,$ and

$P_I(T)$ to be the number of zeros ρ with $\operatorname{Re}\rho \in I,\ 0 < \operatorname{Im}\rho < T,$

where I is an arbitrary interval in $\mathbb{R}$. Obviously,

$$N(T) = P(T) + 2P_J(T) + 2P_{J'}(T).$$

Also, we have $N(T) \sim (2/\pi)\,T\log T$ as $T \to \infty$, see Theorem 9 in §4.

(III) *For some reals $\delta, \gamma, \gamma' > 0$, we have*

$$P(T) \sim \delta T\log T, \quad P_J(T) \sim \gamma T\log T, \quad P_{J'}(T) \sim \gamma' T\log T$$

as $T \to \infty$, and $\delta + 2\gamma + 2\gamma' = 2/\pi$.

Now we consider the distribution of the real parts of the zeros of $L(\tau;\cdot)$ in the intervals J and J' in more detail.

(IV) *Let I be an interval contained in $J = (1/2, 7/12]$. As $T \to \infty$, we have*

$$P_I(T) \sim \Big(\int_I f(x)\,dx\Big) P_J(T)$$

where

$$f(x) = 2^5 3^4 (x - 1/2)(2/3 - x).$$

This conjecture is in good agreement with calculations. We can introduce a probability measure ι on the set of all intervals $I \subset J$ by putting

$$\iota(I) = \lim_{T\to\infty} P_I(T)/P_J(T),$$

and treat f as the density of ι with respect to Lebesgue measure. A few words are in order about where the function f stems from. This function is monotone increasing on J and vanishes at $1/2$, while its derivative vanishes at $7/12$. Our calculations show that these properties must be shared by any function that can, conjecturally, be the density of the measure ι. Imposing the natural normalization condition

$$\int_J f(x)\,dx = 1,$$

we see that our function f is a unique quadratic polynomial with the properties listed above.

(V) *Let I be an open interval contained in $J' = (7/12, \infty)$, say, $I = (\alpha, \beta)$. As $T \to \infty$, we have*

$$P_I(T)/P_{J'}(T) \to 0,$$

provided I is separated away from the point $7/12$, *i.e.*, $\alpha > 7/12$. *On the other hand, if* $I = (7/12, \beta)$ *with some* $\beta > 7/12$, *then as* $T \to \infty$ *we have*

$$P_I(T) \sim P_{J'}(T).$$

This conjecture is a consequence of (II) and (III).

(VI) *The nontrivial zeros of* $L(\tau;\cdot)$ *that lie off the critical line are concentrated most densely near the two "semicritical" lines,* $\operatorname{Re} s = 7/12$ *and* $\operatorname{Re} s = 1 - 7/12$, *while these lines themselves pass through no zeros.*

Here, the first claim follows from (IV), (V), and (5), and the second statement is based on direct calculations.

Now we turn once again to the distribution of the real parts of zeros of $L(\tau;\cdot)$ in the interval $J' = (7/12, \infty)$. The claim stated in (V) is not quite satisfactory. Considering the ratio $P_I(T)/T$ in place of $P_I(T)/P_{J'}(T)$, we find a more substantial statement.

(VII) *Let* I *be an open interval lying in* $J' = (7/12, \infty)$ *and separated away from the point* $7/12$. *As* $T \to \infty$, *we have*

$$P_I(T) \sim \Big(\int_I g(x)\,dx\Big)T,$$

with a real-valued and continuous function g *on* J' (*independent of* I *and* T) *such that* $g(x) \to \infty$ *as* $x \to 7/12$. *Let* q' *be the infimum of the set of all reals* σ *such that* $N(\sigma, T) = o(T)$ *as* $T \to \infty$. *The function* g *is monotone decreasing on the interval* $(7/12, q')$ *and* $g(x) = 0$ *for all* $x > q'$.

We cannot suggest a sharper conjecture. Apparently, q' is very close to 1. However, our calculations do not suffice for determining q' and the behavior of g near the points q' and $7/12$.

§2. Arithmetic of the field $\mathbb{Q}(\sqrt{-3}\,)$

The ring $\mathbb{Z}[\omega]$ of integers of the field $\mathbb{Q}(\sqrt{-3}\,) \subset \mathbb{C}$ is an Euclidean principal ideals ring. Each nonzero element in $\mathbb{Z}[\omega]$ is a product of prime elements, and this product is unique up to multiplication by units. The units of this ring $\mathbb{Z}[\omega]$ are ± 1, $\pm\omega$, $\pm\omega^2$ with $\omega = \exp(2\pi i/3) = (-1 + \sqrt{-3})/2$. Observe that the units are distinct mod3 and represent all classes mod3 relatively prime to 3. It follows that each nonzero element k of the ring $\mathbb{Z}[\omega]$ can be factored uniquely as $k = \lambda(\sqrt{-3})^m c$, where λ is a unit, $m \in \mathbb{Z}$, $m \geq 0$, and $c \in \mathbb{Z}[\omega]$, $c \equiv 1 \pmod 3$.

In a well-known way, we can define the cubic residue symbol $\left(\frac{\cdot}{\cdot}\right)$ on $\mathbb{Z}[\omega]$. For $c \equiv 1 \pmod 3$, we put

$$G(c) = \sum_k \left(\frac{k}{c}\right) e(k/c) \tag{9}$$

with summation on k running over a reduced system of residues $\bmod c$; this is the Gauss cubic sum with module c. In particular, $G(1) = 1$. The main properties of such sums are as follows:

$$|G(c)|^2 = \begin{cases} \|c\| & \text{if } c \text{ is square free,} \\ 0 & \text{otherwise;} \end{cases} \tag{10}$$

$$\overline{G(c)} = G(\bar{c}); \tag{11}$$

$$G(c_1 c_2) = \left(\frac{c_1}{c_2}\right)\left(\frac{c_2}{c_1}\right) G(c_1) G(c_2) \tag{12}$$

if c_1 and c_2 are relatively prime. In the case where c is prime, we also have

$$G(c)^3 = -c^2 \bar{c}. \tag{13}$$

Let p be a positive prime in $\mathbb{Z}$. If $p \equiv 2 \pmod 3$, then $c = -p$ is a prime congruent to $1 \pmod 3$ in $\mathbb{Z}[\omega]$, $G(c) = p$, and $\|c\| = p^2$. If $p \equiv 1 \pmod 3$, then p admits a unique decomposition in $\mathbb{Z}[\omega]$ of the form $p = c\bar{c}$ with $c \equiv \bar{c} \equiv 1 \pmod 3$. In this case, the numbers c and $\bar{c}$ are prime, $\|c\| = \|\bar{c}\| = p$, and

$$G(c) + G(\bar{c}) = \sum_{j=0,1,\dots,p-1} \cos(2\pi j^3/p). \tag{14}$$

For the properties of the cubic residue symbol and the Gauss sums, see [11].

For $\nu \in \mathbb{Q}(\sqrt{-3}\,)$, put

$$\widetilde{\tau}(\nu) = \frac{\overline{G(c)}}{\|c\|^{2/3}} \begin{cases} 3^{-1/3}\left(\frac{3}{c}\right) & \text{if } \nu = \pm(\sqrt{-3})^{3n-1}cd^3, \\ 3^{-1/3}\left(\frac{3\omega}{c}\right)\xi^{-1} & \text{if } \nu = \pm\,\omega(\sqrt{-3})^{3n-1}cd^3, \\ 3^{-1/3}\left(\frac{3\omega^2}{c}\right)\xi & \text{if } \nu = \pm\,\omega^2(\sqrt{-3})^{3n-1}cd^3, \\ \quad 1 & \text{if } \nu = \pm(\sqrt{-3})^{3n-3}cd^3, \\ \quad 0 & \text{for all other } \nu, \end{cases} \tag{15}$$

where $c, d \in \mathbb{Z}[\omega]$, $c \equiv d \equiv 1 \pmod 3$, c is square free, $n \in \mathbb{Z}$, and $\xi = \exp(2\pi i/9)$. The fact that formula (4) determines the coefficients in the expansion (1) of the function $\Theta_{K\text{-}P}$ was proved by Patterson in [2]. It can be seen that $\widetilde{\tau}(\nu)$ does not depend on d and n occurring in (4). We prefer to work with the function τ related to $\widetilde{\tau}$ as in (4). Obviously, $\tau(1) = 1$. Formula (11) and the well-known properties of the residue symbol imply that

$$\tau(\bar{\nu}) = \overline{\tau(\nu)} \tag{16}$$

for all $\nu \in \mathbb{Q}(\sqrt{-3}\,)$. Moreover, we have

$$\tau(ab) = \left(\frac{a}{b}\right)\tau(a)\tau(b), \tag{17}$$

provided $a \in (\sqrt{-3})^{-3}\mathbb{Z}[\omega]$, $b \in \mathbb{Z}[\omega]$, $b \equiv \pm 1 \pmod 3$, and a and b have no common prime divisors. Relation (17) is deduced with the help of the definition (4), the well-known property (12) of the Gauss sums, and the reciprocity law. We see that the function τ is not multiplicative, but τ^3 is multiplicative.

The Dedekind zeta function $\zeta_{\mathbb{Q}(\sqrt{-3}\,)}$ of the field $\mathbb{Q}(\sqrt{-3}\,)$ is holomorphic everywhere on $\mathbb{C}$, except for the point 1, where it has a simple pole. We have $\zeta_{\mathbb{Q}(\sqrt{-3}\,)}(-n) = 0$ for every $n \in \mathbb{Z}$, $n \geq 1$. All other zeros of the Dedekind zeta function $\zeta_{\mathbb{Q}(\sqrt{-3}\,)}$ are not real, lie in the strip $\{s \in \mathbb{C} \mid 0 < \operatorname{Re} s < 1\}$, and, conjecturally, only on the line $\operatorname{Re} s = 1/2$. The function

$$s \mapsto (2\pi/\sqrt{3})^{-s}\Gamma(s)\zeta_{\mathbb{Q}(\sqrt{-3}\,)}(s) \tag{18}$$

is invariant under the change of s by $1 - s$. We have

$$\zeta_{\mathbb{Q}(\sqrt{-3}\,)}(s) = \zeta(s)L(s,\chi), \quad s \in \mathbb{C}, \tag{19}$$

where ζ is the Riemann zeta function and $L(\cdot,\chi)$ is the classical Dirichlet series with the quadratic character $\chi \mod 3$ (see [12]). Sometimes, it is more convenient to deal with the function

$$\zeta_*(s) = \left(1 - \frac{1}{3^s}\right)\zeta_{\mathbb{Q}(\sqrt{-3}\,)}(s), \quad s \in \mathbb{C}, \tag{20}$$

for which, in the domain $\operatorname{Re} s > 1$, we have

$$\zeta_*(s) = \sum_m \frac{1}{\|m\|^s}, \quad \frac{\zeta_*(s)}{\zeta_*(2s)} = \sum_n \frac{1}{\|n\|^s} \tag{21}$$

with summation over $m \equiv 1 \pmod 3$ and over square free $n \equiv 1 \pmod 3$ in the ring $\mathbb{Z}[\omega]$.

For $n > 0$, $n \in \mathbb{Z}$, let $V(n)$ be the number of all $c \in \mathbb{Z}[\omega]$ with $\|c\| = n$, and let $Z(n)$ be the number of all such c satisfying $c \equiv 1 \pmod 3$. It is easily seen that $Z(n) \le V(n)/6$ (because there are 6 units in $\mathbb{Z}[\omega]$ pairwise distinct $\bmod 3$). We write $c \in \mathbb{Z}[\omega]$ as $c = x + y\omega$, $x, y \in \mathbb{Z}$. Then $\|c\| = x^2 - xy + y^2$. Consequently, $V(n)$ is equal to the number of representations of n by the quadratic form $x^2 - xy + y^2$, $x, y \in \mathbb{Z}$, and (see [13]) we have

$$Z(n) \le V(n)/6 = \sum_m \Big(\frac{-3}{m}\Big) \le d(n), \tag{22}$$

where $\big(\frac{\cdot}{\cdot}\big)$ is the quadratic residue symbol on $\mathbb{Z}$, summation extends over all positive divisors m of n, and $d(n)$ is the number of such divisors. If $n \not\equiv 0 \pmod 3$, then we can view $Z(n)$ as the number of ideals of norm n in the ring $\mathbb{Z}[\omega]$. Let $n = 3^m n'$ with $m, n' \in \mathbb{Z}$, $m \ge 0$, $n' \ge 1$, $n' \not\equiv 0 \pmod 3$. The number of ideals of norm n in $\mathbb{Z}[\omega]$ equals $Z(n')$. Thus, for any $n \in \mathbb{Z}$, $n \ge 1$, the number of ideals of norm n in $\mathbb{Z}[\omega]$ does not exceed $d(n)$.

§3. The main properties of the function $L(\tau;\cdot)$

Some properties of $L(\tau;\cdot)$ have been mentioned earlier. Now we give precise formulations and proofs. Put

$$\Omega(s) = (\sqrt{3}/2)\,(2\pi)^{-2s}\Gamma(s - 1/6)\Gamma(s + 1/6), \quad s \in \mathbb{C}. \tag{23}$$

For s with $\operatorname{Re} s > 1/6$, we have

$$\Omega(s) = (6\pi)^{1-2s} \int_0^\infty Ai(x^{2/3}) x^{2s-4/3}\, dx. \tag{24}$$

Information about the Airy function can be found in [14]. In the half-plane $\operatorname{Re} z \ge 0$ we have the asymptotic formula

$$Ai(z^{2/3}) = \frac{1}{2\pi^{1/2} z^{1/6}} \exp(-2z/3)\Big(1 + O\Big(\frac{1}{|z|}\Big)\Big), \quad |z| \to \infty. \tag{25}$$

Theorem 1. *The Dirichlet series* (3) *that determines* $L(\tau;\cdot)$ *converges absolutely and yields a holomorphic function in the domain* $\operatorname{Re} s > 1$. *Next,* $L(\tau;\cdot)$ *extends meromorphically to* $\mathbb{C}$. *The only singularity of* $L(\tau;\cdot)$ *is a simple pole at the point* $5/6$. *The points* $s = -1/6 - n$ *and* $s = -5/6 - n$, $n \ge 0$, *are simple zeros of* $L(\tau;\cdot)$; *these zeros are said to be trivial. The function* $\Lambda(s) = \Omega(s) L(\tau; s)$, $s \in \mathbb{C}$, *satisfies the functional equation* $\Lambda(s) = \Lambda(1-s)$ *and has no singularities except for simple poles with the residues* $1/2$ *and* $-1/2$ *at the points* $5/6$ *and* $1/6$. *For any* $s \in \mathbb{C}$ *we have* $L(\tau;\bar{s}) = \overline{L(\tau;s)}$; *in particular, the zeros of* $L(\tau;\cdot)$ *are located symmetrically relative to the real axis.*

Proof. Each ν in (3) is represented uniquely as the product of one of the six units by $(\sqrt{-3})^n c d^3$ with $n \ge -3$, $n \in \mathbb{Z}$, with a cube free $c \equiv 1 \pmod 3$ in $\mathbb{Z}[\omega]$, and with $d \equiv 1 \pmod 3$ in $\mathbb{Z}[\omega]$. We have $|\tilde{\tau}(\nu)| \le \|c\|^{-1/6}$, see (4) and (10), and $\tau(\nu)$ can be nonzero only if c is square free. Consequently, the series (3) is dominated by the series

$$6 \sum_{n,c,d} \frac{1}{3^{(\sigma-1/6)n} \|c\|^{\sigma} \|d\|^{3\sigma-1/2}} \tag{26}$$

with $\sigma = \operatorname{Re} s$, which converges for $\sigma > 1$. This ensures that the series (3) converges absolutely and that $L(\tau;\cdot)$ is holomorphic in the domain $\operatorname{Re} s > 1$.

Now we consider the integral

$$\Lambda(s) = \int_0^\infty \big\{\Theta_{K\text{-}P}(0, v) - v^{2/3}\big\} v^{2s-2}\, dv, \tag{27}$$

which converges absolutely for $\operatorname{Re} s > 5/6$. Assuming that $\operatorname{Re} s > 1$, we plug the Fourier expansion (27) in (1) and integrate termwise:

$$\begin{aligned}\Lambda(s) &= (6\pi)^{2/3} \sum_{\nu} \widetilde{\tau}(\nu) \int_0^\infty Ai\big((6\pi|\nu|v)^{2/3}\big) v^{2s-4/3}\,dv \\ &= (6\pi)^{1-2s} \sum_{\nu} \frac{\widetilde{\tau}(\nu)}{\|\nu\|^{s-1/6}} \int_0^\infty Ai(x^{2/3}) x^{2s-4/3}\,dx = \Omega(s) L(\tau; s).\end{aligned} \tag{28}$$

On the other hand, the integral in (27) is the sum of the integral from 0 to η and the integral from η to ∞, with an arbitrary $\eta > 0$. We have $\Lambda(s) = X_\eta(s) + Y_\eta(s)$, where

$$X_\eta(s) = \int_\eta^\infty \big\{\Theta_{K\text{-}P}(0, v) - v^{2/3}\big\} v^{2s-2}\,dv, \tag{29}$$

$$Y_\eta(s) = \int_0^\eta \big\{\Theta_{K\text{-}P}(0, v) - v^{2/3}\big\} v^{2s-2}\,dv. \tag{30}$$

In the last-written integral, we put $v = u^{-1}$ and make use of (2), obtaining

$$\begin{aligned}Y_\eta(s) &= \int_{\eta^{-1}}^\infty \big\{\Theta_{K\text{-}P}(0, u) - u^{-2/3}\big\} u^{-2s}\,du \\ &= \int_{\eta^{-1}}^\infty \big\{\Theta_{K\text{-}P}(0, u) - u^{2/3}\big\} u^{-2s}\,du + \int_{\eta^{-1}}^\infty \big\{u^{2/3} - u^{-2/3}\big\} u^{-2s}\,du \\ &= X_{\eta^{-1}}(1-s) + \frac{\eta^{2s-5/3}}{2s-5/3} - \frac{\eta^{2s-1/3}}{2s-1/3}.\end{aligned}$$

Therefore, in the domain $\operatorname{Re} s > 1$ we have

$$\Lambda(s) = \frac{\eta^{2s-5/3}}{2s-5/3} - \frac{\eta^{2s-1/3}}{2s-1/3} + X_\eta(s) + X_{\eta^{-1}}(1-s). \tag{31}$$

The integral on the right in (29) represents an entire function, see (1) and (25). Since X_η and $X_{\eta^{-1}}$ are entire functions, formula (31) yields a meromorphic extension of Λ and, by (28), also of $L(\tau; \cdot)$ to the complex plane $\mathbb{C}$. Clearly, Λ is holomorphic everywhere except the points $5/6$ and $1/6$, where this function has simple poles. Recall that the real number $\eta > 0$ on the right-hand side of (31) is arbitrary. Looking at this right-hand side with $\eta = 1$, we conclude that Λ is invariant under the change $s \mapsto 1 - s$ and that the residues at the poles $5/6$ and $1/6$ are $1/2$ and $-1/2$. Now, observe that the function Ω has no zeros, and has only simple poles only at the points $\pm 1/6 - n$ with integral $n \geq 0$. This justifies our claims about zeros and poles of $L(\tau; \cdot)$. The last statement of the theorem follows from (3) and (16) for s with $\operatorname{Re} s > 1$ and extends to all $s \in \mathbb{C}$ by the analytic continuation principle. □

Let $s, z \in \mathbb{C}$ and $\operatorname{Re} z > 0$. Put

$$\Omega(s, z) = (6\pi)^{1-2s} \int_z^\infty Ai(x^{2/3}) x^{2s-4/3}\,dx, \tag{32}$$

where the integration path runs in the half-plane $\operatorname{Re} x > 0$ from the point z to the real axis and then along the real axis to ∞. In the half-plane $\operatorname{Re} x > 0$ the integrand in (32) is holomorphic, so that the integral is independent of the remaining arbitrariness in the choice of the path. The function Ω is holomorphic everywhere in its domain. Note that the convergence of the integral follows from (25). We fix an arbitrary compact set $V \subset \mathbb{C}$ and a real number $\varepsilon > 0$. If z tends to ∞ within the sector $|\arg(z)| \leq \pi/2 - \varepsilon$, then $|\Omega(s, z)|$ decays faster than $|z|^{-n}$ with any integer n, uniformly in $s \in V$. This is an easy consequence of (25).

Theorem 2. *Let $s, \eta \in \mathbb{C}$, $\operatorname{Re}\eta > 0$. We have*

$$\begin{aligned}\Omega(s)L(\tau; s) = \frac{\eta^{2s-5/3}}{2s-5/3} - \frac{\eta^{2s-1/3}}{2s-1/3} + \sum_{\nu} \frac{\tau(\nu)}{\|\nu\|^s}\,\Omega(s, 6\pi|\nu|\eta) \\ + \sum_{\nu} \frac{\tau(\nu)}{\|\nu\|^{1-s}}\,\Omega(1-s, 6\pi|\nu|\eta^{-1})\end{aligned} \tag{33}$$

where Ω is as in (23), (32), *and summation is over all ν of the form $(\sqrt{-3})^{-3}l$ with $l \in \mathbb{Z}[\omega]$, $l \neq 0$ (i.e., over the same ν as in* (1) *and* (3)). *The series on the right in* (33) *converge absolutely and locally uniformly.*

Proof. Let $s \in \mathbb{C}$ be fixed arbitrarily. The series on the right in (33) converge absolutely and locally uniformly in η and represent holomorphic functions in the domain $\operatorname{Re}\eta > 0$. The functions $\eta \mapsto \eta^{2s-5/3}$ and $\eta \mapsto \eta^{2s-1/3}$ are also holomorphic in that domain. Therefore, if (33) is true for all positive $\eta \in \mathbb{R}$, then so it is for all $\eta \in \mathbb{C}$ with $\operatorname{Re}\eta > 0$.

Now let $\eta \in \mathbb{R}$, $\eta > 0$. By (31), (28), we have

$$\Omega(s)L(\tau; s) = \frac{\eta^{2s-5/3}}{2s-5/3} - \frac{\eta^{2s-1/3}}{2s-1/3} + X_\eta(s) + X_{\eta^{-1}}(1-s). \tag{34}$$

We know that this identity is valid not only for s with $\operatorname{Re} s > 1$, but also for all $s \in \mathbb{C}$. To calculate $X_\eta(s)$, we substitute the Fourier expansion (1) in (29) and integrate termwise:

$$X_\eta(s) = (6\pi)^{2/3} \sum_{\nu} \widetilde{\tau}(\nu) \int_\eta^\infty Ai\big((6\pi|\nu|v)^{2/3}\big) v^{2s-4/3}\,dv.$$

Putting $x = 6\pi|\nu|v$, we obtain

$$X_\eta(s) = \sum_{\nu} \frac{\tau(\nu)}{\|\nu\|^s}\,\Omega(s, 6\pi|\nu|\eta)$$

with Ω as in (32). Also, for $X_{\eta^{-1}}(1-s)$ we have an expansion of the same form with η^{-1} and $1-s$ in place of η and s. To get (34), now it suffices to plug these expansions in (33). □

An important advantage of (33) is the possibility to control the behavior of the series in (33) by choosing the free parameter η. We have used (33) for calculating the values of the function $L(\tau; s)$. By transforming the integral in (32) properly, it can be shown that equation (33) belongs to the class of functional equations described in [15]. After that, the general result obtained in [16] implies the following statement.

Theorem 3. *Let a, b, c, d, ε be positive real numbers. For all $s \in \mathbb{C}$ and all positive real x, y satisfying*

$$|\operatorname{Re} s| \le a, \quad |\operatorname{Im} s| \ge b, \quad xy = (4/\pi^4)|\operatorname{Im} s|^4, \quad c \le x/y \le d,$$

we have

$$L(\tau; s) = \sum_{\|\nu\| \le x} \frac{\tau(\nu)}{\|\nu\|^s} + \frac{\Omega(1-s)}{\Omega(s)} \sum_{\|\nu\| \le y} \frac{\tau(\nu)}{\|\nu\|^{1-s}} + O\big(|\operatorname{Im} s|^{1-2\operatorname{Re} s+\varepsilon}\big), \tag{35}$$

where the constant in O only depends on a, b, c, d, ε, and summation is over all ν as in (1), (3) *with the indicated restrictions on $\|\nu\|$.*

To deduce Theorem 3, the parameter η in Theorem 2 should be chosen so that

$$\arg\eta = \Big(\frac{\pi}{2} - \frac{1}{|\operatorname{Im} s|}\Big)\operatorname{sign}(\operatorname{Im} s), \quad c \le |\eta|^2 \le d,$$

and $x = (2/\pi^2)|\operatorname{Im} s|^2|\eta|$, $y = (2/\pi^2)|\operatorname{Im} s|^2/|\eta|$.

Theorem 4. *For all $s \in \mathbb{C}$ satisfying $0 \le \operatorname{Re} s \le 1$, $|\operatorname{Im} s| \ge 1$, we have*

$$L(\tau; s) \ll |\operatorname{Im} s|^{2-2\operatorname{Re} s+\varepsilon}$$

where $\varepsilon > 0$ is arbitrarily small (the constant in $\ll$ depends on ε only).

Proof. We use Theorem 3 with $x = y = (2/\pi^2)|\operatorname{Im} s|^2$ and $a = b = 1$. Let δ be a positive real number. We know (see Theorem 1) that the series (3), which determines $L(\tau; \cdot)$, converges absolutely on the line $\operatorname{Re} s = 1 + \delta$. Put $\sigma = \operatorname{Re} s$. We estimate the sums on the right in (35):

$$\sum_{\|\nu\| \le x} \frac{|\tau(\nu)|}{\|\nu\|^{\sigma}} = \sum_{\|\nu\| \le x} \frac{|\tau(\nu)|}{\|\nu\|^{1+\delta}} \|\nu\|^{1-\sigma+\delta} \ll x^{1-\sigma+\delta},$$

$$\sum_{\|\nu\| \le y} \frac{|\tau(\nu)|}{\|\nu\|^{1-\sigma}} = \sum_{\|\nu\| \le y} \frac{|\tau(\nu)|}{\|\nu\|^{1+\delta}} \|\nu\|^{\sigma+\delta} \ll y^{\sigma+\delta},$$

where the constants in $\ll$ depend on δ. Next, the Stirling formula [17] yields

$$\frac{\Omega(1-s)}{\Omega(s)} \ll |\operatorname{Im} s|^{2-4\sigma}$$

with an absolute constant in $\ll$. Substituting our estimates in (35) and taking $\delta = \varepsilon/2$, we obtain

$$L(\tau; s) \ll x^{1-\sigma+\delta} + |\operatorname{Im} s|^{2-4\sigma} y^{\sigma+\delta} + |\operatorname{Im} s|^{1-2\sigma+\varepsilon} \ll |\operatorname{Im} s|^{2-2\sigma+\varepsilon},$$

as required. □

The next theorem can be proved by the standard method based on the Perron formula, see [18, 19]. We do not need this theorem, presenting it here without proof only for completeness.

Theorem 5. *The series (3), which determines $L(\tau; s)$, converges for all $s \in \mathbb{C}$ with $\operatorname{Re} s > 5/6$. For any real $\varepsilon > 0$ and $x \ge 1$, we have*

$$\sum_{\|\nu\| \le x} \tau(\nu) = Cx^{5/6} + O(x^{3/5+\varepsilon}),$$

where $C = (2\pi)^{2/3}(6/5)\Gamma(1/3)$, summation is over all ν as in (1), (3) satisfying $\|\nu\| \le x$, and the constant in $\ll$ depends on ε only.

Let $n \ge 1$, $n \in \mathbb{Z}$. If $n \not\equiv 0 \pmod 9$, we put

$$r_n = \sum_c \frac{\overline{G(c)}}{\|c\|^{1/2}} \tag{36}$$

where summation is over all square free $c \equiv 1 \pmod 3$ in $\mathbb{Z}[\omega]$ satisfying $\|c\| = n$. In the case where $n \equiv 0 \pmod 9$, we put

$$r_n = \sum_c \frac{\overline{G(c)}}{\|c\|^{1/2}} \kappa(c), \tag{37}$$

where summation is over all square free $c \equiv 1 \pmod 3$ in $\mathbb{Z}[\omega]$ satisfying $\|c\| = n/9$, and

$$\kappa(c) = \Big(\frac{3}{c}\Big)\Big\{1 + \Big(\frac{\omega}{c}\Big)\xi^{-1} + \Big(\frac{\omega^2}{c}\Big)\xi\Big\}, \quad \xi = \exp(2\pi i/9).$$

Clearly, $|\kappa(c)| \le 3$, and it is easy to check that $\kappa(\bar{c}) = \overline{\kappa(c)}$. Observe also that all r_n are real number, because together with the term corresponding to c, in (36) and (37) we have the complex conjugate term corresponding to $\bar{c}$, see (11). Obviously, $r_1 = 1$. The absolute value of each term in (36) is equal to 1, while in (37) it does not exceed 3, see (10). Now, from (22) we deduce that[3]

$$|r_n| \le d(n) \quad \text{for all} \quad n \ge 1, \quad n \in \mathbb{Z}. \tag{38}$$

For $s \in \mathbb{C}$ with $\operatorname{Re} s > 1$, put

$$E(s) = \sum_n \frac{r_n}{n^s}, \tag{39}$$

where summation is over all $n \ge 1$, $n \in \mathbb{Z}$. We may restrict ourselves to summation over the cube free n, because $r_n = 0$ for all other n. More precisely, if $r_n \ne 0$, then $n = uv^2$, where u is a product of distinct primes congruent to 1 (mod 3), v is square free, and $\gcd(u, v) = 1$ (for all other n the sums in (36) and (37) are empty).

Besides E, we shall need the function $\widehat{E}$ defined by the Dirichlet series inverse to the series (39). Namely, we put

$$\widehat{E}(s) = 1/E(s), \quad s \in \mathbb{C}. \tag{40}$$

We define $f_n \in \mathbb{C}$, $n \ge 1$, $n \in \mathbb{Z}$, by the recurrence relations

$$f_1 = 1, \quad f_n = -\sum_m r_m f_{n/m} \tag{41}$$

with summation over $m \in \mathbb{Z}$ satisfying $m \mid n$, $m \ge 2$. Then

$$\Big(\sum_n \frac{f_n}{n^s}\Big)\Big(\sum_n \frac{r_n}{n^s}\Big) = 1, \quad s \in \mathbb{C}, \tag{42}$$

where summation is over all $n \ge 1$, $n \in \mathbb{Z}$, and the product is understood as the formal product of Dirichlet series. Of course, also we have

$$\widehat{E}(s) = \sum_n \frac{f_n}{n^s} \tag{43}$$

with summation over all $n \ge 1$, $n \in \mathbb{Z}$, provided the series on the right converges absolutely.

Theorem 6. *The Dirichlet series* (39) *converges absolutely and represents a holomorphic function in the domain* $\operatorname{Re} s > 1$. *The function E extends holomorphically to $\mathbb{C}$, and*

$$L(\tau; s) = (2/\sqrt{3})\, 27^s \zeta_{\mathbb{Q}(\sqrt{-3})}(3s - 1/2) E(s), \quad s \in \mathbb{C}. \tag{44}$$

In the domain $\operatorname{Re} s > 1/2$, *the function E is holomorphic everywhere excluding the point* $5/6$, *where it has a simple pole. In the domain* $\operatorname{Re} s > 1/2$, *the zeros of E are the same and have the same orders as those of* $L(\tau; \cdot)$.

Proof. Recall that summation in (3) is over all $\nu = (\sqrt{-3})^{-3} l$ with $l \in \mathbb{Z}[\omega]$, $l \ne 0$. Every such ν can be written uniquely as

$$\nu = \lambda(\sqrt{-3})^{-m+3n} c d^3, \tag{45}$$

where λ is one of the six units of the ring $\mathbb{Z}[\omega]$, $m \in \{1, 2, 3\}$, $n \in \mathbb{Z}$, $n \ge 0$, and $c, d \in \mathbb{Z}[\omega]$, $c \equiv d \equiv 1$ (mod 3), c is cube free. By (4) and (15), the coefficients $\tau(\nu)$ can be nonzero only if the number c is square free and either $m = 3$, $\lambda \in \{1, -1\}$, or $m = 1$. We put

$$R(s) = \sum_\nu \frac{\tau(\nu)}{\|\nu\|^s}, \quad s \in \mathbb{C}, \ \operatorname{Re} s > 1, \tag{46}$$

[3]Recall that $d(n)$ is the number of positive divisors of n.

with summation over all

$$\nu = \lambda(\sqrt{-3})^{-m}c, \tag{47}$$

where c is a square free number in $\mathbb{Z}[\omega]$, $c \equiv 1 \pmod 3$, and ether $\lambda = 1$, $m = 3$, or $\lambda \in \{1, \omega, \omega^2\}$, $m = 1$. We know that the series (3) converges absolutely in the domain $\operatorname{Re} s > 1$. The series (46) is built from (3) by dropping some terms, so that (46) also converges absolutely in this domain. Collecting in (15) the terms that correspond to ν with equal norm and using formulas (15) and (4), we see that

$$R(s) = 3^{3s-1/2}E(s), \quad s \in \mathbb{C}, \ \operatorname{Re} s > 1, \tag{48}$$

and that the series (39), which determines the function E, also converges absolutely for $\operatorname{Re} s > 1$. Let us explain the proof of formula (48). We split the sum on the right in (46) into two sums: the first consists of the terms corresponding to $m = 3$, and the second consists of the terms corresponding to $m = 1$, see (47). Using (15), we get

$$R(\tau; s) = 3^{3s-1/2}\{A(s) + B(s)\}, \tag{49}$$

$$A(s) = \sum_c \frac{\overline{G(c)}}{\|c\|^{s+1/2}}, \quad B(s) = \sum_c \frac{\overline{G(c)}}{9^s\|c\|^{s+1/2}}\kappa(c) \tag{50}$$

with summation over the square free $c \equiv 1 \pmod 3$ in $\mathbb{Z}[\omega]$. It remains to compare (49) and (50) with (36) and (37).

Next, by (3) and (4) we have

$$L(\tau; s) = \sum_\nu \frac{\widetilde{\tau}(\nu)}{\|\nu\|^{s-1/6}}, \quad s \in \mathbb{C}, \ \operatorname{Re} s > 1,$$

where summation is over all ν as in (45). Turning to (15), note that $\widetilde{\tau}(\nu)$ is independent of d, n and that $\widetilde{\tau}(-\nu) = \widetilde{\tau}(\nu)$. We obtain the identity

$$L(\tau; s) = 2\Big(\sum_\nu \frac{\widetilde{\tau}(\nu)}{\|\nu\|^{s-1/6}}\Big)\Big(\sum_n \frac{1}{3^{3n(s-1/6)}}\Big)\Big(\sum_d \frac{1}{\|d\|^{3s-1/2}}\Big),$$

where summation is over ν as in (47), $n \in \mathbb{Z}$, $n \geq 0$, $d \in \mathbb{Z}[\omega]$, $d \equiv 1 \pmod 3$. Consequently (see (21)), we have

$$L(\tau; s) = 2R(s)\Big(1 - \frac{1}{3^{3s-1/2}}\Big)^{-1}\zeta_*(3s - 1/2).$$

Combined with (20) and (48), this yields (44) for s with $\operatorname{Re} s > 1$. The analytic continuation principle delivers the meromorphic extension of E, with preservation of (44), to the entire complex plane $\mathbb{C}$. The function $s \mapsto \zeta_{\mathbb{Q}(\sqrt{-3})}(3s - 1/2)$ is holomorphic and has no zeros in the domain $\operatorname{Re} s > 1/2$. Therefore, in this domain the zeros and poles of E are the same and of the same orders as the zeros and poles of $L(\tau; \cdot)$. □

Theorem 7. *Let $s \in \mathbb{C}$ and $\operatorname{Re} s > 1$. Then*

$$L(\tau; s) = \frac{2}{\sqrt{3}}\, 27^s \sum_m \frac{c_m}{m^s}, \tag{51}$$

where summation is over all $m \in \mathbb{Z}$, $m \geq 1$, and $c_m \in \mathbb{R}$ is the sum of the coefficients $\tau(\nu)$ in (3) *over all ν satisfying $\|\nu\| = m/27$. Suppose that $a, b \in \mathbb{Z}$, $a, b \geq 1$, a is cube free, and let λ_b be the number of ideals of norm b in the ring $\mathbb{Z}[\omega]$. Then*

$$c_{ab^3} = r_a\lambda_b\sqrt{b} \quad \text{and} \quad |c_{ab^3}| \leq d(a)d(b)\sqrt{b}, \tag{52}$$

where r_a is as in (36), (37).

Proof. Formula (51) is a direct consequence of (3), and the fact that all c_m are real follows from (16). Observe that each $m \in \mathbb{Z}$, $m \geq 1$, can uniquely be written in the form ab^3 with $b \in \mathbb{Z}$, $b \geq 1$, and with a cube free $a \in \mathbb{Z}$, $a \geq 1$. Comparing (51) with the decomposition (44) in Theorem 6, we see that

$$\sum_m \frac{c_m}{m^s} = \zeta_{\mathbb{Q}(\sqrt{-3})}\Big(3s - \frac{1}{2}\Big)E(s), \tag{53}$$

with summation over $m \in \mathbb{Z}$, $m \geq 1$. Next, using (39) and (38), we obtain

$$E(s) = \sum_a \frac{r_a}{a^s}, \qquad |r_a| \leq d(a), \tag{54}$$

with summation over the cube free $a \in \mathbb{Z}$, $a \geq 1$. Also, we have

$$\zeta_{\mathbb{Q}(\sqrt{-3})}\Big(3s - \frac{1}{2}\Big) = \sum_b \frac{\lambda_b}{b^{3s}}\sqrt{b}, \quad \lambda_b \leq d(b), \tag{55}$$

where summation is over $b \in \mathbb{Z}$, $b \geq 1$. Multiplying (54) and (55) and comparing with (53), we get (52). □

§4. On the zeros of $L(\tau; \cdot)$

In this section we prove several claims about zeros of the function $L(\tau; \cdot)$. We start with a theorem that bounds the domain where the zeros can lie.

Theorem 8. *If* $L(\tau; s) = 0$*, then* $\operatorname{Re} s < 1.2$*.*

Proof. Instead of $L(\tau; \cdot)$, consider the function E. By Theorem 6, E has the same zeros as $L(\tau; \cdot)$ in the domain $\operatorname{Re} s > 1/2$.

Suppose $\operatorname{Re} s > 1$. We have

$$E(s) = 1 + \sum_n \frac{r_n}{n^s}, \tag{56}$$

where summation is over all $n \geq 2$, $n \in \mathbb{Z}$. If for some real σ we have

$$\sum_n \frac{|r_n|}{n^\sigma} < 1, \tag{57}$$

where summation is as in (56), then the function E has no zeros in the half-plane $\operatorname{Re} s \geq \sigma$. Calculations show that inequality (57) is valid for $\sigma = 1.26$, but not for $\sigma = 1.25$. To get a better result, consider the product of the series (56) by the sum of several terms of the series inverse to the series (56) (see (40)–(43) above) and expand this product in a Dirichlet series. With some $h_k \in \mathbb{C}$ we have

$$\Big\{1 + \sum_m \frac{f_m}{m^s}\Big\}\Big\{1 + \sum_n \frac{r_n}{n^s}\Big\} = 1 + \sum_k \frac{h_k}{k^s}, \tag{58}$$

where summation is over $m, n, k \in \mathbb{Z}$ that satisfy $Q \geq m \geq 2$, $n \geq 2$, $k > Q$. Here Q is a free parameter to be chosen later. If for some real σ we have

$$\sum_m \frac{|f_m|}{m^\sigma} < 1, \quad \sum_k \frac{|h_k|}{k^\sigma} < 1 \tag{59}$$

with summation as in (58), then the function E has no zeros in the half-plane $\operatorname{Re} s \geq \sigma$. We take $Q = 18$. We have

$$r_4 = 1, \quad r_7 = 1.7919064\ldots, \quad r_9 = 2.5320888\ldots, \quad r_{13} = 0.5052408\ldots,$$

and $r_n = 0$ for all other n with $2 \leq n \leq 18$. Relation (41) implies that

$$f_4 = -r_4, \quad f_7 = -r_7, \quad f_9 = -r_9, \quad f_{13} = -r_{13}, \quad f_{16} = -r_4 f_4 = r_4^2,$$

and $f_n = 0$ for the other n with $2 \le n \le 18$. The coefficients r_n were defined for $n \ge 1$, $n \in \mathbb{Z}$, in (36) and (37). We extend this definition: let $r_n = 0$ for $n \notin \mathbb{Z}$. Then

$$h_k = r_k + f_4 r_{k/4} + f_7 r_{k/7} + f_9 r_{k/9} + f_{13} r_{k/13} + f_{16} r_{k/16} \tag{60}$$

for every $k \in \mathbb{Z}$, $k > 18$. Our calculations show that inequalities (59) are valid for $\sigma = 1.2$, as required. Of course, when calculating the sum over k in (59), we restrict ourselves to the terms that correspond to $k \le X$ for some large X. We explain how to control the discrepancy, i.e., the sum

$$\sum_k \frac{|h_k|}{k^\sigma} \tag{61}$$

over $k \in \mathbb{Z}$ with $k > X$. With the help of (60), estimation of the sum (61) reduces to that of the sum

$$\sum_k \frac{|r_k|}{k^\sigma} \tag{62}$$

over all $k \in \mathbb{Z}$ that satisfy $k > Y = X/q$ with $q = 1, 4, 7, 9, 13, 16$. By (36) and (37), estimation of (62) reduces to that of the sum

$$\sum_c \frac{|G(c)|}{\|c\|^{\sigma+1/2}} \tag{63}$$

over all square free c congruent to 1 (mod 3) in $\mathbb{Z}[\omega]$ such that $\|c\| > Y$. Finally, see (10) and (21), the sum (63) is equal to

$$\frac{\zeta_*(\sigma)}{\zeta_*(2\sigma)} - \sum_c \frac{1}{\|c\|^\sigma}, \tag{64}$$

where summation is over all square free c congruent to 1 (mod 3) in $\mathbb{Z}[\omega]$ such that $\|c\| \le Y$. Here the sum over c is finite, and the calculation of the values of ζ_* reduces, by (20) and (19), to calculating certain values of the Riemann zeta function and the Dirichlet L-function $L(\cdot, \chi)$ with a quadratic character χ mod 3. Thus, we have a method to obtain a good estimate for the discrepancy in question. We omit the details for brevity. $\square$

Consider the function $\widehat{E}$ and the corresponding Dirichlet series, see (40) and (43). If the series (43) converges absolutely in the half-plane $\operatorname{Re} s > 1$, then E has no zeros in this half-plane. Then (by Theorem 6) the function $L(\tau; \cdot)$ also has no zeros ρ with $\operatorname{Re} \rho > 1$, as claimed in conjecture (I). The supposition that (43) converges absolutely for $\operatorname{Re} s > 1$ looks fairly plausible. Direct calculations with the help of (41) do not detect any fast growth of $|f_n|$ as n grows. It is not hard to calculate f_n explicitly in the case of $n = p^k$, where p is a prime number and $k \ge 0$, $k \in \mathbb{Z}$. If $p \equiv 2 \pmod 3$, then $f_n = 0$ for k odd, and $f_n = (-1)^{k/2}$ for k even. If $p \equiv 1 \pmod 3$, then $f_n = x_p^k + x_p^{k-2} + \cdots + x_p^{-k}$, where $x_p = -2/\bigl(r_p + \sqrt{r_p^2 - 4}\bigr)$. Since $|x_p| = 1$, we have $|f_n| \le (k+1) = d(n)$. If $p = 3$, then $f_n = 0$ for k odd and $f_n = (-r_9)^{k/2}$ for k even ($r_9 = 2.53\ldots$). However, it is unclear how to estimate $|f_n|$ for all $n \ge 1$, $n \in \mathbb{Z}$.

For a real number $T > 0$, we define $N(T)$ as the number of the zeros ρ of $L(\tau; \cdot)$ (counted with multiplicities) such that $0 < \operatorname{Im} \rho < T$. In the next theorem we give an asymptotic formula for $N(T)$ similar to the Riemann–Mangoldt formula. In the proof, we apply the Bachlund method.

Theorem 9. *Let $A = 2/\pi$, and let $B = -\bigl(2 + \log(4\pi^2/27)\bigr)/\pi$. For $T \ge 2$ we have*

$$N(T) = AT \log T + BT + S(T) \quad \textit{and} \quad S(T) \ll \log T.$$

Proof. Like in Theorem 1, put $\Lambda(s) = \Omega(s)L(\tau; s)$, where $\Omega(s)$ is as in (23) and $s \in \mathbb{C}$. We have

$$L(\tau; s) = (2/\sqrt{3})27^s D(s) \quad \text{for all} \quad s \in \mathbb{C},$$

$$D(s) = 1 + \sum_m \frac{c_m}{m^s} \quad \text{for} \quad s \quad \text{with} \quad \operatorname{Re} s > 1, \tag{65}$$

$$\sum_m \frac{|c_m|}{m^2} < 1 \tag{66}$$

where $c_m \in \mathbb{R}$ are as in Theorem 7, and summation is over all $m \geq 2$, $m \in \mathbb{Z}$. To deduce (66), we observe that $c_2 = c_3 = 0$, $c_4 = 1$, $c_5 = c_6 = 0$ and apply (52) to estimate $|c_m|$ with $m \geq 7$. We omit technical details for brevity.

Suppose that none of the zeros of $L(\tau; \cdot)$ has imaginary part equal to T. Consider the closed contour C that consists of the line segment from the point 2 to the point $2 + iT$, the line segment from $2 + iT$ to $-1 + iT$, the line segment from $-1 + iT$ to -1, and some curve passing from the point -1 to the point 2 slightly above the real axis so that there are no zeros of $L(\tau; \cdot)$ between the real axis and this curve. By Theorem 1, the function Λ is holomorphic everywhere except for the points $5/6$ and $1/6$, which lie neither on C nor in the domain bounded by C. Note also that the nonreal zeros of Λ are the same and of the same multiplicity as those of $L(\tau; \cdot)$. By Theorem 8, these zeros lie in the strip $-0.2 < \operatorname{Re} s < 1.2$. By the argument principle, the increment $\Delta_C \arg \Lambda(s)$ of the argument of Λ along the contour C is equal to $2\pi N(T)$. Let P be the part of C that consists of the line segment from 2 to $2 + iT$ and the line segment from $2 + iT$ to $1/2 + iT$, and let P' be the part of C that consists of the line segment from $1/2 + iT$ to $-1 + iT$ and the line segment from $-1 + iT$ to -1. We have $\Delta_{P'} \arg \Lambda(s) = \Delta_P \arg \Lambda(s)$, because $\Lambda(s) = \overline{\Lambda(1 - \overline{s})}$ for all $s \in \mathbb{C}$, see Theorem 1. Next, the argument increment along the lower part of the contour, i.e., along the curve connecting -1 and 2, is a constant independent of T. Thus,

$$\pi N(T) = \Delta_P \arg \Lambda(s) + O(1)$$

with an absolute constant in O. The increment of the argument of Λ is equal to the sum of the corresponding increments for the function D, the functions $s \mapsto (2\pi)^{-2s}$ and $s \mapsto 27^s$, and the functions $s \mapsto \Gamma(s \pm 1/6)$, see (23). The argument increments for the functions $s \mapsto (2\pi)^{-2s}$ and $s \mapsto 27^s$ along P are equal, respectively, to $-2T \log(2\pi)$ and $T \log(27)$. For the functions $s \mapsto \Gamma(s \pm 1/6)$, the argument increments are equal to $\operatorname{Im} \log \Gamma\big((1/2 + iT) \pm 1/6\big)$ and can be found with the help of the Stirling formula [17]. Thus, we get

$$\pi N(T) = \pi\,(A\,T \log T + B\,T) + \Delta_P \arg D(s) + O(1).$$

Now we consider $\Delta_P \arg D(s)$. Recall that P consists of two line segments from 2 to $2 + iT$ and from $2 + iT$ to $1/2 + iT$. By (65), (66), the increment of the argument of D along the first segment is within $-\pi/2$ and $\pi/2$. Next, let Q be the number of points s on the second segment at which $\operatorname{Re} D(s)$ vanishes. The increment of the argument of D along this segment lies between $-(Q+1)\pi$ and $(Q+1)\pi$, this is a well-known fact. The number Q can be viewed as the number of zeros of the function

$$F_T(s) = D(s + iT) + D(s - iT)$$

on the segment I of the real axis from the point 2 to the point $1/2$. Let $K_r = \{s \in \mathbb{C} \mid |s - 2| \leq r\}$ be the disk of radius r centered at 2, and let $n(r)$ be the number of zeros (counted with multiplicities) for the function F_T in K_r. Obviously, $I \subset K_{3/2}$, $Q \leq n(3/2)$, and to prove the theorem it suffices to show that $n(3/2) \ll \log T$. Take a real number

R slightly larger than $3/2$ and such that F_T has no zeros on the boundary of K_R. To estimate $n(3/2)$, we apply the Jensen formula:

$$\int_0^R \frac{n(r)}{r}\, dr = \frac{1}{2\pi}\int_0^{2\pi} \log|F_T(2+Re^{i\vartheta})|\, d\vartheta - \log|F_T(2)|. \tag{67}$$

Note that $F_T(2) = 2\operatorname{Re} D(2+iT)$. Formula (65) implies that $F_T(2) \neq 0$ and that $\log|F_T(2)| \ll 1$ with an absolute constant in $\ll$. For some real $h > 0$, we have

$$\left|F_T(2+Re^{i\vartheta})\right| \le \left|D(2+Re^{i\vartheta}+iT)\right| + \left|D(2+Re^{i\vartheta}-iT)\right| \le T^h,$$

whence the integrand in (67) satisfies

$$\log|F_T(2+Re^{i\vartheta})| \le h\log T.$$

Thus, we arrive at the inequality

$$\int_0^R \frac{n(r)}{r}\, dr \ll \log T. \tag{68}$$

On the other hand,

$$\int_0^R \frac{n(r)}{r}\, dr \ge \int_{3/2}^R \frac{n(r)}{r}\, dr \ge n(3/2)\int_{3/2}^R \frac{1}{r}\, dr. \tag{69}$$

Estimates (68) and (69) show that $n(3/2) \ll \log T$, which completes the proof. □

For real σ and T, define $N(\sigma, T)$ as the number of the zeros ρ of $L(\tau;\cdot)$ (counted with multiplicities) such that $\operatorname{Re}\rho \ge \sigma$ and $0 < \operatorname{Im}\rho < T$. There is a certain relationship between $N(\sigma, T)$ and the behavior in the mean of the functions $t \mapsto |L(\tau;\sigma+it)|^2$. In the general context of finite order functions given by Dirichlet series, this issue was considered, e.g., in [18]. The following theorem is "extracted" from [18, 6.2.3].

Theorem 10. *For any real $\sigma \ge 0$, if*

$$\int_1^T |L(\tau;\sigma+it)|^2\, dt \ll T, \ \ \textit{then}\ \ N(\sigma,T) \ll T \ \ \textit{as}\ \ T\to\infty \tag{70}$$

(*the constants meant in the symbols $\ll$ depend on σ only*).

If the estimate on the right in (70) is valid for some σ, then so it is with any larger σ. We do not know the infimum of the set of all σ for which estimates in (70) are valid.

Theorem 11. *If $\sigma > 3/4$, then $N(\sigma,T) \ll T$ as $T \to \infty$.*

Proof. It suffices to prove the claim for $\sigma < 1$. We apply Theorem 10. We need to check that the estimate

$$\int_{T/2}^T |L(\tau;\sigma+it)|^2\, dt \ll T \tag{71}$$

is valid with a constant in $\ll$ depending only on σ, provided $1 > \sigma > 3/4$, $T \ge 4$. Clearly, if we sum the estimates (71) that correspond to T, $T/2$, $T/4$, ..., then we get an estimate of the form $\ll T$ for the integral on the left in (70). It remains to prove (71).

In the calculations below, it is assumed that $1 > \sigma > 1/2$ and $t \ge 2$. By the Stirling formula [17], we obtain

$$\left|\frac{\Omega(1-\sigma-it)}{\Omega(\sigma+it)}\right| \ll t^{2-4\sigma} \tag{72}$$

with an absolute constant in $\ll$. Let $\varepsilon > 0$. Theorem 3 and estimate (72) imply

$$|L(\tau;\sigma+it)|^2 \ll \left|\sum_\nu \frac{\tau(\nu)}{\|\nu\|^{\sigma+it}}\right|^2 + t^{4-8\sigma}\left|\sum_\nu \frac{\tau(\nu)}{\|\nu\|^{1-\sigma-it}}\right|^2 + t^{2-4\sigma+2\varepsilon}, \tag{73}$$

where summation over ν is as in (1) and (3), under the condition $\|\nu\| \le (2/\pi^2)\, t^2$, and the constant in $\ll$ depends only on ε. Consequently,

$$\int_{T/2}^{T} |L(\tau;\sigma+it)|^2\, dt \ll A + T^{4-8\sigma} B + T^{3-4\sigma+2\varepsilon}, \tag{74}$$

where the constant in $\ll$ depends only on ε, and

$$A = \int_{T/2}^{T} \Big| \sum_{\nu} \frac{\tau(\nu)}{\|\nu\|^{\sigma+it}} \Big|^2 dt, \quad B = \int_{T/2}^{T} \Big| \sum_{\nu} \frac{\tau(\nu)}{\|\nu\|^{1-\sigma-it}} \Big|^2 dt$$

with summation as in (73). Consider the integral A. Theorem 7 yields

$$A = \frac{2}{\sqrt{3}}\, 27^{\sigma} \int_{T/2}^{T} \Big| \sum_{a,b} \frac{c_{ab^3}}{(ab^3)^{\sigma+it}} \Big|^2 dt, \tag{75}$$

where summation is over all positive cube free $a \in \mathbb{Z}$ and all positive $b \in \mathbb{Z}$ satisfying $ab^3 \le z_t = (54/\pi^2)t^2$. Let $\eta \in \mathbb{R}$. The integrand in (75) can be written as

$$\Big| \sum_{b} \frac{1}{b^{\eta}} \Big(\frac{1}{b^{3\sigma+3it-\eta}} \sum_{a} \frac{c_{ab^3}}{a^{\sigma+it}} \Big) \Big|^2 \tag{76}$$

where summation is over all positive integers $b \le z_t^{1/3}$ and all positive cube free integers $a \le z_t/b^3$. Applying the Cauchy inequality, we see that the expression in (76) does not exceed the quantity

$$\Big(\sum_{b} \frac{1}{b^{2\eta}} \Big) \Big(\sum_{b} \frac{1}{b^{6\sigma-2\eta}} \Big| \sum_{a} \frac{c_{ab^3}}{a^{\sigma+it}} \Big|^2 \Big), \tag{77}$$

with summation over a and b as in (76). Let $\eta > 1/2$. Then the first sum over b in (77) is dominated by a constant depending only on η (not on z_t). This leads to the estimate

$$A \ll \sum_{b} \frac{1}{b^{6\sigma-2\eta}} \int_{T/2}^{T} \Big| \sum_{a} \frac{c_{ab^3}}{a^{\sigma+it}} \Big|^2 dt, \tag{78}$$

where summation is over all positive integers $b \le z_T^{1/3}$ and all positive cube free integers $a \le z_t/b^3$. The integral in (78) is equal to

$$\sum_{a,a'} \frac{c_{ab^3} c_{a'b^3}}{(aa')^{\sigma}} \int_{\gamma_{T,b}^{a,a'}}^{T} \Big(\frac{a'}{a} \Big)^{it} dt, \tag{79}$$

where $\gamma_{T,b}^{a,a'} = \max\{\pi\sqrt{ab^3/54}, \pi\sqrt{a'b^3/54}, T/2\} \le T$, and summation is over $a, a' \in \mathbb{Z}$ from 1 to z_T/b^3. Computing the integrals and using estimate (52) in Theorem 7, we find that (79) is dominated by the expression

$$T \sum_{a} \frac{c_{ab^3}^2}{a^{2\sigma}} + \sum_{a \ne a'} \frac{|c_{ab^3} c_{a'b^3}|}{(aa')^{\sigma} |\log(a'/a)|} \ll d(b)^2 b \Big\{ T \sum_{a} \frac{d(a)^2}{a^{2\sigma}} + \sum_{a \ne a'} \frac{d(a)d(a')}{(aa')^{\sigma} |\log(a'/a)|} \Big\} \tag{80}$$

with summation over $a, a' \in \mathbb{Z}$ running from 1 to z_T/b^3, $a' \ne a$, and with an absolute constant in $\ll$. On the right in (80), the first sum (over a) is $\ll 1$ because $2\sigma > 1$ and, for any real $\delta > 0$, we have $d(m) \ll m^{\delta}$ as $m \to \infty$. To estimate the second sum (over a and a'), we use the standard estimate[4]

$$\sum_{1 \le m < n \le x} \frac{d(m)d(n)}{(mn)^{\vartheta} \log(n/m)} \ll x^{2-2\vartheta+\delta}, \tag{81}$$

[4]Note that, in (81), summation is over all $m, n \in \mathbb{Z}$ with $1 \le m < n \le x$, and that the constant in $\ll$ only depends on ϑ and δ, but not on x. See [19, 20].

which is valid for all real x, ϑ, and δ satisfying $\vartheta \le 1$, $\delta > 0$. We apply (81) with $\vartheta = \sigma$ and $x = z_T/b^3$. Recall that $z_T = (54/\pi^2)T^2$. As a result, for the expression (80) and, with it, for the integral (78), we get an estimate of the form

$$\ll d(b)^2 b \left\{T + (T^2/b^3)^{2-2\sigma+\delta}\right\}. \tag{82}$$

Plugging (82) in (78), we obtain

$$A \ll \sum_b \frac{d(b)^2 b}{b^{6\sigma-2\eta}} \left\{T + \left(\frac{T^2}{b^3}\right)^{2-2\sigma+\delta}\right\} = \left(\sum_b \frac{d(b)^2}{b^{6\sigma-1-2\eta}}\right) T + \left(\sum_b \frac{d(b)^2}{b^{5-2\eta+3\delta}}\right) T^{4-4\sigma+2\delta} \tag{83}$$

where $\delta > 0$ is arbitrary and summation is over all positive integers $b \le z_T^{1/3}$. Taking $\eta > 1/2$ sufficiently small, we see that $6\sigma - 1 - 2\eta > 1$, $5 - 2\eta + 3\delta > 1$; consequently, the sums over b in (83) are $\ll 1$. Thus, we arrive at the final estimate

$$A \ll T + T^{4-4\sigma+\delta} \tag{84}$$

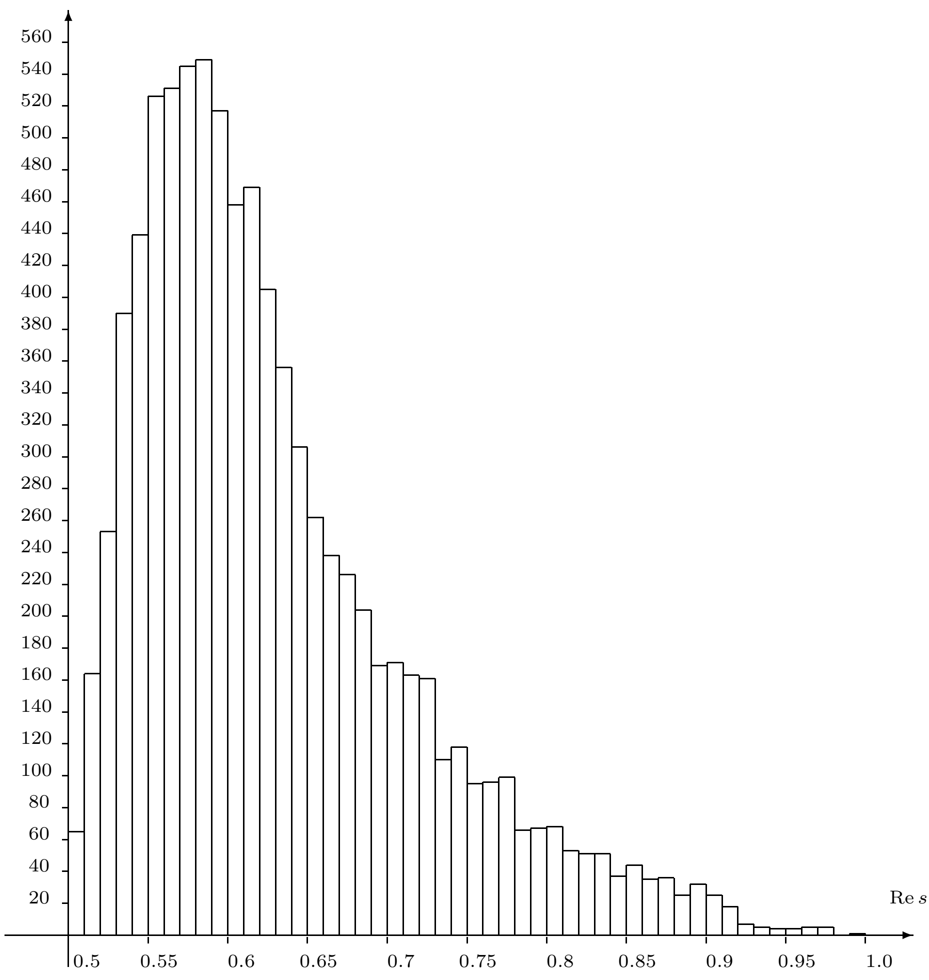

FIGURE 1. The histogram of the distribution of the points σ_1, σ_2, $\ldots$, σ_{8724}. The segment $[0.5, 1]$ is split into 50 segments of length 0.01. If $[a, b]$ is one of them, then the height of the column above it is equal to the number of n such that $a < \sigma_n \le b$.

with an arbitrary real $\delta > 0$. Similarly, for B in (74) we find the estimate

$$B \ll T^{4\sigma+\delta} \tag{85}$$

with an arbitrary real $\delta > 0$. Note that the constants in $\ll$ in (84) and (85) only depend on σ and δ, but not on T. Now, to obtain (71) it remains to substitute (84) and (85) in (74) and to take δ sufficiently small. □

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