

ON THE CUSPIDAL SPECTRUM FOR FINITE VOLUME SYMMETRIC SPACES

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1. Introduction

Let $K \backslash G/\Gamma$ be a noncompact locally symmetric space of finite volume. Here G is a semisimple Lie group and Γ is an arithmetic subgroup. Moreover, K is a maximal compact subgroup.

If Δ is the Laplacian on $K \backslash G/\Gamma$, we consider Δ acting on the cuspidal functions $L^2_{\text{cusp}}(K \backslash G/\Gamma)$ in the sense of Langlands [14]. Our main result is the following:

Theorem 1.1. *Let $N(\lambda)$ be the number of linearly independent cuspidal eigenfunctions with eigenvalue less than λ . Then $N(\lambda)$ is finite for each fixed $\lambda > 0$.*

Moreover, one has the asymptotic upper bound:

$$(1.2) \quad \overline{\lim}_{\lambda \rightarrow \infty} \frac{N(\lambda)}{\lambda^{d/2}} \leq (4\pi)^{-d/2} \frac{\text{vol}(K \backslash G/\Gamma)}{\Gamma(d/2 + 1)}.$$

Here d is the dimension of $K \backslash G/\Gamma$ and vol denotes the volume. Also, $\Gamma(d/2 + 1)$ is the ordinary Gamma function.

The fact that $N(\lambda)$ is finite for fixed $\lambda > 0$ was announced by Borel and Garland [2], [10].

If $G = \text{SL}(2, R)$, then Theorem 1.1 has apparently been well known for some time. It certainly follows from the scattering theory of [15], although the explicit estimate is not stated there. Several authors [21] have given more detailed information for particular discrete subgroups Γ of $\text{SL}(2, R)$. In the case $\Gamma = \text{SL}(2, Z)$, equality holds in (1.2) and the limit on the left-hand side exists [15], [20].

When G is a real rank one, Gangolli and Warner [9] obtained the estimate $N(\lambda) \leq C\lambda^n$, for some C and n . However, their method did not give a good estimate of n .

Theorem 1.1 was proved for real rank one in the author's earlier paper [6]. The arguments given below are a natural development of the approach initiated in this earlier work. Note that for the present paper, $K \backslash G/\Gamma$ may have arbitrary rank.

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2. Basic facts concerning arithmetic groups

This section summarizes some standard facts concerning semisimple Lie groups G and arithmetic subgroups Γ . For more details the reader is referred to [2] and [14].

Let $P = MAN$ be a parabolic subgroup of G . The parabolic subgroups P_θ belonging to P are in one-one correspondence with subsets θ of the simple roots Ψ of \mathfrak{a} , the Lie algebra of A . We may write $P_\theta = M_\theta A_\theta N_\theta$ where $N_\theta \subset N$, $A_\theta \subset A$, and $M_\theta \supset M$. The Lie algebra of N_θ consists of those positive roots containing at least one simple root not belonging to θ . We denote $S_\theta = M_\theta N_\theta$ and $S = MN$.

We denote the simple roots of \mathfrak{a} by $\alpha_1, \alpha_2, \dots, \alpha_k$. Set $A_c = \exp\{v \in \mathfrak{a} \mid \alpha_i(v) \geq c, \text{ for all } i\}$. Here c is a real number and $\exp: \mathfrak{a} \rightarrow A$ is the diffeomorphism induced by the exponential map.

Suppose that P is a parabolic in the sense of Langlands [14]. In particular, $\Gamma \cap P \subseteq S$ and $S/\Gamma \cap S$ is compact. Moreover, for any parabolic P_θ belonging to P one has $\Gamma \cap P_\theta \subseteq S_\theta$, $N_\theta/\Gamma \cap N_\theta$ is compact, and $S_\theta/\Gamma \cap S_\theta$ has finite volume. All parabolic subgroups are conjugate in G .

If $P = MAN$ is any parabolic, then set $\mathcal{S}_c(P) = K \backslash MA_c N/\Gamma \cap P$, for any real number c . One may choose a finite set Ω of parabolic subgroups P so that $K \backslash G/\Gamma$ is covered by $\bigcup_{P \in \Omega} \mathcal{S}_c(P)$, for some real number c ,

3. The metric on the cusp

Let $P = MAN$ be a parabolic. The manifold with boundary $\mathcal{S}_c(P)$ will be referred to as the cusp.

By proper choice of base point, we may assume that $K \cap P = K \cap M$, as is done in [4, p. 246]. We denote $K \backslash P_c = K \backslash MA_c N = ZA_c N$, where $Z = K \backslash M$. Then $K \backslash P_c$ is contained in $K \backslash G$ and the Killing form of G induces a right invariant metric on $K \backslash P_c$.

For each $(z, a) \in ZA_c$, the metric of $K \setminus P_c$ restricts to a metric on N . It is well known [4, p. 246] that this metric has uniformly bounded dependence on z , so the metric will be denoted by g_a . The crucial point is to understand the dependence of g_a upon a . One obtains a flat metric \hat{g}_a , on the Lie algebra \mathfrak{n} of N , by identifying \mathfrak{n} with the tangent space of N at the identity. Since N is a simply connected nilpotent Lie group, the exponential map $\exp: \mathfrak{n} \rightarrow N$ is a diffeomorphism. Here we mean the group exponential map of N , which does not depend upon a choice of metric. Pulling back the metric \hat{g}_a by $(\exp)^{-1}$ one may define a metric h_a on N .

It will be useful to employ a comparison of the metric g_a and h_a .

Lemma 3.1. *For ϵ sufficiently small, one has, in a g_a ball of radius ϵ about the identity element, $g_a \geq C_1 h_a$. Here C_1 is independent of a .*

Proof. For a fixed value a_0 of a one has, for some $\epsilon > 0$, $g_{a_0} \geq C_1 h_{a_0}$, since \exp is a diffeomorphism with differential the identity map. However, for any a , $zan = za_0(b^{-1}nb)b^{-1}$, where $b = a^{-1}a_0 \in A$. Since the Killing metric of $K \setminus P_c = ZA_cN$ is right invariant, $g_a = \text{Ad}_b g_{a_0}$ and $\hat{g}_a = \text{Ad}_b \hat{g}_{a_0}$. Notice that A normalizes N . The lemma now follows from the commutative diagram:

$$\begin{array}{ccc} \mathfrak{n} & \xrightarrow{\exp} & N \\ \text{Ad}_b \downarrow & & \downarrow \text{Ad}_b \\ \mathfrak{n} & \xrightarrow{\exp} & N \end{array}$$

The metric $(d\omega)^2$ on $K \setminus P_c$ is described very explicitly in [4, p. 247]. In fact, one may write:

$$(3.2) \quad (d\omega)^2 = dz^2 + dr^2 + \sum_{\beta \in \Phi} e^{-2\beta(r)} (d\omega_\beta(z))^2.$$

Here $r = (r_1, r_2, \dots, r_k)$ are coordinates on A_c , obtained from the exponential map of A , $\exp: \mathfrak{a} \rightarrow A$. In fact, $r_i(x) = \alpha_i(x)$, for $x \in \mathfrak{a}$, where α_i are the simple positive roots. Note that $\exp: \mathfrak{a} \rightarrow A$ is a diffeomorphism, which allows us to identify \mathfrak{a} with A . We may assume that A_c is parameterized by $r_i \geq c$, for all $1 \leq i \leq k$. The β belong to the set of positive roots Φ of \mathfrak{a} .

As given by (3.2), g_a is the right invariant metric on N which satisfies $g_a = \sum e^{-2\beta(r)} (d\omega_\beta(z))^2$ at the identity. It is difficult to obtain estimates on g_a directly since the distributions defined by the root spaces, i.e. the $d\omega_\beta(z)$ are not integrable. Thus g_a is not a product metric.

However, the metric h_a is a product metric, along the root spaces in \mathfrak{n} , which agrees with g_a at the identity. Of course, h_a is not right invariant with respect to N . Nevertheless, it is easier to estimate geometric quantities in h_a . This explains the utility of Lemma 3.1.

A key technical lemma is:

Lemma 3.3. *Let $\rho(x, y)$ denote the geodesic distance in the metric $(d\omega)^2$. Then one has, for ε sufficiently small, and any x, y , points in a fundamental domain for $\Gamma \cap N$:*

$$\sum_{\substack{\rho(x, y\gamma) < \varepsilon \\ 1 \neq \gamma \in \Gamma \cap N}} \frac{1}{\rho(x, y\gamma)} \leq C_2 \left(\max_{\substack{\alpha \in \Psi \\ n_\alpha = 1}} \alpha(r) \right) \prod_{\beta \in \Phi} e^{n_\beta \beta(r)},$$

where n_β is the dimension of the root space corresponding to β . Here α runs over all simple positive roots of multiplicity one. The product in β runs over all positive roots. Moreover, $r = r(x)$, or if desired $r = r(y)$.

Proof. By Lemma 3.1 and formula (3.2), it suffices to obtain the analogous estimate for the Euclidean product metric h_α .

However, if ρ is the geodesic distance in h_α , one has

$$(3.4) \quad \sum_{\substack{\rho(x, y\gamma) < \varepsilon \\ 1 \neq \gamma \in \Gamma \cap N}} \frac{1}{\rho(x, y\gamma)} \leq C_3 \sum_{\substack{\rho_\beta(x, y\gamma) \leq C_4 e^{\beta(r)} \\ \beta \in \Phi}} \left(\sum_{\beta} e^{-\beta(r)} \rho_\beta(x, y\gamma) \right)^{-1},$$

where β are the positive roots of α in \mathfrak{n} and ρ_β is a fixed Euclidean metric on the root space corresponding to β . Thus ρ_β is independent of r .

A result of Moore [17, p. 155], states that the preimage of $\Gamma \cap N$ under $\exp: \mathfrak{n} \rightarrow N$ is commensurable to a Euclidean lattice in the Lie algebra \mathfrak{n} . Using this fact, one obtains Lemma 3.3 after replacing the right sum in (3.4) by an integral:

$$\begin{aligned} \sum_{\substack{\rho(x, y\gamma) < \varepsilon \\ 1 \neq \gamma \in \Gamma \cap N}} \frac{1}{\rho(x, y\gamma)} &\leq C_5 \sum_{\alpha \in \Psi} \left(e^{\alpha(r)} \int_1^{C_4 e^{\alpha(r)}} t^{n_\alpha - 2} dt \right) \\ &\times \prod_{\beta \in \Phi - \alpha} \int_1^{C_4 e^{\beta(r)}} t^{n_\beta - 1} dt. \end{aligned}$$

4. Neumann bracketing

Let $\phi \in L^2(K \backslash G / \Gamma)$ be a square integrable function. Suppose that P is a parabolic and $P_\theta = M_\theta A_\theta N_\theta$ is any associated parabolic.

We may define

$$(4.1) \quad T(P, \theta)\phi(x) = \int_{N_\theta / \Gamma \cap N_\theta} \phi(xn) dn$$

for $x \in K \backslash G$. Here one has identified ϕ with a Γ invariant function on $K \backslash G$. Notice that the integral in (4.1) is well defined since $N_\theta / \Gamma \cap N_\theta$ is compact. If

$T(P, \theta)\phi = 0$ for all (P, θ) , then ϕ is said to be cuspidal. If in addition $\Delta\phi = \mu\phi$, for some $\mu \geq 0$, then ϕ is a cuspidal eigenfunction and μ belongs to the cuspidal spectrum.

Choose a finite set P_1, P_2, \dots, P_r of percuspidal parabolics so that the collection $\mathfrak{S}_c(P_i)$, $1 \leq i \leq r$, covers $K \backslash G/\Gamma$. A function ψ on $\mathfrak{S}_c(P_i)$ is said to be cuspidal if $T(P_i, \theta)\psi = 0$, for the fixed parabolic P_i and all θ . Denote $\pi_i: \mathfrak{S}_c(P_i) \rightarrow K \backslash G/\Gamma$.

Now select a sequence of smooth compact manifolds with boundary $B_k \subset K \backslash G/\Gamma$ with $B_k \subset B_{k+1}$ and $UB_k = K \backslash G/\Gamma$. For each i and k , let $X_{i,k} \subset \mathfrak{S}_c(P_i)$ be a smooth manifold with boundary which contains $\mathfrak{S}_c(P_i) - \pi_i^{-1}B_k$. Suppose that $X_{i,k}N = X_{i,k}$, to guarantee that the cuspidal condition still makes sense in $L^2(X_{i,k})$. Eventually, we will wish to choose $X_{i,k}$ so that the volume of $X_{i,k}$ is sufficiently close to the volume of $\mathfrak{S}_c(P_i) - \pi_i^{-1}B_k$.

Let $\Delta_{i,k}$ be the Laplacian Δ acting on the cuspidal functions in $L^2(X_{i,k})$ which satisfy Neumann boundary conditions. Denote $N_{i,k}(\lambda)$ to be the number of cuspidal eigenfunctions in $L^2(X_{i,k})$ with eigenvalue less than λ . Similarly, we define $N_k(\lambda)$ to be the number of eigenvalues less than λ for the usual Neumann problem of the compact manifold with boundary B_k . It is not necessary to impose any cuspidal side condition in B_k .

The principle of modified Neumann bracketing developed in [6] and [15] now gives:

Proposition 4.2. *Let k be a fixed integer and suppose that $\Delta_{i,k}$ has pure point spectrum for all $1 \leq i \leq r$. If $N(\lambda)$ is the number of linearly independent cuspidal eigenfunctions on $K \backslash G/\Gamma$ with eigenvalue less than λ , then, for any value of λ :*

$$N(\lambda) \leq N_k(\lambda) + \sum_{i=1}^r N_{i,k}(\lambda).$$

A priori, $\Delta_{i,k}$ might have nonempty essential spectrum so that Proposition 4.1 would not apply. However, we will show presently that $\Delta_{i,k}$ does indeed have pure point spectrum for all i and k .

5. Interior parametrix

Let P be a fixed percuspidal parabolic. If $P_\theta = M_\theta A_\theta N_\theta$ is a cuspidal parabolic associated to P , then denote $T_\theta = T(P, \theta)$, where $T(P, \theta)$ is the cuspidal projection given by (4.1). We will normalize Haar measure on N_θ so that $\int_{N_\theta/\Gamma \cap N_\theta} dn_\theta = 1$. Recall that θ is a subset of the positive roots Ψ . It is convenient to set $\mathcal{L}_\theta = T_{\Psi-\theta}$.

The following algebraic lemma is well known [11, p. 12]:

Lemma 5.1. (i) For any $\theta \subset \Psi$, one has $\mathcal{L}_\theta = \prod_{\alpha \in \theta} \mathcal{L}_\alpha$. Here the product runs over simple positive roots contained in the subset θ .

(ii) For any $\theta \subset \Psi$ one has $\mathcal{L}_\theta^2 = \mathcal{L}_\theta$.

(iii) For any two subsets $\theta_1, \theta_2 \subset \Psi$, the associate projections commute, $\mathcal{L}_{\theta_1} \mathcal{L}_{\theta_2} = \mathcal{L}_{\theta_2} \mathcal{L}_{\theta_1}$.

Now let $X \supset \mathcal{S}_c(P) - \pi^{-1}B$ be a smooth manifold with boundary as chosen in §4. Recall that X depend upon integer parameters i, k . However, in the next two sections, both P and B are fixed so we will suppress the dependence upon i and k . Our eventual goal is to construct the fundamental solution of the heat equation problem with cuspidal interior conditions and Neumann boundary conditions on X . In this section, a parametrix satisfying the interior cuspidal conditions will be obtained. Lemma 5.1 is vital for this purpose.

Suppose $E(t, x, y)$ is the fundamental solution for the heat equation on the simply connected space $K \setminus G$. Then E is smooth on $(0, \infty) \times K \setminus G \times K \setminus G$ and satisfies the estimates [5]:

$$\begin{aligned}
 |E(t, x, y)| &\leq C_1 t^{-d/2} \exp\left(\frac{-\rho^2(x, y)}{4t}\right), \\
 \left|\frac{\partial E}{\partial \rho}(t, x, y)\right| &\leq C_2 t^{-d/2} (\rho/t) \exp\left(\frac{-\rho^2(x, y)}{4t}\right) \\
 &\quad + C_3 t^{-d/2} \exp\left(\frac{-\rho^2(x, y)}{4t}\right)
 \end{aligned}
 \tag{5.2}$$

uniformly for $0 < t \leq \tau$, any $\tau > 0$. Here $\rho(x, y)$ is the geodesic distance from x to y in $K \setminus G$ and d is the dimension of $K \setminus G$.

Let $P = MAN$. Then $G = KMAN$, and by proper choice of base point one has $K \setminus G = (K \cap M \setminus M)AN = ZAN$. Set $Y = K \setminus G / \Gamma \cap P = ZAN / \Gamma \cap P$. Then Y is a complete Riemannian manifold. Moreover, Y contains $\mathcal{S}_c P = ZA_c N / \Gamma \cap P$, and therefore Y also contains the manifold with boundary X . In fact, X is an open set in Y .

Consider the infinite sum:

$$F(t, x, y) = \sum_{\gamma \in \Gamma \cap P} E(t, x, y\gamma).
 \tag{5.3}$$

By the results of [5], this sum converges uniformly on compact sets in $(0, \infty) \times K \setminus G \times K \setminus G$. Moreover, $F(t, x, y)$ represents the fundamental solution of the heat equation problem on Y .

Of course, $F(t, x, y)$ must be modified by projection onto the cuspidal conditions (4.1). Set

$$(5.4) \quad \begin{aligned} \bar{F}(t, x, y) &= \prod_{\alpha \in \Psi} (1 - \mathcal{L}_\alpha(y))F(t, x, y) \\ &= \sum_{\theta \subset \Psi} (-1)^{|\theta|} \mathcal{L}_\theta(y)F(t, x, y). \end{aligned}$$

Here the product runs over all simple roots and the sum runs over subsets θ of the simple roots. The projectors $\mathcal{L}_\theta(y)$ act on the third argument y of $F(t, x, y)$. It is immediate, from Lemma 5.1, that for all subsets $\psi \subset \Psi$, one has $\mathcal{L}_\psi(y)\bar{F}(t, x, y) = 0$. Thus F satisfies the cuspidal condition (4.1) and is suitable for an interior parametrix. By symmetry and isometry invariance of the heat kernel, one also has $\mathcal{L}_\psi(x)\bar{F}(t, x, y) = 0$, for all $\psi \subset \Psi$.

It is crucial to estimate the parametrix $\bar{F}(t, x, y)$ as a function of x and y for small $0 < t \leq \tau$, any fixed τ . For this purpose, we identify $x, y \in Y$ with points x, y in the universal cover $K \setminus G$, which realize the geodesic distance from x to y in Y .

Our basic technical estimate is:

Lemma 5.5. *For any fixed simple root α , let $F_\alpha(t, x, y) = (1 - \mathcal{L}_\alpha(y))F(t, x, y)$. Suppose that $0 < t \leq \tau$, where τ is fixed. One has the inequality:*

$$\begin{aligned} |F_\alpha(t, x, y)| &\leq B_1 t^{-d/2} \min(e^{-ar(x)}, e^{-ar(y)}), \\ &\times \max \left(\prod_{\beta \in \Phi} e^{n_\beta \beta(r(x))}, \prod_{\beta \in \Phi} e^{n_\beta \beta(r(y))} \right) \\ &\times \max_{\substack{\sigma \in \Psi \\ n_\sigma = 1}} \max_{x, y} (\sigma(r(x)), \sigma(r(y))) \exp(-\rho^2(x, y)/32t) \end{aligned}$$

uniformly for $x, y \in \mathcal{S}_c(P)$, any given c . Here r are the coordinates given by (3.2) and one uses the notation of Lemma 3.3.

Proof. Let $\psi = \Psi - \alpha$ be the complement of α in Ψ . Then by definition:

$$F_\alpha(t, x, y) = \sum_{\gamma \in \Gamma \cap P} E(t, x, y\gamma) - \int_{N_\psi/N_\psi \cap \Gamma} E(t, xn, y\gamma) dn.$$

Using (5.2), we estimate the term coming from the identity element $\gamma = 1$:

$$\begin{aligned} F_\alpha(t, x, y) &= \sum_{\substack{\gamma \in \Gamma \cap P \\ \gamma \neq 1}} E(t, x, y\gamma) - \int_{N_\psi/N_\psi \cap \Gamma} E(t, xn, y\gamma) \\ &\quad + O(t^{-d/2} \exp(-\rho^2(x, y)/4t)). \end{aligned}$$

The mean value theorem combined with (5.2) yields

$$|F_\alpha(t, x, y)| \leq B_2 t^{-d/2} \sum_{\substack{\gamma \in \Gamma \cap P \\ \gamma \neq 1}} \left[\frac{\rho(x, y\gamma)}{t} \exp\left(\frac{-\rho^2(x, y\gamma)}{8t}\right) + \exp\left(\frac{-\rho^2(x, y\gamma)}{8t}\right) \right] \\ \times \min(\text{diam}(x), \text{diam}(y)) + O(t^{-d/2} \exp(-\rho^2(x, y)/4t)).$$

Here $\text{diam}(x)$ is the diameter of $N_\psi/N_\psi \cap \Gamma$ at x . By formula (3.2), one has $\text{diam}(x) = O(e^{-\alpha(r)})$, where $r = r(x)$ are the coordinates on A_c used in (3.2).

It is an elementary lemma that $w e^{-w}$ is uniformly bounded for real $w \geq 0$. Consequently,

$$|F_\alpha(t, x, y)| \leq B_3 \min(e^{-\alpha(r(x))}, e^{-\alpha(r(y))}) t^{-d/2} \\ \times \sum_{\substack{\gamma \in \Gamma \cap P \\ \gamma \neq 1}} \left[\frac{1}{\rho(x, y\gamma)} \exp\left(\frac{-\rho^2(x, y\gamma)}{16t}\right) + \exp\left(\frac{-\rho^2(x, y\gamma)}{8t}\right) \right] \\ + O(t^{-d/2} \exp(-\rho^2(x, y)/4t)).$$

For any fixed $\epsilon > 0$, we employ the estimate of [5, p. 491] to obtain:

$$|F_\alpha(t, x, y)| \leq B_4 t^{-d/2} \min(e^{-\alpha(r(x))}, e^{-\alpha(r(y))}) \\ \times \left[\sum_{\substack{\gamma \in \Gamma \cap P \\ \gamma \neq 1 \\ \rho(x, y\gamma) < \epsilon}} \frac{1}{\rho(x, y\gamma)} \exp\left(\frac{-\rho^2(x, y\gamma)}{16t}\right) + \max(\text{Vol}^{-1}(x), \text{Vol}^{-1}(y)) \exp\left(\frac{-\rho^2(x, y)}{32t}\right) \right] \\ + O(t^{-d/2} \exp(-\rho^2(x, y)/4t)).$$

Here $\text{Vol}^{-1}(x) = 1/\text{Vol}(x)$, and $\text{Vol}(x)$ is the volume of $N/\Gamma \cap N$ at x .

By formula (3.2), one has $\text{Vol}^{-1}(x) = O(\prod_{\beta \in \Phi} e^{n_\beta \beta(r(x))})$. Here Φ is the collection of positive roots of \mathfrak{a} in \mathfrak{n} . Thus

$$|F_\alpha(t, x, y)| \leq B_5 t^{-d/2} \min(e^{-\alpha(r(x))}, e^{-\alpha(r(y))}) \\ \times \left[\sum_{\substack{\gamma \in \Gamma \cap P \\ \gamma \neq 1 \\ \rho(x, y\gamma) < \varepsilon}} \frac{1}{\rho(x, y\gamma)} \exp\left(\frac{-\rho^2(x, y\gamma)}{16t}\right) \right. \\ \left. + \max\left(\prod_{\beta \in \Phi} e^{n_\beta \beta(r(x))}, \prod_{\beta \in \Phi} e^{n_\beta \beta(r(y))}\right) \exp\left(\frac{-\rho^2(x, y)}{32t}\right) \right].$$

If ε is sufficiently small, then referring to (3.2) we see that for $\gamma \in \Gamma \cap P$ and $\rho(x, y\gamma) < \varepsilon$, one must have $\gamma \in \Gamma \cap N$. Therefore Lemma 3.3 applies to yield:

$$|F_\alpha(t, x, y)| \leq B_1 t^{-d/2} \min(e^{-\alpha(r(x))}, e^{-\alpha(r(y))}) \\ \times \max\left(\prod_{\beta \in \Phi} e^{n_\beta \beta(r(x))}, \prod_{\beta \in \Phi} e^{n_\beta \beta(r(y))}\right) \\ \times \max_{\substack{\sigma \in \Psi \\ n_\sigma = 1}} \max_{x, y} (\sigma(r(x)), \sigma(r(y))) \exp(-\rho^2(x, y)/32t).$$

Here σ runs over the simple roots of multiplicity one.

Using Lemma 5.5, it is easy to deduce:

Proposition 5.6. *If $\bar{F}(t, x, y)$ is the cuspidal parametrix defined by (5.4), then one has the estimate:*

$$|\bar{F}(t, x, y)| \leq B_7 t^{-d/2} \min_{\alpha \in \Psi} \min_{x, y} (e^{-\alpha(r(x))/2}, e^{-\alpha(r(y))/2}), \\ \times \max\left(\prod_{\beta \in \Phi} e^{n_\beta \beta(r(x))}, \prod_{\beta \in \Phi} e^{n_\beta \beta(r(y))}\right) \cdot \exp(-\rho^2(x, y)/32t)$$

uniformly for $x, y \in \mathcal{S}_c(P)$, $0 < t \leq \tau$, for any given c and $\tau > 0$.

Proof. For any simple root α , we have

$$\bar{F}(t, x, y) = \prod_{\substack{\beta \in \Psi \\ \beta \neq \alpha}} (1 - \mathcal{L}_\beta(y)) F_\alpha(t, x, y).$$

Moreover, the projections $\mathcal{L}_\beta(y)$, defined by (4.1), are L^∞ -bounded.

Using the definition (4.1) and Lemma 5.5, one obtains immediately:

$$\begin{aligned}
 |\bar{F}(t, x, y)| &\leq B_6 t^{-d/2} \min(e^{-\alpha(r(x))}, e^{-\alpha(r(y))}) \\
 &\quad \times \max\left(\prod_{\beta \in \Phi} e^{n_\beta \beta(r(x))}, \prod_{\beta \in \Phi} e^{n_\beta \beta(r(y))}\right) \\
 &\quad \times \max_{\substack{\sigma \in \Psi \\ n_\sigma = 1}} \max_{x, y} (\sigma(r(x)), \sigma(r(y))) \exp(-\rho^2(x, y)/32t).
 \end{aligned}$$

Here α is arbitrary.

Proposition 5.6 now follows by taking a minimum over α .

The same method gives estimates for the higher order derivatives of $F(t, x, y)$:

Proposition 5.7. *If $\bar{F}(t, x, y)$ is the cuspidal parametrix defined by (5.4), then one has the estimate:*

$$\begin{aligned}
 &\left| \left(\frac{\partial}{\partial t}\right)^i \nabla_x^j \nabla_y^k \bar{F}(t, x, y) \right| \\
 &\quad \leq B_8 t^{-d/2-i-j-k} \min_{\alpha \in \Psi} \min_{x, y} (e^{-\alpha(r(x))/2}, e^{-\alpha(r(y))/2}) \\
 &\quad \quad \times \max\left(\prod_{\beta \in \Phi} e^{n_\beta \beta(r(x))}, \prod_{\beta \in \Phi} e^{n_\beta \beta(r(y))}\right) \exp(-\rho^2(x, y)/32t).
 \end{aligned}$$

Proof. First observe that the averaging process (4.1) commutes with covariant differentiation, since N acts isometrically. One then follows routinely through the proof of Proposition 5.6 and Lemma 5.5 by using the higher order estimates of [5]

6. Boundary corrections

Let W_1 be a complete Riemannian manifold without boundary containing a submanifold W_2 with boundary ∂W_2 . We assume that W_1 and W_2 have the same dimension, i.e. the interior of W_1 is an open set in W_2 . If W_1 and W_2 are compact, then given a fundamental solution F of the heat equation on W_1 , the method of single layer potentials [19, pp. 175–194] allows one to modify F to obtain a fundamental solution of the heat equation on W_2 with Neumann boundary conditions. If the universal cover of W_1 has bounded geometry, i.e. the curvature is absolutely bounded and the injectivity radius is bounded below, and if ∂W_2 is compact, one can employ [5] to generalize the single layer potential construction given in [19]. However, when ∂W_2 is noncompact, further hypotheses are required.

We will use the single layer potential construction to modify $\bar{F}(t, x, y)$, given by (5.4), yielding a fundamental solution to the heat equation problem with Neumann boundary conditions on ∂X and cuspidal conditions on the interior of X . Here X is defined as in §5. Even though ∂X may be noncompact, its topology and geometry are precisely known outside a compact set. Thus, no serious difficulty arises when applying the methods of [19].

The basic estimates are the following:

Proposition 6.1. *Let $\bar{F}(t, x, y)$ be given by (5.4). Then set*

$$Q^{(0)}(t, x, y) = \bar{F}(t, x, y),$$

$$Q^{(m+1)}(t, x, y) = \int_0^t ds \int_{\partial X} \bar{F}(x, u, s) \frac{\partial}{\partial \nu} Q^{(m)}(u, y, t-s) du.$$

Here, the unit normal derivative $\partial/\partial \nu$ is applied to the argument u of $Q^{(m)}$.

One has the estimates, for $m \geq 1$:

$$\begin{aligned} \left| \left(\frac{\partial}{\partial t} \right)^i \nabla_x^j \nabla_y^k Q^{(m)}(t, x, y) \right| &\leq C_1^m (\Gamma(m/2))^{-1} t^{-d/2-i-j-k} \\ &\times \exp(-C_2(\sigma^2(x) + \sigma^2(y))/t) \exp(-C_3 \rho^2(x, y)/t) \\ &\times \min_{\alpha \in \Psi} \min_{x, y} (e^{-\alpha(r(x))/2}, e^{-\alpha(r(y))/2}) \\ &\times \max \left(\prod_{\beta \in \Phi} e^{n_\beta \beta(r(x))}, \prod_{\beta \in \Phi} e^{n_\beta \beta(r(y))} \right). \end{aligned}$$

The notation is that of Proposition 5.7. Moreover, $\sigma(x)$ is the distance from x to ∂X .

Proof. The argument proceeds by induction starting from Proposition 5.7. One uses the method of [19] combined with the precise description of the metric on X given in (3.2). Since ∂X is given outside a compact set by $r_i = c$, for some i , in the coordinates of §3, the details are quite straightforward.

The fundamental solution is obtained as in [19].

Theorem 6.2. *Let $\bar{E}(t, x, y) = \sum_{m=0}^{\infty} (-2)^m Q^{(m)}(t, x, y)$, where $Q^{(m)}$ are given by Proposition 6.1. Then \bar{E} is the fundamental solution of the heat equation with Neumann boundary conditions and cuspidal interior conditions on X .*

One has the estimate:

$$\begin{aligned} |\bar{E}(t, x, y)| &\leq C_4 t^{-d/2} \min_{\alpha \in \Psi} \min_{x, y} (e^{-\alpha(r(x))/2}, e^{-\alpha(r(y))/2}) \\ &\times \max \left(\prod_{\beta \in \Phi} e^{n_\beta \beta(r(x))}, \prod_{\beta \in \Phi} e^{n_\beta \beta(r(y))} \right) \exp \left(\frac{-C_3 \rho^2(x, y)}{t} \right) \\ &\times (1 + C_3 \exp(-C_2(\sigma^2(x) + \sigma^2(y))/t)). \end{aligned}$$

Here C_3 and C_4 depend only upon $Y = K \setminus G/\Gamma \cap P$. However, C_2 and C_5 are dependent upon the submanifold X .

Proof. From §5 and Proposition 6.1, it is clear that $\bar{E}(t, x, y)$ satisfies the heat equation and cuspidal interior conditions on X .

To show that $\bar{E}(t, x, y)$ satisfies Neumann boundary conditions one establishes the jump relations [19, p. 187] for the $Q^{(m)}$. This is primarily a local computation, which is undisturbed by the noncompactness of X .

The upper bound for \bar{E} follows by writing $\bar{E} = Q^{(0)} + (\bar{E} - Q^{(0)})$ and quoting the estimates of Propositions 5.6 and 6.1.

7. Spectral function on the cusp

In this section we give an asymptotic upper bound for $N_X(\lambda)$. Here $N_X(\lambda)$ denotes the number of eigenvalues less than λ for the Laplacian with cuspidal interior conditions and Neumann boundary conditions on X , defined as in §6.

We begin with the following elementary lemma [7]:

Lemma 7.1. *Let B denote a nonnegative self adjoint operator acting on a Hilbert space. Suppose that the associated heat operator $\exp(-tB)$ is trace class, for all $t > 0$. Then B has pure point spectrum, so we may define $N_B(\lambda)$ as the number of eigenvalues of B less than λ . If, for some positive integer d ,*

$$\overline{\lim}_{t \rightarrow 0} t^{d/2} \text{Tr}(e^{-tB}) \leq D_1$$

then

$$\overline{\lim}_{\lambda \rightarrow \infty} \lambda^{-d/2} N_B(\lambda) \leq D_1 e.$$

Consider the Laplacian Δ_X acting on $L^2 X$ with Neumann boundary conditions and cuspidal interior conditions. The associated heat kernel $\bar{E}(t, x, y)$ for $\exp(-t\Delta_X)$ is estimated in Theorem 6.2. One may deduce:

Theorem 7.2. *The heat kernel $\bar{E}(t, x, y)$ defines a trace class operator $\exp(-t\Delta_X)$. Moreover, one has the estimate:*

$$\text{Tr}(e^{-t\Delta_X}) \leq D_2 t^{-d/2} \int_{r(X)} \min_{\alpha \in \Psi} (e^{-\alpha(r)/2}) dr + O(t^{-d/2+1/2}).$$

The constant D_2 depends only upon $Y = K \setminus G/\Gamma \cap P$. Here $r(X)$ is the set of r coordinates, as in (3.2), for points in X .

Proof. By the spectral theory of self adjoint operators, $\bar{E}(t, x, y)$ satisfies the semigroup property:

$$(7.3) \quad \bar{E}(t, x, y) = \int_X \bar{E}(t, x, z) \bar{E}(t, z, y) dz$$

and symmetry $\bar{E}(t, x, y) = \bar{E}(t, y, x)$.

Setting $x = y$, and integrating we find that

$$(7.4) \quad \int_X |\bar{E}(t, x, y)|^2 dx dy = \int_X \bar{E}(t, x, x) dx.$$

The key estimate of Theorem 6.2 now gives, for small $t > 0$:

$$(7.5) \quad \int_X \bar{E}(t, x, x) dx \leq D_3 t^{-d/2} \int_{r(X)} \min_{\alpha \in \Psi} (e^{-\alpha(r)/2}) dr + O(t^{-d/2+1/2}).$$

The integral on the right-hand side of (7.5) converges, so \bar{E} is Hilbert-Schmidt by (7.4). However, the semigroup property (7.3) now shows that \bar{E} is trace class. Then (7.5) gives the required upper bound for $\text{Tr}(e^{-t\Delta_X})$.

It is convenient to denote $\mathfrak{N}(X) = \int_{r(X)} \min_{\alpha \in \Psi} (e^{-\alpha(r)/2}) dr$.

From Lemma 7.1 and Theorem 7.2, one has immediately:

Corollary 7.6. *Let X be as in the first paragraph of this section. Then*

$$\overline{\lim}_{\lambda \rightarrow \infty} \lambda^{-d/2} N_X(\lambda) \leq D_4 \mathfrak{N}(X).$$

The constant D_4 depends only upon $Y = K \setminus G/\Gamma \cap P$. Otherwise, D_4 is independent of the particular choice of submanifold X .

8. Proof of the main theorem

It is now a straightforward matter to complete the proof of Theorem 1.1 of the introduction. Let B_k denote an exhaustion of $K \setminus G/\Gamma$ as in §4. Suppose $X_{i,k}$ are smooth manifolds in $\mathfrak{S}_c(P_i)$ as chosen there.

One has the asymptotic estimate of Minakshisundaram-Pleijel [1]:

Proposition 8.1. *Let W be a compact Riemannian manifold with boundary. If $N_W(\lambda)$ denotes the number of eigenvalues less than λ for the Neumann problem on W , then*

$$\lim_{\lambda \rightarrow \infty} \frac{N_W(\lambda)}{\lambda^{d/2}} = (4\pi)^{-d/2} \frac{\text{Vol}(W)}{\Gamma(d/2 + 1)}.$$

Here d is the dimension of W and $\text{Vol}(W)$ is the volume of W .

If $N(\lambda)$ is the number of cuspidal eigenvalues on $K \setminus G/\Gamma$ which are less than λ , then by Proposition 4.2, for any k :

$$\overline{\lim}_{\lambda \rightarrow \infty} \frac{N(\lambda)}{\lambda^{d/2}} \leq \overline{\lim}_{\lambda \rightarrow \infty} \frac{N_k(\lambda)}{\lambda^{d/2}} + \sum_{i=1}^r \overline{\lim}_{\lambda \rightarrow \infty} \frac{N_{i,k}(\lambda)}{\lambda^{d/2}}.$$

Here $N_k(\lambda)$ is the number of eigenvalues less than λ for the Neumann problem on the compact Riemannian manifold B_k .

Using Proposition 8.1 one obtains

$$\overline{\lim}_{\lambda \rightarrow \infty} \frac{N(\lambda)}{\lambda^{d/2}} \leq (4\pi)^{-d/2} \frac{\text{Vol}(B_k)}{\Gamma(d/2 + 1)} + \sum_{i=1}^r \overline{\lim}_{\lambda \rightarrow \infty} \frac{N_{i,k}(\lambda)}{\lambda^{d/2}}.$$

Applying Corollary 7.6, one may deduce:

$$(8.2) \quad \overline{\lim}_{\lambda \rightarrow \infty} \frac{N(\lambda)}{\lambda^{d/2}} \leq (4\pi)^{-d/2} \frac{\text{Vol}(B_k)}{\Gamma(d/2 + 1)} + \sum_{i=1}^r C_i \mathfrak{N}_i(X_{i,k}).$$

However, $\lim_{k \rightarrow \infty} \text{Vol}(B_k) = \text{Vol}(K \setminus G/\Gamma)$. Moreover, with a sensible choice of $X_{i,k}$, $\lim_{k \rightarrow \infty} \mathfrak{N}_i(X_{i,k}) = 0$, for all i .

Theorem 1.1 of the introduction follows by letting $k \rightarrow \infty$ in (8.2).

9. Coefficients in a bundle

The results derived above may be extended in a routine way to suitable differential operators acting on sections of equivariant vector bundles. In fact, the constructions of [5] are valid for any second order operator, which is G -invariant, and has leading symbol given by the metric tensor. Consequently, one may follow the previous sections of the present paper line by line to obtain:

Theorem 9.1. *Let ρ be any irreducible unitary representation of K , acting on a finite dimensional space of dimension $\dim(\rho)$. Suppose that $N(\lambda)$ is the number of cuspidal eigenfunctions less than λ for the Casimir operator acting on sections of the associated vector bundle $V_\rho \rightarrow G/K$. Then one has the asymptotic upper bound:*

$$\overline{\lim}_{\lambda \rightarrow \infty} \frac{N(\lambda)}{\lambda^{d/2}} \leq (4\pi)^{-d/2} \frac{\text{Vol}(K \setminus G/\Gamma)}{\Gamma(d/2 + 1)} \dim(\rho).$$

By the argument of Matsushima-Murakami [16, p. 385], we may identify the Hodge Laplacian on p -forms with the Casimir operator on the bundle associated to the p th exterior power of the isotropy representation of K . Thus, a special case of Theorem 9.1 is:

Corollary 9.2. *Let $N(\lambda)$ be the number of cuspidal eigenfunctions with eigenvalue less than λ for the Hodge Laplacian acting on differential p -forms. Then one has the asymptotic upper bound:*

$$\overline{\lim}_{\lambda \rightarrow \infty} \frac{N(\lambda)}{\lambda^{d/2}} \leq (4\pi)^{-d/2} \frac{\text{Vol}(K \setminus G/\Gamma)}{\Gamma(d/2 + 1)} \binom{d}{p}.$$

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