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ABSTRACT

In [1], we showed that the minimum probability p° for which pairwise enabling is an optimal group testing algorithm under an unbounded Bernoulli arrival sequence model satisfies $0.430 \leq p^\circ \leq 0.568$, defined by the threshold probabilities at which certain triple enabling algorithms (operating with and without the aid of a helpful "genie", respectively) become more efficient. By deriving constructive results using the powerful but seemingly non-constructive upper bounding technique introduced by Mikhailov and Tsybakov in [2], we now sharpen this result by proving that $p^\circ \leq 0.5$ in the unbounded arrival sequence model, and that $p^\circ = 0.545$ in the finite arrival sequence model considered in [3]. Our results for unbounded arrival sequences also extend to the reservation schemes considered in [3], where we show that $0.386 \leq p_I^\circ \leq 0.387$ under the intermediate reservation model and $0.436 \leq p_G^\circ \leq 0.466$ under the Gudjohnsen reservation model, respectively.

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On the Cutoff Point for Pairwise Enabling in Multiple Access Systems*

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Abstract — **In [1], we showed that the minimum probability p° for which pairwise enabling is an optimal group testing algorithm under an unbounded Bernoulli arrival sequence model satisfies $0.430 \leq p^\circ \leq 0.568$, defined by the threshold probabilities at which certain triple enabling algorithms (operating with and without the aid of a helpful "genie", respectively) become more efficient. By deriving constructive results using the powerful but seemingly non-constructive upper bounding technique introduced by Mikhailov and Tsybakov in [2], we now sharpen this result by proving that $p^\circ \leq 0.5$ in the unbounded arrival sequence model, and that $p^\circ = 0.545$ in the finite arrival sequence model considered in [3]. Our results for unbounded arrival sequences also extend to the reservation schemes considered in [3], where we show that $0.386 \leq p_I^\circ \leq 0.387$ under the intermediate reservation model and $0.436 \leq p_G^\circ \leq 0.466$ under the Gudjohnsen reservation model, respectively.**

I. Introduction

Consider the problem of scheduling message transmissions over a slotted collision-type multiaccess channel that is shared by a set of users. For a given message transmission to be successful, there can be no others taking place in the same slot; otherwise all are lost in a *collision* and must be repeated in some other slot. We assume that at the end of each slot, the channel outcome in that slot (i.e., "idle", "success" or "collision") is revealed to the scheduling algorithm, and that the set of messages to be sent can be modelled as a *Bernoulli arrival sequence* in which each point in the sequence has either one or zero messages associated with it, independently, with respective probabilities p and $1-p$. Here various *tree conflict resolution* algorithms, or, more generally *group testing* algorithms are useful, since for $p < 1/\sqrt{2}$ they can transmit successfully each of the messages in an M-point sequence using fewer than M steps on average.

The Bernoulli probability $1/\sqrt{2}$ is a critical value under the above "Capetanakis-type" channel model because it can be shown [1, 3] that for $p \geq 1/\sqrt{2}$, no group testing algorithm uses on average fewer than the M steps required by "roll call" polling to transmit all the messages contained in an M-point sequence, while for all $p < 1/\sqrt{2}$ and all $M \geq 2$, using "pairwise enabling" reduces the average number of steps required. It does this by selecting points two at a time, and in the event of a collision, using two more slots to transmit the corresponding pair of messages, thus either saving one step or using one extra step (with probabilities $1-p^2$ and p^2, respectively) compared to roll call polling. Thus $p^\circ = 1/\sqrt{2}$ is called the *cutoff point* for roll

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call polling.

II. $p^{\circ} \leq 0.5$ for Pairwise Enabling on Unbounded Sequences

In [3], Hwang and Chang defined the cutoff point for an optimal group testing algorithm in terms of the expected number of steps required to process a finite Bernoulli arrival sequence. However, since the throughput (or efficiency) of an algorithm is inversely proportional to the required number of steps, one can as easily define the cutoff point in terms of throughput. That is, the cutoff point for an algorithm is the minimum Bernoulli arrival probability p for which its efficiency in processing Bernoulli arrival sequences is at least as high as that for any other algorithm. Under the finite arrival sequence model, the cutoff point is obtained by taking the maximum p over all possible sequence lengths, thus imposing the worst possible stopping condition on the algorithm. Under the unbounded sequence model, however, there is no stopping condition to contend with. Thus to show that the cutoff point for a given algorithm is below some threshold p^*, it is sufficient to show that its asymptotic throughput forms an upper bound to the efficiency of *all* conflict resolution algorithms under the unbounded Bernoulli arrival sequence model when $p > p^*$.

In [2], Mikhailov and Tsybakov proved that 0.587 is an upper bound to the highest attainable channel efficiency, or *capacity*, under the unbounded Poisson arrival model. The modifications to adapt their proof to the unbounded Bernoulli arrival sequence model are straightforward and appear in [4], and also in [5] as a special case of a more general result. In the Bernoulli arrival sequence model, this result may be stated in the following lemma and theorem:

Lemma 1:
If, at the tth algorithmic step, n points are selected for transmission, none of which were selected at a previous step that resulted in either an idle slot or a successful transmission, then, no matter what history led up to the selection of those n points,

$$P\,[\text{none of the } n \text{ points are busy}] \triangleq I_n^{(t)} \leq (1-p)^n$$

$$P\,[\text{exactly one of the } n \text{ points is busy}] \triangleq S_n^{(t)} \leq \frac{np\,(1-p)^{n-1}}{1-(1-p)^n}\cdot(1-I_n^{(t)}) \qquad (1)$$

Theorem 1:
Define

$$\gamma_{t,q} \triangleq D_t + q\cdot s_t,$$

where D_t and s_t represent running totals up to the end of the tth step of the number of arrival points that were selected at an idle slot or a successful transmission and of the number of successful transmissions, respectively, and q is an arbitrary non-negative weight. Then

$$E\,[\gamma_{t,q} - \gamma_{t-1,q}] \leq \max_{n>0} \{H(n,q)\},$$

where

$$H(n,q) \triangleq \max_{0 \le I_n^{(t)} \le (1-p)^n} \left\{ I_n^{(t)} \cdot [n] + (1 - I_n^{(t)}) \cdot \frac{np\,(1-p)^{n-1}}{1-(1-p)^n} \cdot [n+q] \right\}$$

$$= n(1-p)^{n-1} \cdot \max \left\{ 1 - p + (n+q)p,\ \frac{(n+q)p}{1-(1-p)^n} \right\}, \qquad (2)$$

from which it follows that

$$C_p \le \frac{p \cdot \min_q \max_{n>0} H(n,q)}{1 + p \cdot q}, \qquad (3)$$

where C_p is the capacity under the unbounded Bernoulli arrival sequence model at the given p.

It is important to note that since $H(n,q)$ is an upper bound to $E[\gamma_{i,q} - \gamma_{i-1,q}]$ given that exactly n Bernoulli points were enabled at that step, the maximization with respect to n in Eq. (3) is obviously necessary, but the minimization with respect to q is optional: every q gives us an upper bound on C_p, but by considering all $q \ge 0$, we hope to find the tightest bound.

In this section, we shall use this seemingly non-constructive method to demonstrate that the expression for the asymptotic throughput of pairwise enabling is actually an upper bound to capacity when $0.5 \le p \le 1/\sqrt{2}$. However, before continuing, we wish to establish the following useful lemmas.

Lemma 2:

For $0 \le q \le 1$, $p \ge q/(1+q)$, and $n \ge 2$,

$$H(n,q) = H_1(n,q) \triangleq n(1-p)^{n-1}[1 - p + (n+q)p]$$

Proof:

To prove the result, it is sufficient to show that under the conditions above, the first term in the maximization within Eq. (2) dominates the second. But

$$\frac{(n+q)p}{1-(1-p)^n} = \frac{(n+q)p\,(1-p)^n}{1-(1-p)^n} + (n+q)p$$

$$\le \frac{(n+q)p\,(1-p)^n}{\left(\binom{n}{2} \frac{p^2}{1-p} + np \right)(1-p)^{n-1}} + (n+q)p$$

$$\le \frac{(n+q)p\,(1-p)^n}{(n+q)p\,(1-p)^{n-1}} + (n+q)p$$

$$= 1 - p + (n+q)p,$$

where the second inequality follows because $n \ge 2$ implies $\binom{n}{2} \ge 1$, and because the inequalities $p \ge \frac{q}{1+q}$ and $\frac{p}{1-p} \ge q$ are equivalent.

Lemma 3:

For all $n \ge 1$, $p \ge 1/4$ and $0 \le q \le 1$, $H_1(n,q)$ attains its maximum at the least n for which $H_1(n,q) \ge H_1(n+1,q)$.

Proof:

It is sufficient to prove that the first order difference series, $H_1(n+1, q) - H_1(n, q)$, $n=1, 2, \cdots$, is monotonically decreasing. But, after some algebraic manipulations, we have

$$H_1(n+1, q) - H_1(n, q) = -p\ H_1(n, q) + (1-p)^n\ [1 + (2n+q)p]$$
$$= (1-p)^{n-1}\,[1 + (n-1+q)p - p^2(n+q)[n+1]]. \quad (4)$$

Since $(1-p)^{n-1}$ is a positive, monotonically decreasing function of n, and since the incremental change to the remaining term in Eq. (4), namely $p - p^2[2n+2+q]$, is clearly negative for $p \geq 1/4$ and $q \geq 0$, the result is proven.

Using the results above, we can now show that $p^\circ \leq 0.5$ for pairwise enabling. The basic idea is to guess a solution (n^*, q^*) which, if $H(n^*, q^*)$ was indeed a maximum over all n for the given q^* in the range $0.5 \leq p \leq 1/\sqrt{2}$, would imply the optimality of pairwise enabling, and then to verify that the proposed solution is indeed a maximum. The key step is to equate the right-hand side of Eq. (3), evaluated at n^*, with the expression for the asymptotic throughput of pairwise enabling, namely $2p\ /\ (1+2p^2)$, and then solve for q^*.

Suppose $n^* = 1$. Then, since Eq. (2) simplifies to

$$H(1, q) = \max\{\,1 + q\cdot p\,,\, 1 + q\,\} = 1 + q\,, \quad (5)$$

we obtain from Eq. (3) that

$$q^* = \frac{1 - 2p^2}{1-2p\,(1-p)}. \quad (6)$$

Notice that q^* decreases monotonically from 1 to 0 as p increases from 0.5 to $1/\sqrt{2}$, and thus that our proposed solution is feasible over the range of interest.

Now, in view of Eq. (5) and Lemma 2,

$$\max_{n\geq 1} H(n, q^*) = \max\Big\{\,H(1, q^*), \max_{n\geq 2} H_1(n, q^*)\,\Big\} = \max\Big\{\,1 + q^*, H_1(2, q^*)\,\Big\},$$

where the right-most equality holds because of Lemma 3 and observation that Eq. (4) gives us a negative difference at $n=2$. Remarkably, the expressions for $H(1, q^*)$ and $H_1(2, q^*)$ simplify to exactly the same form, so that

$$\max_{n\geq 1} H(n, q^*) = 1 + q^* = \frac{2(1-p)}{1 - 2p\,(1-p)}, \qquad 0.5 \leq p \leq 1/\sqrt{2}. \quad (7)$$

Thus, using Eq. (3), we obtain

$$C_p \leq \frac{p\cdot H(1, q^*)}{1 + p\cdot q^*} = \frac{2p}{1 + 2p^2} \qquad 0.5 \leq p \leq 1/\sqrt{2}.$$

Thus pairwise enabling must be an optimal group testing algorithm in the range $0.5 \leq p \leq 1/\sqrt{2}$ under the unbounded Bernoulli arrival sequence model. It is important to note, however, that we *cannot* say that $p^\circ = 0.5$ for pairwise enabling, since we have merely observed that an upper bound to capacity is tight for $p \geq 0.5$, but not for $p < 0.5$. But since no known conflict resolution algorithm has a higher asymptotic throughput than pairwise enabling for $0.430 < p < 0.5$, it follows that $0.430 \leq p^\circ \leq 0.5$ in the unbounded Bernoulli arrival sequence model.

III. $p^\circ = 0.545$ for Pairwise Enabling on Finite Sequences

Following Hwang and Chang's definition of p° for roll call polling under the finite Bernoulli arrival sequence model [3], we define p° for the following pairwise enabling strategy to be that probability for which it is always optimal for all $p^\circ \leq p \leq 1/\sqrt{2}$ and every sequence length $M \geq 1$, but if $p < p^\circ$ there exists an M for which this policy is not optimal. This strategy consists of $\lfloor M/2 \rfloor$ applications of pairwise enabling, followed by a single application of roll call to handle the remaining point (if any) when M is finite and odd.

Clearly, if p° for pairwise enabling is to *increase* as we move from infinite to finite arrival sequences, it must be a consequence of the end conditions that arise when M is finite and odd. This is because the class of infinite sequence algorithms subsumes the class of finite sequence algorithms, since we can always construct an infinite sequence algorithm from any algorithm for M-point sequences by executing the algorithm repeatedly on successive blocks of M points, and because pairwise enabling attains its same efficiency, which is optimal on unbounded sequences by assumption, whenever M is even.

Now consider the case where $M=3$. One application of pairwise enabling to handle the first two points, and followed by a single application of roll call to handle the remaining point, uses $2 + 2p^2$ steps on average. However, one application of the "triple enabling" algorithm defined in [1, Eq. (8)] is sufficient to handle all three points, using $1 + 7p^2 - 3p^3$ steps on average. Thus, since the inequality

$$\left(2 + 2p^2\right) > \left(1 + 7p^2 - 3p^3\right)$$

holds for all $p \leq 0.545$, we see that $p^\circ \geq 0.545$ for pairwise enabling under the finite arrival sequence model.

To show that $p^\circ \leq 0.545$ in this model, we can once again apply the proof technique used in the previous section. Recall that for all $0.545 < p < 1/\sqrt{2}$, $E[\gamma_{t,q^*} - \gamma_{t-1,q^*}] \leq H(1, q^*)$ must hold for all t from Eqs. (2) and (7), with equality if either one known busy point or two unexamined points are selected. Since it is easy to see by induction that any strategy using only those two "optimal" rules must terminate after having examined an *even* number of arrival points, every algorithm that terminates after having examined an *odd* number of arrival points must have made a "sub-optimal" selection of points at least once. Thus, to complete the proof, it is sufficient to show that if A^* makes its first suboptimal selection, B_{t_1}, at step t_1, and thereafter A^* makes further suboptimal selections at steps $t_2, \ldots, t_\zeta$ for which $B_{t_1} \cap B_{t_j} \neq \emptyset$, then

$$E\left[\sum_{i=1}^{\zeta} \left(\gamma_{t_i,q^*} - \gamma_{t_i-1,q^*}\right)\right] \leq H_1(1, q^*) + (\zeta-1)\cdot H(1, q^*), \tag{8}$$

which is satisfied with equality by the above pairwise enabling strategy. We do this by expanding all possible decision sequences at $t_1, \ldots, t_\zeta$ until we have shown that Eq. (8) is satisfied for every one.

Because of the way we selected $t_1, \ldots, t_\zeta$, we will in each case have additional information about some of the points in $B_{t_1}, \ldots, B_{t_\zeta}$ that is not accounted for in Eq. (2). Thus let $H^*(n, m, \iota, \sigma, q)$ be a stronger upper bound to $E[\gamma_{t,q} - \gamma_{t-1,q}]$ when it is known that for m of the n points in B_t, the probability of finding no messages or exactly one message among them is exactly ι and σ, respectively. Then, in view of Lemma 1, we have

$$H^*(n, m, \iota, \sigma, q) \triangleq \max_{0 \le I_{n-m}^{(t)} \le (1-p)^{n-m}} \Big\{ \iota \cdot I_{n-m}^{(t)} \cdot [n] + \Big[\sigma \cdot I_{n-m}^{(t)} + \iota \cdot (1 - I_{n-m}^{(t)}) \cdot \frac{np\ (1-p)^{n-1}}{1-(1-p)^n}\Big] \cdot [n+q] \Big\}. \tag{9}$$

Since the expression being maximized is clearly an increasing function of $I_{n-m}^{(t)}$ whenever $\sigma \ge \iota$, which will always be the case below, Eq. (9) may be simplified to

$$H^*(n, m, \iota, \sigma, q) = (1-p)^{n-m-1} \cdot (\,(1-p) \cdot [\iota \cdot n + \sigma \cdot (n+q)] + (n-m) \cdot p \cdot \iota(n+q)), \tag{10}$$

and, in particular, if $\iota = 0$ then it can be shown that $H^*(n, m, 0, \sigma, q)$ is a monotonically decreasing function of n for fixed values of m and σ.

Continuing with the proof of Eq. (8), we see that since B_{t_1} represents the first suboptimal selection by A^*, it must consist of either u unexamined points, or else one known busy point along with v unexamined points, for some $u \ne 2$ and $v > 0$. But in the latter case Eq. (8) would already be satisfied after t_1, since

$$H^*(v+1, 1, 0, 1, q^*) \le H^*(2, 1, 0, 1, q^*) \le H_1(1, q^*), \qquad v > 0.$$

Thus, since $H_1(u, q^*) \le H_1(1, q^*)$ unless $u = 2,3$ (and $u = 2$ is not a suboptimal selection), assume that B_{t_1} consists of 3 unexamined points. But in the event of a collision, which happens with probability $\phi \triangleq p^3 + 3p^2(1-p)$, the points in B_{t_1} require further processing at step t_2. Clearly B_{t_2} must be a suboptimal selection because the points in $B_{t_1} \cap B_{t_2}$ are neither unexamined nor known to be busy. If $|\, B_{t_1} \cap B_{t_2} \,| = 2$, then $B_{t_2} \cap B_{t_1}$ has either one or two messages, and we see that $\iota = 0$, $\sigma = (2-2p)/(3-2p) \triangleq \psi$, so that $H^*(n, 2, 0, \psi, q^*)$ attains its maximum at $n = 2$. But since

$$H_1(3, q^*) + \phi \cdot H^*(2, 2, 0, \psi, q^*) < H_1(1, q^*) + \phi \cdot H(1, q^*)$$

holds, Eq. (8) would be satisfied after step t_2 unless $|\, B_{t_1} \cap B_{t_2} \,| = 1$. In this case, $B_{t_1} \cap B_{t_2}$ contains at most one message, $\iota = (1-p)/(3-2p)$, $\sigma = (2-p)/(3-2p) \triangleq \theta$, and we find that

$$\max_{n>1} \Big\{ H_1(3, q^*) + \phi \cdot H^*(n, 1, 1-\theta, \theta, q^*) \Big\} < H_1(1, q^*) + \phi \cdot H(1, q^*).$$

Thus Eq. (8) would again be satisfied after step t_2 unless B_{t_2} consists of a single point from B_{t_1}. In this case, however, consider $B_{t_1} - B_{t_2}$ after step t_2 in the event that B_{t_2} was found contain a message. Since these last two points from B_{t_1} are neither unexamined nor known to contain two messages, further processing is required at step t_3, and B_{t_3} must be a suboptimal selection. But if $|\, B_{t_1} \cap B_{t_3} \,| = 2$ we see that $\iota = 0$ and thus that the best we can do is complete the last suboptimal step in a conflict resolution epoch for triple enabling—which we have already found to be less efficient than the above pairwise enabling strategy. Thus, since $\iota = 1/(2-p)$,

$\sigma = (1-p)/(2-p)$, and

$$\max_{n>0} \{H_1(3, q^*) + \phi\cdot(H^*(1, 1, 1-\theta, \theta, q^*) + \theta\cdot H^*(n, 1, \iota, \sigma, q^*))\}$$
$$< H_1(1, q^*) + \phi\cdot(1+\theta)\cdot H(1, q^*),$$

Eq. (8) is proven. Thus, $p^\circ = 0.545$ for pairwise enabling on finite arrival sequences.

IV. p° for Pairwise Enabling on a Reservation Channel

Our results for p° on unbounded arrival sequences can also be extended to the class of *reservation schemes* discussed in [3]. Here it is assumed that there are separate channels for conflict resolution and message transmission, and that only the conflict resolution steps (using the reservation channel) contribute to the 'cost' of running the algorithm.

To complete a reservation under the *Gudjohnsen* reservation model, an arrival point must be identifiable as containing a message possibly using the complete channel history up to the present algorithmic step. Under the *intermediate* reservation model, however, we restrict the reservation process (but not the conflict resolution algorithm) to making inferences based on the outcome from a single algorithmic step, namely those steps where there was either a single request on the reservation channel or there was a "collision" on the reservation channel when *exactly two* arrival points were selected. Notice that under both models, pairwise enabling always requires exactly one step to process two arrival points, so that the "throughput" of pairwise enabling (measured in reservations completed per algorithmic step) is simply $2p$. Thus in [3], Hwang and Chang consider pairwise enabling to be (the generalization of) "roll call" polling for reservation channels, and on finite arrival sequences they found that $p_I^\circ = 0.5$ for the intermediate reservation model and that $p_G^\circ = 0.597$ for the Gudjohnsen reservation model. In the remainder of this section, we will show that on unbounded arrival sequences, $0.386 \le p_I^\circ \le 0.387$ under the intermediate reservation model and $0.436 \le p_G^\circ \le 0.466$ under the Gudjohnsen reservation model.

Consider the following "triple enabling" algorithm under the Gudjohnsen reservation model. Initially, three unexamined arrival points are selected. If the result is not a collision, the algorithm terminates after using one step to complete at most one reservation. Otherwise, two of the three arrival points from the collision are selected. Clearly this second step must result in either a success or another collision on the reservation channel. Following a success, we can infer that the third arrival point must have requested a reservation, thus using two steps to complete two reservations. And following another collision, we have no information about the third arrival point and it is left behind for the next epoch, again using two steps to complete two reservations. Since the efficiency of this algorithm, namely $3p - p^3$ completed reservations per $2 - (1-p)^2[1+2p]$ steps, is higher than pairwise enabling whenever $p < 0.436$, we see that $p_G^\circ \ge 0.436$.

Now consider the following "quadruple enabling" algorithm under the intermediate reservation model. Initially, four unexamined arrival points are selected. If the result is not a collision, the algorithm terminates after using step to complete at most one reservation. Otherwise, two of the four arrival points from the collision are selected. If this second step results in a non-collision on the reservation channel, then the remaining two points are enabled at the third step, resulting in the completion of at least two reservations in the epoch. If the second step results in a collision, however, the algorithm terminates, since we have no information about the remaining two points. Since the efficiency of this algorithm, namely $4p - 2p^3$ completed reservations per $1 + p^2[5-4p+6(1-p)^2]$ steps, is higher than pairwise enabling whenever $p < 0.386$, we see that $p_I^\circ \geq 0.386$.

Having thus obtained lower bounds to p° on unbounded arrival sequences under both reservation models, it remains to find some corresponding upper bounds to p° by extending our results from section II. That is, we wish to find (n^*, q^*) such that

$$\hat{H}(n^*, q^*) \geq \hat{H}(n, q^*) \qquad n = 1, 2, \cdots \tag{11}$$

and

$$2p = \frac{p \cdot \hat{H}(n^*, q^*)}{1 + p \cdot q^*} \tag{12}$$

hold, where $\hat{H}(n^*, q^*)$ stands for $\hat{H}_I(n, q)$ and $\hat{H}_G(n, q)$, which are extensions of Eq. (2) to account for the "extra" reservations that can be completed via inference rules under the intermediate and Gudjohnsen reservation models, respectively.

Handling the intermediate reservation model is straightforward, since only one extra inference rule is introduced, namely that a collision when exactly two arrival points are selected implies that *both* of the selected points contain messages. Thus $\hat{H}_I(n, q) = H(n, q)$ for all $n \neq 2$, and

$$\hat{H}_I(2, q) = \max_{\substack{0 \leq I_2^{(t)} \leq (1-p)^2 \\ 0 \leq \frac{S_2^{(t)}}{1 - I_2^{(t)}} \leq \frac{2p\,(1-p)}{1-(1-p)^2}}} \left\{ I_2^{(t)} \cdot [2] + S_2^{(t)} \cdot [2 + q] + (1 - I_2^{(t)} - S_2^{(t)}) \cdot [2 + 2q] \right\}$$

$$= 2 + 2q \ .$$

Since $\hat{H}_I(2, q) = 2 \cdot \hat{H}_I(1, q)$ for every q, we must let $n^* = 2$, in which case Eq. (12) holds if $q^* = 0$. To find the minimum p for which Eq. (11) holds, we note that Lemmas 2 and 3 still apply for all $n \geq 3$. Thus, since

$$2 \geq H_1(3, 0) > H_1(4, 0)$$

holds for all $p \geq 0.387$, we have immediately that $0.386 \leq p_I^\circ \leq 0.387$ for the intermediate reservation model on unbounded arrival sequences.

Handling the Gudjohnsen reservation model, however, is more subtle, since without keeping track of the complete state of the algorithm, we must account for the possibility that a *non*-collision at the ith step could have revealed the identities of up to two arrival points requesting reservations from each previously unresolved collision, depending on the number of points in $B_j - B_i$, $j = i-1, i-2, \cdots$. Fortunately, Hwang and Chang introduced a simpler way to account for these inferences in [3, Theorem 3]. They suggested that these "extra" reservations should be attributed to the jth step, where the existence of some reservation requests in B_j was first revealed via a collision, rather than to the subsequent non-collision at the i th step, where the identities of those active points is ultimately revealed via inferences. Furthermore, they observed that such inferences are only possible when B_j contains *exactly two* requests, and that even in this case, no requests can be inferred unless the *last* point in B_j to be examined by the algorithm contains *one* of the messages, and two requests can be inferred only if the *last two* points in B_j to be examined by the algorithm contain *both* of the messages. It is important to note, however, that because of Lemma 1 it is impossible to infer that an arrival point does *not* contain a message, and thus that even under the Gudjohnsen reservation model an arrival point can only be added to D_t if it was selected at a step that resulted in an idle slot or a success on the reservation channel, or if it can be inferred from the channel history that it must contain a message.

If the observations above are used in constructing $\hat{H}_G(n, q)$, then we see that $\hat{H}_G(n, q) = \hat{H}_I(n, q)$ for $n = 1, 2$. Thus, to show that Eqs. (11–12) hold under the Gudjohnsen reservation model, we can restrict our attention to $n^* = 2$, and $q^* = 0$.

To find $\hat{H}_G(n, q)$ for $n > 2$, Eq. (2) must be modified to account for the points added to D_t as a result of inferences. In this case, the n points in B_t may be ordered by the step number where the status of that point (i.e., message / no message) becomes known to the algorithm (using the arrival point number in the event of a tie). Notice that B_t may be partitioned into three disjoint subsets $B_{t,1}$ containing the last point, $B_{t,2}$ containing the second-last point, and $B_{t,3}$ containing all remaining points, respectively, and that Lemma 1 may be applied to each subset individually. Let α_0 and $\alpha_1 = 1-\alpha_0$ be the probabilities that $B_{t,1}$ contains zero or one message, respectively, and let β_0, $\beta_1 = 1-\beta_0$, δ_0 and δ_1 be the corresponding probabilities for $B_{t,2}$ and $B_{t,3}$. Then clearly

$$\hat{H}_G(n, 0) = \max_{\substack{0 \le \alpha_0, \beta_0 \le (1-p) \\ 0 \le \delta_0 \le (1-p)^{n-2} \\ 0 \le \frac{\delta_1}{1-\delta_0} \le \frac{(n-2)p(1-p)^{n-3}}{1-(1-p)^{n-2}}}} \Big\{ \alpha_0\beta_0\delta_0[n] + (\alpha_1\beta_0\delta_0 + \alpha_0\beta_1\delta_0 + \alpha_0\beta_0\delta_1)[n] + \alpha_1\beta_0\delta_1[1] + \alpha_1\beta_1\delta_0[2] \Big\},$$

which, after some simplifications, may be rewritten as

$$\hat{H}_G(n, 0) = \max_{\substack{0 \le \alpha_0, \beta_0 \le (1-p) \\ 0 \le \delta_0 \le (1-p)^{n-2} \\ 0 \le \frac{\delta_1}{1-\delta_0} \le \frac{(n-2)p(1-p)^{n-3}}{1-(1-p)^{n-2}}}} \Big\{ \delta_0\big(n\beta_0 + ((n-2)\alpha_0 + 2)\beta_1\big) + \delta_1\big((n-1)\alpha_0 + 1\big)\beta_0 \Big\} \tag{13}$$

Notice that α_1 does not appear in Eq. (13), that $2 + (n-2)\alpha_0 \le n$, and that the first term in the coefficient of δ_0 dominates the entire coefficient of δ_1, Thus $\hat{H}_G(n, 0)$ attains its maximum for $\delta_0 = (1-p)^{n-2}$, $\delta_1 = (n-2)p(1-p)^{n-3}$, and $\alpha_0 = \beta_0 = (1-p)$. After substitution in Eq. (13) and some simplifications, we obtain

$$\hat{H}_G(n, 0) = n\ (1-p)^{n-2}\left[1 + p(1-p)(n-2)\right] \qquad n \ge 2. \tag{14}$$

Computations reveal that for all $p > 0.466$, $\hat{H}_G(n, 0)$ attains its maximum at $n = 2$, and thus that $0.436 \le p_G^o \le 0.466$ for the Gudjohnsen reservation model on unbounded arrival sequences.

V. Concluding Remarks

In the final section of [3], Hwang and Chang discuss the relationships among the cutoff points for pairwise enabling under the standard Capetanakis-type channel model and the two reservation channel models considered above. After stating that we found in [1] that pairwise enabling is optimal for unbounded sequences under the Capetanakis model when $0.568 \le p \le 1/\sqrt{2}$, they incorrectly concluded that our result meant that $p^\circ = 0.568$, rather than $p^\circ \le 0.568$, in this case. Furthermore, the subsequent discussion includes several additional misconceptions.

First, by their own definition for the cutoff point, p° is chosen on the basis of the *maximum* over all values of M of the probabilities at which the crossover point occurs. Thus, showing the existence of a case where pairwise enabling uses fewer steps on average than triple enabling when $p = 0.5$ and $M = 4$ gives no information about p° for finite arrival sequences.

Second, the authors seem to be basing their surprise that p° for pairwise enabling under the Capetanakis model is higher than p° for roll call under their intermediate reservation model on the following monotonicity principle. Since it is clear that every algorithm A_C for the Capetanakis model is also valid under the intermediate and Gudjohnsen reservation models, and since every algorithm A_I for the intermediate reservation model is also valid under the Gudjohnsen reservation model, and indeed it may even be possible to combine or eliminate some steps during one of these conversions without invalidating the algorithm, then the minimum number steps required to process an M-point Bernoulli arrival sequence must be monotonically decreasing under the Capetanakis, intermediate reservation, and Gudjohnsen reservation models. Note in particular that when all the conflict resolution steps in pairwise enabling are eliminated (since they are unnecessary under the two reservation models), the resulting algorithm is effectively roll call polling under the two reservation models. However, just because one can trivially convert an algorithm A_C for the Capetanakis model into an algorithm A_I for the intermediate reservation model without increasing its running time on any sequence of arrival points does not mean that *every* algorithm for the Capetanakis model will benefit to the same extent by the conversion process. Thus, there is no reason to believe that a similar monotonicity property should also hold for the cutoff points of pairwise enabling under the Capetanakis, intermediate reservation, and Gudjohnsen reservation models. And indeed, the following table that summarizes our results demonstrates that this property does *not* hold for either finite or unbounded arrival sequences.

And finally, the following argument provides a counterexample to the suggestion in [3] that p° for pairwise enabling might be smaller on finite arrival sequences than on unbounded arrival sequences. Consider a p between the values of p° for pairwise enabling on finite and infinite arrival sequences. If the suggestion were true, then there must be an optimal group testing algorithm, $A^*(\infty)$, that attains a strictly higher efficiency on infinite arrival sequences than is possible using any algorithm (including pairwise enabling) on arrival sequences of any finite length. But it is easy to construct unbounded sequences of algorithms, $\{A^*(M) : M = M_1, M_2, \cdots ; M_1 < M_2 < \cdots\}$, from $A^*(\infty)$ by restricting each selection to the first M points from the arrival sequence, and including only those steps from $A^*(\infty)$ in $A^*(M)$ where $B_{(t)} \cap \{1, \ldots, M\} \ne \emptyset$. Note that each $A^*(M)$ is a valid conflict resolution algorithm for some $M < \infty$ and thus must be considered in the computation of p° for pairwise enabling on finite arrival sequences. Furthermore, each point from the arrival sequence can only

	Finite Sequences	Unbounded Sequences
Capetanakis Model	$p^\circ = 0.545$	$0.430 \le p^\circ \le 0.5$
Intermediate Model	$p_I{}^\circ = 0.5$ (from [3])	$0.386 \le p_I{}^\circ \le 0.387$
Gudjohnsen Model	$p_G^\circ = 0.597$ (from [3])	$0.436 \le p_G^\circ \le 0.466$

Table 1: Cutoff points for pairwise enabling under various models.

be selected a finite number of times on average if the capacity of $A^*(\infty)$ is to be greater than zero. Thus, since the proportion of steps where the actions of $A^*(M)$ differ from the corresponding actions of $A^*(\infty)$ must decrease monotonically to zero as $M \to \infty$, there must be a sequence $M_1, M_2, \cdots$ for which the efficiencies of $A^*(M_1)$, $A^*(M_2)$, $\cdots$ are converging monotonically to that of $A^*(\infty)$. Now since the efficiency of $A^*(\infty)$ is strictly greater than that of pairwise enabling by hypothesis, there must be an $M_j < \infty$ for which the efficiency of $A^*(M_j)$ exceeds that of pairwise enabling, thus contradicting the optimality of pairwise enabling on finite arrival sequences for the given value of p.

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