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ON THE DEAD-CORE PROBLEM FOR THE p-LAPLACE EQUATION WITH A STRONG ABSORPTION

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Abstract. We study an initial boundary value problem for the p-Laplace equation with a strong absorption. We are concerned with the dead-core behavior of the solution. First, some criteria for developing dead-core are given. Also, the temporal dead-core rate for certain initial data is determined. Then we prove uniqueness theorem for the backward self-similar solutions.

1. Introduction. In this paper, we consider the following initial boundary value problem

$$\begin{cases} u_t = (|u_x|^{p-2}u_x)_x - u^q\,, & 0 < x < 1\,,\ t > 0\,,\\ u_x(0,t) = 0\,, \quad u(1,t) = k\,, & t > 0\,,\\ u(x,0) = u_0(x)\,, & x \in [0,1]\,, \end{cases} \tag{1.1}$$

where parameters $q \in (0,1)$, $p > 2$ and $k > 0$. We shall always assume that $u_0(x) > 0$ for all $x \in [0,1]$. The local existence and uniqueness of classical solution of (1.1) is trivial. Let $[0,T)$, $T = T(u_0) > 0$, be the maximal time interval for the existence of a positive solution u to the problem (1.1). In the case when $T < \infty$, we have

$$\liminf_{t \nearrow T}\{\min_{x\in[0,1]} u(x,t)\} = 0$$

so that the solution reaches zero at some point in a finite time. We call such a property a dead-core phenomenon. In fact, in the study of chemical reaction, the chemical is inactive when the chemical concentration u vanishes and so the set of vanishing concentration is called the dead-core.

Problem (1.1) with $p = 2$ arises from the modeling of an isothermal reaction-diffusion process (cf. [2, 9]). See also [1, 3]. In problem (1.1), there is no flux on the boundary $x = 0$ and it is injected with a fixed amount of reactant on the boundary $x = 1$. Owing to the strong absorption (the reaction rate $-u^{q-1} \to -\infty$ as $u \searrow 0$, since $q < 1$), a dead-core is expected to be developed for certain initial data. On the other hand, it is also very interesting to study the temporal dead-core rate. For this, we refer the reader to [7, 6, 5, 4] for the study of dead-core for the equation

$$u_t = (u^m)_{xx} - u^q \tag{1.2}$$

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in which $0 < q < 1$ and $m > q$. The case for $m = 1$ corresponds to the case for $p = 2$ in (1.1). The dead-core rate can be either of self-similar or of non-self-similar type for different range of m. We refer to [7, 5, 4] for more details. For $m = 1$, the exact dead-core rates for different initial data is addressed in [6] by applying a braid group theory. The other cases are largely open.

The main purpose of this paper is to study the dead-core problem for the p-Laplace equation with a strong absorption. Some criteria of developing dead-core is provided in the next section. Now suppose that u develops a dead-core in finite time T. If we further assume that $u_0' \geq 0$ on $[0, 1]$, then $u_x > 0$ in $(0, 1] \times (0, T)$ by the strong maximum principle. Therefore, in this case we may re-write the first equation in (1.1) as

$$u_t = (p-1)u_x^{p-2}u_{xx} - u^q .$$

In particular, at $x = 0$ we have

$$u_t(0, t) = -u(0, t)^q \quad \text{for all } t > 0 . \tag{1.3}$$

An integration of (1.3) from $t < T$ to T gives that

$$u(0, t) = \alpha^{-\alpha}(T - t)^{\alpha} , \quad \alpha := 1/(1-q) .$$

This determines the dead-core rate at the point $x = 0$. Notice that the dead-core time T is uniquely determined by $u_0(0) = \alpha^{-\alpha}T^{\alpha}$.

Finally, we study the self-similar solutions of problem (1.1). For this, we introduce the following standard self-similar transformation for (1.1):

$$v(y, s) = \frac{u(x, t)}{(T - t)^{\alpha}} , \quad y = \frac{x}{(T - t)^{\beta}} , \quad s = -\ln(T - t) , \quad \beta := \frac{p - 1 - q}{p(1 - q)} . \tag{1.4}$$

Then v satisfies

$$v_s = (|v_y|^{p-2}v_y)_y - \beta y v_y + \alpha v - v^q \quad \text{for } 0 < y < e^{\beta s}, s > s_0 := -\ln T . \tag{1.5}$$

We shall only consider the classical solutions, i.e., solutions in C^2 class. For a classical stationary solution V of (1.5) for $y \in [0, \infty)$, i.e., V satisfies

$$(|V_y|^{p-2}V_y)_y - \beta y V_y + \alpha V - V^q = 0 , \quad y \in [0, \infty) , \tag{1.6}$$

in the classical sense, there corresponds a self-similar solution u of (1.1) in the form

$$u(x, t) = V\left(\frac{x}{(T - t)^{\beta}}\right)(T - t)^{\alpha} .$$

In particular, let

$$V_*(y) := c_0 y^{\gamma} , \; y \geq 0 , \tag{1.7}$$

where

$$\gamma := \frac{p}{p - 1 - q} = \frac{\alpha}{\beta} > 1 , \quad c_0 := [(p-1)\gamma^{p-1}(\gamma - 1)]^{\frac{1}{q-p+1}} .$$

Then $V_*(y)$ is a classical solution of (1.6) when $\gamma \geq 2$. Another trivial nonzero self-similar solution is the constant function $V_0 \equiv \alpha^{-\alpha}$. It is interesting to see whether there are any other self-similar solutions. We prove in the last section of this paper that these two functions are the only nonzero classical solutions of (1.6) under certain conditions. Similar results were proved in [7, 5, 4].

This paper is organized as the follows. We give some criteria of developing dead-core in §2. Then we prove the uniqueness (under certain conditions to be specified below) of self-similar solutions in §3. Although the method of proving the uniqueness theorem is quite similar to the one given in [4], the analysis here is more delicate and involved due to the p-Laplace diffusion term.

2. Occurrence of dead-core. In this section, we provide some criteria for the occurrence of dead-core. Similar results for the (1.2) can be found in [5, 4]. The proof of the following theorem is based on an idea of [8].

THEOREM 2.1. *For any $k > 0$, $\delta \in (0,1)$ and $M > 0$, there is a constant $\sigma > 0$, depending on δ and M, such that $T(u_0) < \infty$ for any initial datum u_0 with $0 < u_0 \le M$ in $[0,1]$ and $u_0 \le \sigma$ in $[0,\delta]$.*

PROOF. Set $U(x,t) = \varepsilon(T-t)^\alpha(1+y^2)^\eta$, where

$$y := \frac{x}{(T-t)^\beta}, \quad \beta := \frac{p-1-q}{p(1-q)}, \quad \frac{p}{2(p-1-q)} < \eta < \frac{p}{2(p-2)}.$$

Then

$$\begin{aligned} Q[U] &:= U_t - (|U_x|^{p-2}U_x)_x + U^q \\ &= \varepsilon(T-t)^{\alpha-1}\{-\alpha(1+y^2)^\eta + 2\beta\eta y^2(1+y^2)^{\eta-1} \\ &\quad -\varepsilon^{p-2}(2\eta)^{p-1}(p-1)[2(\eta-1)y^2+(1+y^2)](1+y^2)^{(\eta-1)(p-1)-1}y^{p-2} \\ &\quad +\varepsilon^{q-1}(1+y^2)^{\eta q}\}. \end{aligned}$$

Note that from the choice of η, $2\beta\eta > \alpha$ and so $Q[U] \ge 0$, if $y \ge L \gg 1$. On the other hand, choose ε sufficiently small, we have $Q[U] \ge 0$, if $y \in [0, L]$. Hence

$$U_t - (|U_x|^{p-2}U_x)_x + U^q \ge 0 \quad \text{for } x \in (0,1),\ t \in (0,T).$$

Moreover,

$$U(x,t) \ge \varepsilon x^{2\eta} T^{\alpha-2\beta\eta}.$$

Suppose $u_0 \le \sigma$ in $[0,\delta]$ and $0 < u_0 \le M$ in $[0,1]$, where $\sigma := \min_{[0,\delta]} U(x,0)$. Choosing T small enough such that $\varepsilon\delta^{2\eta}T^{\alpha-2\beta\eta} \ge M$, we have $u_0(x) \le U(x,0)$ for all $x \in [0,1]$. Also, we have $U_x(0,t) = 0$ and $U(1,t) \ge M \ge u_0(1)$ $\forall t \in (0,T)$. Hence by the Comparison Principle, $u \le U$ in $[0,1]\times[0,T)$, where u is the solution of (1.1). Notice that $U(0,t) = \varepsilon(T-t)^\alpha \to 0$ as $t \to T^-$. Therefore, u attains a dead-core in a finite time ahead of T. The theorem is proved. □

Next, we study the stationary solutions of (1.1). For this, we define the following quantities

$$K(p,q) := \left[\frac{p}{(p-1)(q+1)}\right]^{1/p}, \quad k_0 := \left[\frac{p-q-1}{p}K(p,q)\right]^{p/(p-q-1)}.$$

Let $U = U_\mu$ be the solution of the following initial value problem:

$$(|U'(x)|^{p-2}U'(x))' = U^q(x),\ x \geq 0, \quad U'(0) = 0,\ U(0) = \mu \geq 0.$$

Then it is easy to see that $U' > 0$ on $(0, 1]$ if $U > 0$ on $(0, 1]$. Moreover, $U_{\mu_1} < U_{\mu_2}$ on $[0, 1]$ if $0 \leq \mu_1 < \mu_2$. Suppose that $U(1) = k > \mu$. Then, by an integration, we can easily deduce that

$$I_k(\mu) := \int_\mu^k \frac{du}{(u^{q+1} - \mu^{q+1})^{1/p}} = K(p, q).$$

Note that the integral $I_k(\mu)$ is integrable such that $I_k(k) = 0$ and

$$I_k(0) = \int_0^k u^{-(q+1)/p} du = \frac{p}{p-q-1} k^{(p-q-1)/p}.$$

Therefore, a (unique) solution U with $U > 0$ on $(0, 1]$ to the problem

$$(|U'(x)|^{p-2}U'(x))' = U^q(x),\ x \in [0, 1], \quad U'(0) = 0,\ U(1) = k > 0, \tag{2.1}$$

exists if and only if $k \geq k_0$. We denote this solution by U_k. On the other hand, for $k \in (0, k_0)$, there is a unique solution U_k to (2.1) such that $U_k = 0$ on $[0, \delta_k]$ and $U_k > 0$ on $(\delta_k, 1]$ for some positive constant $\delta_k < 1$. Indeed, the constant δ_k is determined by

$$(1 - \delta_k)K(p, q) = \frac{p}{p-q-1} k^{(p-q-1)/p}.$$

Actually, the solution u to (1.1) exists globally with $u \geq 0$. We introduce the energy

$$E[u](t) := \frac{1}{p} \int_0^1 |u_x(x, t)|^p dx + \frac{1}{q+1} \int_0^1 u^{q+1}(x, t) dx.$$

Then it is easy to check that

$$E[u]'(t) = -\int_0^1 u_t^2(x, t) dx \leq 0.$$

Therefore, the standard energy argument shows that $u \to U_k$ uniformly on $[0, 1]$ as $t \to \infty$. This together with Theorem 2.1 imply the following dead-core criterion.

COROLLARY 2.2. *For $k \in (0, k_0)$, any solution of* (1.1) *develops a dead-core in finite time.*

3. Uniqueness of self-similar solutions. This section is devoted to the study of the self-similar solutions of (1.1). In order to study the self-similar solution of (1.1), following [4] we introduce

$$r = \frac{Bx}{(T-t)^\beta}, \quad s = -[\ln(T-t)]/\gamma, \quad z^\gamma(r, s) = \frac{u(x, t)}{A(T-t)^\alpha}, \tag{3.1}$$

where

$$A := \alpha^{-\alpha}, \quad B := [\alpha^{1-2\alpha+p\alpha}(\gamma-1)^{-1}(p-1)^{-1}\gamma^{1-p}]^{1/p}.$$

Then the equation (1.1) is transformed to

$$z^{\gamma-1} z_s/\alpha = a z^{\gamma-\sigma} z_r^{p-2} z_{rr} + z^{\gamma-\sigma-1}(z_r^p - 1) + z^{\gamma-1}(z - r z_r), \tag{3.2}$$

where

$$a := (\gamma - 1)^{-1}, \quad \sigma := -(\gamma - 1)(p-2) + 1 .$$

Note that $\gamma > 1$ and $\sigma \in (-1, 1)$, since

$$\sigma - 1 = \frac{(1+q)(2-p)}{p-1-q} < 0 < \frac{p(1-q)}{p-1-q} = \sigma + 1 .$$

Also, we have $\gamma - \sigma - 1 = pq/(p-1-q) > 0$.

Suppose that u is a solution of (1.1) with the maximal existence time interval $[0, T)$. If we assume that $u_0'(x) \geq 0$ and $(|u_0'|^{p-2}u_0')' - u_0^q \leq 0$ for all $x \in [0, 1]$, then it follows from the strong maximum principle that $u_t < 0 < u_x$ for all $x \in (0, 1)$ and $t \in (0, T)$. From the fact that $u_t < 0$, we obtain

$$(|u_x|^{p-2}u_x)_x - u^q < 0 .$$

Multiplying this inequality by u_x and integrating it from 0 to x with $x > 0$, we obtain

$$0 < u_x(x,t) < C_{p,q} u^{\frac{q+1}{p}}(x,t), \quad 0 < x < 1,\ 0 < t < T , \tag{3.3}$$

where $C_{p,q} = [\frac{p}{(p-1)(q+1)}]^{1/p}$. It follows from (3.3) that

$$0 < v_y < C_{p,q} v^{\frac{q+1}{p}}, \quad 0 < y < e^{\beta s},\ s > s_0 . \tag{3.4}$$

Due to the estimate (3.3) (and (3.4)), we have

$$z(r,s) > 0, \quad 0 < z_r(r,s) \leq 1$$

for $0 < r < Be^{\beta\gamma s}$, $s > -\ln T/\gamma$. Therefore, any ω-limit Z of z (if it exists) satisfies

$$Z \geq 0, \quad 0 \leq Z_r \leq 1, \quad r \geq 0 . \tag{3.5}$$

Also, Z is a (nonnegative) steady-state solution of (3.2) in $[0, \infty)$, i.e., Z satisfies the equation

$$aZ^{\gamma-\sigma} Z_r^{p-2} Z_{rr} + Z^{\gamma-\sigma-1}(Z_r^p - 1) + Z^{\gamma-1}(Z - rZ_r) = 0, \quad r \geq 0 . \tag{3.6}$$

Hence hereafter we only consider solutions of (3.6) satisfying (3.5) such that $Z^\gamma \in C^2([0,\infty))$. Note that $Z \equiv 0$ and $Z \equiv 1$ are two trivial (constant) solutions of (3.6).

For a given nontrivial solution Z of (3.6) satisfying (3.5) with $Z^\gamma \in C^2([0, \infty))$, we set

$$r_0 := \inf\{r \geq 0 \mid Z(r) > 0\} .$$

Then r_0 is well-defined and $r_0 \in [0, \infty)$. Note that $Z(r) > 0$ for all $r > r_0$. On the other hand, we observe from (3.6) that $Z(\bar{r}) = 1$, if $Z_r(\bar{r}) = 0$ for some point $\bar{r} > r_0$. Also, $Z \equiv 1$ on $[\bar{r}, \tilde{r}]$, if $Z_r(\bar{r}) = Z_r(\tilde{r}) = 0$ with $r_0 < \bar{r} < \tilde{r}$. Indeed, since $Z(\bar{r}) = Z(\tilde{r}) = 1$, if $Z(r) \neq 1$ for some $r \in (\bar{r}, \tilde{r})$, then the mean value theorem implies that there is a point r_0 in either $(\bar{r}, r)$ or $(r, \tilde{r})$ such that $Z'(r_0) < 0$, a contradiction to (3.5).

REMARK 3.1. For $\{Z > 0, Z_r > 0\}$, we may write (3.6) as the system

$$Z_r := W ,$$
$$W_r = \{Z^{-1}(W^{2-p} - W^2) + Z^{\sigma-1}(rW - Z)W^{2-p}\}/a .$$

Due to the singularity of the nonlinearity, the standard uniqueness theory for initial value problem cannot be applied for the initial value $(Z, W) = (1, 0)$.

REMARK 3.2. Note that, due to the boundary condition, we may assume that $(Z^\gamma)_r(0) = 0$. This implies that $Z_r(0) = 0$, if $Z(0) > 0$ (so that $r_0 = 0$). Note that $Z(r_0) = Z_r(r_0) = 0$, if $r_0 > 0$. Therefore, in any case we have either $Z(r_0) = 0$ or $Z_r(r_0) = 0$. Also, it is trivial that $Z_{rr}(r_0)$ is finite, when $r_0 > 0$, due to (3.5). On the other hand, for $r_0 = 0$, we have $Z^{\gamma-1}(0)Z_{rr}(0) < \infty$ due to $Z^\gamma \in C^2([0,\infty))$. In particular, if $Z(0) > 0$, then we have the finiteness of $Z_{rr}(0)$.

LEMMA 3.3. *Suppose that Z is a nontrivial solution of* (3.6) *satisfying* (3.5) *and*

(A) $Z_r > 0$ *on* (r_0, r_1) *and* $Z_r(r_1) = 0$ *for some* $r_1 \in (r_0, \infty)$.

Then $Z_{rr}(r)$ has the same sign as that of $-\sigma$ for all $r \in (r_0, r_1)$.

PROOF. For $r \in (r_0, r_1)$, we define the quantity

$$\rho(r) := \int_{r_*}^{r} \frac{\xi Z^{\sigma-1}(\xi)}{a Z_\xi^{p-2}(\xi)} d\xi\,, \quad r_* := (r_0 + r_1)/2\,,$$

which is well-defined and nonnegative for all $r \in (r_0, r_1)$. Writing (3.6) as

$$a Z^{1-\sigma} Z_r^{p-2} Z_{rr} + Z^{-\sigma}(Z_r^p - 1) + (Z - rZ_r) = 0\,, \quad r_0 < r < r_1\,,$$

and differentiating it once in r, we obtain

$$\begin{aligned} &a Z^{1-\sigma} Z_r^{p-2} Z_{rrr} + \sigma Z^{-\sigma-1} Z_r (1 - Z_r^p) \\ &\quad + Z^{-\sigma} Z_r^{p-1}\big[a(1-\sigma) + p + a(p-2) Z Z_r^{-2} Z_{rr} - r Z^\sigma Z_r^{1-p}\big] Z_{rr} = 0,\ r_0 < r < r_1. \end{aligned}$$

It follows that Z satisfies

$$a \frac{d}{dr}\big[Z_{rr} Z^{1-\sigma+\frac{p}{a}} Z_r^{p-2} e^{-\rho}\big] = -\sigma Z^{-1-\sigma+\frac{p}{a}} Z_r (1 - Z_r^p) e^{-\rho} \tag{3.7}$$

for all $r \in (r_0, r_1)$.

For any $r \in (r_0, r_1)$, by integrating (3.7) from $\tilde{r} \in (r_0, r)$ to r and sending $\tilde{r}$ to r_0, we obtain

$$Z_{rr}(r) = -\frac{Z(r)^{-1+\sigma-\frac{p}{a}} Z_r(r)^{-p+2} e^{\rho(r)}}{a} \int_{r_0}^{r} \sigma Z^{-1-\sigma+\frac{p}{a}} Z_\xi (1 - Z_\xi^p) e^{-\rho} d\xi \tag{3.8}$$

for all $r \in (r_0, r_1)$. Here we have used Remark 3.2 and the fact that the number

$$-1 - \sigma + \frac{p}{a} = \frac{2pq}{p - 1 - q}$$

is positive so that the integral in (3.8) is well-defined. Notice that

$$\big[Z_{rr} Z^{1-\sigma+\frac{p}{a}} Z_r^{p-2} e^{-\rho}\big](r_0^+) = 0\,. \tag{3.9}$$

Indeed, (3.9) holds trivially when $Z_{rr}(r_0)$ is finite. It is left to check the case when $r_0 = 0$ and $Z(0) = 0$. In this case, we have $Z^{\gamma-1}(0)Z_{rr}(0) < \infty$. Writing

$$Z_{rr} Z^{1-\sigma+\frac{p}{a}} = Z_{rr} Z^{\gamma-1} Z^{1-\gamma+1-\sigma+\frac{p}{a}}$$

and noting that $1-\gamma+1-\sigma+\frac{p}{a}$ is positive, we obtain (3.9). Therefore, (3.8) holds and we conclude that $Z_{rr}(r)$ has the same sign as that of $-\sigma$ for all $r \in (r_0, r_1)$, if the condition (A) holds. □

LEMMA 3.4. *Suppose that Z is a nontrivial solution of* (3.6) *satisfying* (3.5) *and*
(B) $Z_r(r) > 0$ *for all $r > r_2$ for some $r_2 \ge r_0$.*
Then $Z_{rr}(r)$ has the same sign as that of σ for all $r \in (r_2, \infty)$.

PROOF. Similar to Lemma 3.3, we define

$$\rho_1(r) := \int_{r_2+1}^{r} \frac{\xi Z^{\sigma-1}(\xi)}{a Z_\xi^{p-2}(\xi)} d\xi\,, \quad r > r_2\,. \tag{3.10}$$

We first claim that

$$\rho_1(r) \ge c r^{\sigma+1}\,, \quad r \ge Z(0) + 2(r_2+1)\,, \tag{3.11}$$

for some constant $c > 0$. Indeed, since $0 \le Z_r(r) \le 1$ on $(0, \infty)$, we have

$$Z(r) = Z(0) + \int_0^r Z_\xi(\xi) d\xi \le Z(0) + r\,.$$

Hence $Z(r) \le 2r$, if $r \ge Z(0) + 2(r_2+1)$. It follows that

$$\rho_1(r) \ge \int_{r/2}^{r} \frac{\xi(2\xi)^{\sigma-1}}{a} dr = \frac{2^{\sigma+1}-1}{4a(\sigma+1)} r^{\sigma+1}\,, \quad r \ge Z(0) + 2(r_2+1)$$

and so (3.11) holds.

Next, we claim that

$$Z_{rr}(r) = \frac{Z^{-1+\sigma-\frac{p}{2}} Z_r^{-p+2} e^{\rho_1(r)}}{a} \int_r^\infty \sigma Z^{-1-\sigma+\frac{p}{a}} Z_\xi (1 - Z_\xi^p) e^{-\rho_1} d\xi \tag{3.12}$$

for all $r > r_2$. To see this, we note that (3.7), with ρ replacing by ρ_1, holds for $r > r_2$. Using the estimate

$$\begin{aligned} a|Z_{rr}| Z^{1-\sigma+\frac{p}{a}} Z_r^{p-2} &= Z^{-\sigma+\frac{p}{a}} (a|Z_{rr}| Z Z_r^{p-2}) \\ &\le Z^{-\sigma+\frac{p}{a}} (1 - Z_r^p) + Z^{\frac{p}{a}} |(r Z_r - Z)| \\ &\le |Z(0) + r|^{-\sigma+\frac{p}{a}} + |Z(0) + r|^{\frac{p}{a}+1} \end{aligned}$$

for $r > r_2$, by integrating (3.7), with ρ replacing by ρ_1, from $r > r_2$ to ∞, (3.12) follows. Hence the lemma is proved. □

LEMMA 3.5. *Suppose that Z is a nontrivial solution of* (3.6) *satisfying* (3.5) *and* $\sigma \le 0$. *Then $Z_r > 0$ for all $r > r_0$.*

PROOF. Suppose for contradiction that there exists $\bar{r} > r_0$ such that $Z_r(\bar{r}) = 0$. Then $Z(\bar{r}) = 1$. Set

$$r_1 := \inf\{r > r_0 \mid Z_r(r) = 0\}\,, \quad r_2 := \sup\{r > r_0 \mid Z_r(r) = 0\}\,.$$

Then $r_i \in [r_0, \infty]$, $i = 1, 2$, are well-defined such that $r_1 \le r_2$ and $Z \equiv 1$ on $[r_1, r_2]$. Furthermore, we have either $r_1 > r_0$ or $r_2 < \infty$. Otherwise, $r_1 = r_0$ implies that $Z(r_0) = 1$

and so $r_0 = 0$. If we also have $r_2 = \infty$, then $Z \equiv 1$ on $[0, \infty)$, a contradiction to Z is nontrivial.

Suppose that $r_1 > r_0$. Then $Z_r > 0$ in (r_0, r_1) and $Z_r(r_1) = 0$. Hence the condition (A) holds. It follows from the mean value theorem that $Z_{rr}(\hat{r}) < 0$ for some $\hat{r} \in ((r_0+r_1)/2, r_1)$, a contradiction to Lemma 3.3 due to the assumption $\sigma \leq 0$.

On the other hand, suppose that $r_2 < \infty$. Then $Z_r(r_2) = 0$ and $Z_r(r) > 0$ for all $r > r_2$. It follows from Lemma 3.4 that $Z_{rr}(r) \leq 0$ for all $r > r_2$. But, the mean value theorem implies that $Z_{rr}(\hat{r}) > 0$ for some $\hat{r} \in (r_2, \infty)$, a contradiction. Thus the lemma follows. □

LEMMA 3.6. *Suppose that Z is a nontrivial solution of* (3.6) *satisfying* (3.5) *and $\sigma \leq 0$. Then $r_0 = 0$ and $Z_r(r) > 0$ for all $r > 0$.*

PROOF. Note that $Z(r), Z_r(r) > 0$ for $r > r_0$, by Lemma 3.5. Set

$$J(r) := rZ_r - Z\,. \tag{3.13}$$

Then we compute

$$\begin{aligned} J_r = rZ_{rr} &= \frac{r}{aZZ_r^{p-2}}(1 - Z_r^p + Z^\sigma J) \\ &= \frac{r}{aZZ_r^{p-2}}(1 - Z_r^p) + (\rho_1)_r J \end{aligned}$$

for all $r > r_0$. It follows that

$$\frac{d}{dr}(e^{-\rho_1} J) = \frac{re^{-\rho_1}}{aZZ_r^{p-2}}(1 - Z_r^p)\,, \quad r > r_0\,.$$

Integrating from $r > r_0$ to $R > r$ and letting $R \to \infty$, we obtain

$$J(r) = -e^{\rho_1(r)} \int_r^\infty \frac{\xi e^{-\rho_1(\xi)}}{aZZ_\xi^{p-2}}(1 - Z_\xi^p)d\xi\,, \quad r > r_0\,, \tag{3.14}$$

by using (3.11) and the fact that

$$-(Z(0) + R) \leq -Z(R) \leq J(R) \leq R\,.$$

Notice that, by (3.14), $J(r) \leq 0$ for all $r > r_0$.

Suppose for contradiction that $r_0 > 0$. Then $Z(r) = 0$ on $[0, r_0]$. Integrating $[Z(r)/r]_r = J/r^2 \leq 0$ over $(\hat{r}, r)$, $r_0 < \hat{r} < r$, and sending $\hat{r}$ to r_0, we have $Z(r)/r \leq Z(r_0)/r_0 = 0$ for all $r > r_0$. This implies that $Z \equiv 0$ on $[0, \infty)$, a contradiction. Consequently, $r_0 = 0$ and $Z_r(r) > 0$ for all $r > 0$. The proof is completed. □

Now, we divide our discussions into two cases. First, we deal with the case that $Z(0) = 0$.

LEMMA 3.7. *Suppose that Z is a nontrivial solution of* (3.6) *satisfying* (3.5) *and $\sigma \leq 0$. If $Z(0) = 0$, then $Z_r(0^+) = 1$.*

PROOF. First, from (3.14), we have $J(r) \le 0$ and $(Z/r)_r = J/r^2 \le 0$ for all $r > 0$. Hence the limit

$$l := \lim_{r\to 0^+} \frac{Z(r)}{r}$$

exists and $l \in [Z(1), 1]$.

Next, we claim that

$$l = \lim_{r\to 0^+} Z_r(r) = Z_r(0^+)\,. \tag{3.15}$$

For this, from

$$\limsup_{r\to 0^+} Z_r(r) \le \lim_{r\to 0^+} Z(r)/r = l = \lim_{r\to 0^+} \int_0^1 Z_r(r\xi)d\xi \le \limsup_{r\to 0^+} Z_r(r)\,,$$

it follows that $\limsup_{r\to 0^+} Z_r(r) = l$. On the other hand, from (3.6) it follows that

$$aZ_r^{p-1} Z_{rr} \ge -Z^\sigma Z_r$$

for all $r > 0$. Integrating each side from r_1 to r with $0 < r_1 < r$, we obtain

$$\frac{a}{p} Z_r^p(r) - \frac{a}{p} Z_r^p(r_1) \ge -\frac{Z(r)^{\sigma+1}}{\sigma+1} + \frac{Z(r_1)^{\sigma+1}}{\sigma+1} \ge -\frac{Z(r)^{\sigma+1}}{\sigma+1}\,.$$

Sending r_1 to 0 along a subsequence on which $Z_r(r_1) \to l$, we have

$$Z_r^p(r) \ge l^p - \frac{p}{a(\sigma+1)} Z^{\sigma+1}(r) \quad \text{for all } r > 0\,.$$

It follows $\liminf_{r\to 0^+} Z_r(r) \ge l$ and thus (3.15) is proved.

Now, we rewrite (3.6) as

$$aZZ_r^{p-2} Z_{rr} + Z_r^p - 1 = Z^\sigma (rZ_r - Z)\,.$$

Then integrating it from 0 to $r > 0$ yields

$$\begin{aligned}
&\frac{a}{p-1}(ZZ_r^{p-1})(r) + \frac{p-1-a}{p-1} \int_0^r Z_\xi^p d\xi - r \\
&= \frac{r}{\sigma+1} Z^{\sigma+1}(r) - \frac{\sigma+2}{\sigma+1} \int_0^r Z^{\sigma+1} d\xi = O(r^{\sigma+2})\,.
\end{aligned}$$

Dividing the above equation by r and sending r to 0^+, we deduce that

$$\frac{a}{p-1} l^p + \frac{p-1-a}{p-1} l^p - 1 = 0\,.$$

Here the fact $\sigma + 1 > 0$ is used. Hence we have $l^p = 1$ and so $l = 1$. □

With this lemma, we are ready to prove the first main theorem of this section.

PROPOSITION 3.8. *Let Z be a nontrivial solution of* (3.6) *satisfying* (3.5) *and let $\sigma \le 0$. If $Z(0) = 0$, then $Z(r) = r$ for all $r \ge 0$.*

PROOF. From Lemma 3.7, $Z_r(0^+) = 1$, and so $Z(r) = r(1 + o(1))$ for $0 < r \ll 1$. Recall (3.10) with $r_2 = 0$. Then

$$\rho_1(0) = \lim_{r\to 0^+} \rho_1(y) = -\int_0^1 \frac{\xi Z^{\sigma-1}(\xi)}{aZ_\xi^{p-2}(\xi)} d\xi \,,$$

since $\sigma > -1$. Moreover, from (3.14),

$$\lim_{r\to 0^+} J(r) = -\int_0^\infty \frac{\xi e^{\rho_1(0)-\rho_1(\xi)}}{aZZ_\xi^{p-2}}(1 - Z_\xi^p) d\xi \,.$$

On the other hand, $\lim_{r\to 0^+} J(r) = \lim_{r\to 0^+}\{rZ_r - Z\} = 0$. It follows $Z_r^p \equiv 1$ and so $Z_r \equiv 1$ on $[0, \infty)$. Hence $Z(r) = r$ for all $r \geq 0$ and the proposition is proved. □

For the case $Z(0) > 0$, we have

PROPOSITION 3.9. *Let Z be a solution of* (3.6) *satisfying* (3.5) *and let $\sigma \leq 0$. If $Z(0) > 0$, then $Z(r) \equiv 1$ for all $r \geq 0$.*

PROOF. Note that $Z_r(0) = 0$, since $Z(0) > 0$. Suppose that $Z \not\equiv 1$. Then, by Lemma 3.6, $Z_r(r) > 0$ for all $r > 0$. But, by Lemma 3.4, $Z_{rr}(r) \leq 0$ for all $r > 0$. This implies that $Z_r(r) \leq 0$ for all $r \geq 0$, a contradiction. Hence the proposition follows. □

Returning to the original variable, we have the following uniqueness theorem.

THEOREM 3.10. *Suppose that $\sigma \leq 0$ and V is a nonzero classical solution of*

$$(|V_y|^{p-2}V_y)_y - \beta y V_y + \alpha V - V^q = 0\,, \quad y \geq 0\,, \tag{3.16}$$

and $V_y(0) = 0$ such that

$$0 \leq V_y \leq C_{p,q} V^{\frac{q+1}{p}}(x, t)\,, \quad y > 0\,. \tag{3.17}$$

Then either $V \equiv \alpha^{-\alpha}$ or $V = V_*$, where V_* is defined by (1.7).

PROOF. Let V be a nonzero classical solution of (3.16) and $V_y(0) = 0$ such that (3.17) holds. From (1.4) and (3.1), we can see that Z defined by

$$Z(r) := A^{-1/\gamma} V^{1/\gamma}(y)\,, \quad r := By\,,$$

is a nonzero solution of (3.6) satisfying (3.5). Hence the theorem follows by combining Propositions 3.8 and 3.9. □

Note that $\sigma \leq 0$ if and only if $p \geq 1 + 1/q$. Due to $q \in (0, 1)$, $1 + 1/q > 2$.

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