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On the Debarre–de Jong and Beheshti–Starr Conjectures on Hypersurfaces with Too Many Lines

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1. Introduction

Let K be an algebraically closed field of characteristic 0. Write X_{sing} for the singular points of a variety X, $\mathbb{P}^n = K\mathbb{P}^n$, and $\mathbb{G}(\mathbb{P}^1, \mathbb{P}^n) = G(2, n+1)$ for the Grassmannian.

The following conjecture essentially states that if $X^{n-1} \subset \mathbb{P}^n$ has "too many" lines then, for any point $x \in X$ that has (too many) lines going through it, one of the lines through x will contain a singular point of X.

Conjecture 1.1. *Let $X^{n-1} \subset \mathbb{P}^n$ be a hypersurface of degree $d \geq n$ and let $\mathbb{F}(X) \subset G(2, n+1)$ denote the Fano scheme of lines on X. Let $B \subset \mathbb{F}(X)$ be an irreducible component of dimension at least $n-2$. Let $\mathcal{I}_B := \{(x, E) \mid x \in X, E \in B, x \in \mathbb{P}E\}$, and let π and ρ denote (respectively) the projections to X and B. Let $X_B = \pi(\mathcal{I}_B) \subseteq X$ and let $\tilde{\mathcal{C}}_x = \pi\rho^{-1}\rho\pi^{-1}(x)$.*

Then, for all $x \in X_B$, $\tilde{\mathcal{C}}_x \cap X_{\text{sing}} \neq \emptyset$.

If we take hyperplane sections in the case $d = n$, then Conjecture 1.1 would imply the following, which was conjectured independently by Debarre and de Jong.

Conjecture 1.2 (Debarre–de Jong conjecture). *Let $X^{n-1} \subset \mathbb{P}^n$ be a smooth hypersurface of degree $d \leq n$. Then the dimension of the Fano scheme of lines on X equals $2n - d - 3$.*

Our conjecture extends to smaller degrees as follows.

Conjecture 1.3. *Let $X^{n-1} \subset \mathbb{P}^n$ be a hypersurface of degree $n - \lambda$. Let $B \subset \mathbb{F}(X)$ be an irreducible component of dimension $n-2$ with $\mathcal{I}_B, X_B, \dots$ as before. If* $\operatorname{codim}(X_B, X) \geq \lambda$ *and $\mathcal{C}_x$ is reduced for general $x \in X_B$, then for all $x \in X_B$, $\tilde{\mathcal{C}}_x \cap X_{\text{sing}} \neq \emptyset$.*

The cases $X_B = X$ and $\operatorname{codim}(X_B, X) = n/2$ are known; for example, they appear in Debarre's unpublished notes containing Conjecture 1.2. In [9], Harris,

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Mazur, and Pandharipande proved Conjecture 1.2 when d is small with respect to n. Debarre also proved the case $d = n \leq 5$, and Collino [3] had earlier proven the case $d = n = 4$. In [2], Beheshti proved the case $d = n \leq 6$, and a different proof was also given in [10].

Conjecture 1.1 would also imply that a smooth hypersurface of degree $d \geq n$ in $\mathbb{P}^n$ cannot contain an $(n-2)$-dimensional family of lines. This is a special case of a conjecture of Beheshti and Starr (Question 1.3 of [1]) about $\mathbb{P}^k$s on hypersurfaces, which, in the same paper, Beheshti proved for $k \geq (n-1)/4$ and Conjecture 1.1 would prove for $k = 1$.

Central to our work is finding additional structure on the tangent space to $B \subset \mathbb{F}(X)$ at a general point. This structure gives rise to vector bundles on the cone $\tilde{\mathcal{C}}_x$ swept by the B-lines passing through x. These vector bundles come endowed with a canonical section whose zero locus is $X_{\mathrm{sing}} \cap \tilde{\mathcal{C}}_x$. In particular, this translates the problem of finding singular points of X on the cone $\tilde{\mathcal{C}}_x$ into proving that the intersection number of certain top Chern classes of vector bundles is nonzero. We exploit this approach to prove Conjecture 1.1 when our construction gives rise to exactly one vector bundle; see Theorem 3.6.

OVERVIEW. The statement of Conjecture 1.1 indicates how one should look for singular points. Say $y \in X$ and we want to determine whether $y \in X_{\mathrm{sing}}$. Let $v_0, v_1, \dots, v_n$ be a basis of W with $y = [v_0]$ and P an equation for X; to show $y \in X_{\mathrm{sing}}$ we would need that all partial derivatives of local coordinates in y vanish. This is expressed by the n equations $dP_y(v_1) = \cdots = dP_y(v_n) = 0$. Say we fix a line $\mathbb{P}E$ and look for a singular point y of X on $\mathbb{P}E$. Let e_1, e_2 be a basis of E that we expand to a basis $e_1, e_2, w_1, \dots, w_{n-1}$ of W; then the equations $dP_y(e_1) = dP_y(e_2) = 0$ come for free, so we have one less equation to satisfy.

A further simplification is obtained by a study of $T_E B \subset T_E G(2, W) = E^* \otimes W/E$. We observe that $T_E B$ is the kernel of the map $\alpha \otimes w \mapsto \alpha \circ (w \lrcorner P)|_E$ described in Proposition 2.1. Moreover, we identify the tangent space $T_E \mathcal{C}_x \subset T_E B \subset E^* \otimes W/E$ to the Fano scheme of B-lines through x as a subspace $\hat{x}^{\perp E} \otimes \Pi$, where $\Pi \subset W/E$ is independent of $x \in \mathbb{P}E$; see Proposition 2.2. In the same proposition we remark that $E^* \otimes \Pi \subset T_E B$ is the intersection of $T_E B$ with the locus of rank-1 homomorphisms in $T_E G(2, W) = E^* \otimes W/E$. As a consequence, $T_E B/(E^* \otimes \Pi)$ corresponds to a linear subspace of the space of $2 \times m$ matrices of constant rank 2 for which there are normal forms. The normal forms allow us to reduce the number of equations defining the singular locus on a given line even further; see Section 3. The new number of equations will depend on the dimension of Π but is always bounded by $\dim \tilde{\mathcal{C}}_x$, where $\tilde{\mathcal{C}}_x$ is the cone swept by the lines of B passing through a general point x. For this reason, one expects to find at least a finite number of singular points of X lying on $\tilde{\mathcal{C}}_x$.

Using this description, we observe an elementary case where X must be singular (Theorem 3.2) and show that $X_{\mathrm{sing}} \cap \tilde{\mathcal{C}}_x$ is the zero locus of a section of a vector bundle, which yields a sufficient condition (5) for the nonemptiness of X_{sing} in terms of top Chern classes of vector bundles. We conclude this first part of the paper by illustrating how the construction of the equations defining the singular points on a given line works by revisiting some known examples.

In the second part of the paper, we determine certain positivity properties of the vector bundles in Lemma 5.1; we also prove Theorem 3.6, the special case of Conjecture 1.1 in which all local equations have the same degree. Another case in which the conjecture holds is considered in Section 7.

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2. The Tangent Space to *B*

In this section we study $\mathbb{P}^k$s on an arbitrary projective variety $X \subset \mathbb{P}W$. Let W denote a vector space over an algebraically closed field K of characteristic 0. For algebraic subsets $Z \subset \mathbb{P}W$, we let $\hat{Z} \subset W$ denote the affine cone and let $\mathbb{F}_k(Z) \subset G(k+1, W) = \mathbb{G}(\mathbb{P}^k, \mathbb{P}W)$ denote the Fano scheme of $\mathbb{P}^k$s on Z. Let $X \subset \mathbb{P}W$ be a variety. Let $B \subset \mathbb{F}_k(X)$ be an irreducible component. Let $\mathcal{I}_B := \{(x, E) \mid E \in B, x \in \mathbb{P}E\}$ be the incidence correspondence, and let π and ρ denote the projections to X and B. Let $X_B = \pi(\mathcal{I}_B)$. Let $\mathcal{C}_x = \rho\pi^{-1}(x)$ and let $\tilde{\mathcal{C}}_x = \pi\rho^{-1}(\mathcal{C}_x)$, so $\tilde{\mathcal{C}}_x \subset X \subset \mathbb{P}W$ is a cone with vertex x and base isomorphic to $\mathcal{C}_x$.

For a vector space V, $v \in V$, and $q \in S^k V^*$, we let $v \lrcorner q \in S^{k-1}V^*$ denote the contraction. We also write $q(v^a, w^{k-a}) = q(v, \dots, v, w, \dots, w)$ et cetera when we consider q as a multilinear form. We denote the symmetric product by $\circ$; for example, $v \circ w \in S^2 V$ for $v, w \in V$. The following proposition is essentially a rephrasing of the discussion in [6, p. 273]. We include a short proof for the sake of completeness.

Proposition 2.1. *Let $X \subset \mathbb{P}W$ be a projective variety, and let $E \in \mathbb{F}_k(X)$. Then $T_E\mathbb{F}_k(X) = \ker \sigma_{(X,E)}$, where*

$$\begin{aligned} \sigma_{(X,E)} \colon T_E G(k+1, W) = E^* \otimes W/E &\to \textstyle\bigoplus_d \operatorname{Hom}(I_d(X), S^d E^*), \\ \alpha \otimes w &\mapsto \{P \mapsto \alpha \circ (w \lrcorner P)|_E\}. \end{aligned} \tag{1}$$

Proof. We first note that $(w \lrcorner P)|_E$ is well-defined because $P|_E = 0$. Without loss of generality, we can restrict to the case where X is a hypersurface defined by a degree-d polynomial P. The general case follows by considering intersections.

Let $e_0, \dots, e_k$ be a basis of E and let $\alpha_0, \dots, \alpha_k$ be the dual basis. A tangent vector $\eta = \alpha_0 \otimes \bar{w}_0 + \dots + \alpha_k \otimes \bar{w}_k \in T_E G(k+1, W)$ corresponds to the first-order deformation $E_t = \langle e_0 + tw_0, \dots, e_k + tw_k \rangle$ of E in W, where the w_j are arbitrary liftings of the $\bar{w}_j$ to W. Recall that $E = \langle e_0, \dots, e_k \rangle$ belongs to $\mathbb{F}_k(X)$ if and only if P vanishes on all points of $\mathbb{P}E$—that is, if and only if $P(e_0^{b_0}, \dots, e_k^{b_k}) = 0$ for all $b_0, \dots, b_k$ such that $b_0 + \dots + b_k = d$. Therefore, the condition $\eta \in T_E\mathbb{F}_k(X)$ is equivalent to the vanishing, in $K[t]/(t^2)$, of

$$
\begin{aligned}
&P((e_0+tw_0)^{b_0},(e_1+tw_1)^{b_1},\dots,(e_k+tw_k)^{b_k})\\
&\quad= P(e_0^{b_0},\dots,e_k^{b_k})\\
&\qquad+t[P(e_0^{b_0-1},w_0,e_1^{b_1},\dots,e_k^{b_k})+\cdots+P(e_0^{b_0},e_1^{b_1},\dots,e_k^{b_k-1},w_k)]\\
&\quad= 0+t[(\alpha_0\circ P)(w_0,e_0^{b_0},\dots,e_k^{b_k})\\
&\qquad\qquad+(\alpha_1\circ P)(w_1,e_0^{b_0},\dots,e_k^{b_k})+\cdots+(\alpha_k\circ P)(w_k,e_0^{b_0},\dots,e_k^{b_k})]\\
&\quad= t[(\sigma_{(X,E)}(\eta))(e_0^{b_0},\dots,e_k^{b_k})]
\end{aligned}
$$

for every choice of $b_0,\dots,b_k$. This implies the claim. □

Proposition 2.2. *With notation as before, let x be a general point of X_B and E a general point of B with $x\in\mathbb{P}E$.*

(i) *If there exist a $w\in W/E$ and an $\alpha\in E^*\backslash 0$ such that $\sigma_{(X,E)}(\alpha\otimes w)=0$, then $E^*\otimes w\subset\ker\sigma_{(X,E)}$.*

(ii) *If $\Pi\subset W/E$ is maximal such that $E^*\otimes\Pi\subset\ker\sigma_{(X,E)}$, then for $x\in\mathbb{P}E$ we have $T_E\mathcal{C}_x=\hat{x}^{\perp E}\otimes\Pi$.*

Proof. $\sigma_{(X,E)}(\alpha\otimes w)=0$ means $\alpha(u)P(w,u^{d-1})=0$ for all $u\in E$ and for all $P\in I(X)$. If $P(w,u^{d-1})=0$ for all u with $\alpha(u)\neq 0$, then $P(w,u^{d-1})=0$ for all $u\in E$; thus $E^*\otimes w\subset\ker\sigma_{(X,E)}$. The second assertion is clear. □

3. How to Find Singular Points on X

We now specialize to the case where $k=1$ and X is a hypersurface in $\mathbb{P}^n=\mathbb{P}W$. In this case $T_EB/(E^*\otimes\Pi)$ is a linear subspace of $K^2\otimes K^m$ of constant rank 2, where $m=n-1-\dim T_E\mathcal{C}_x$ in view of Proposition 2.2. There is a normal form for linear subspaces L of $K^2\otimes K^m$ containing no decomposable vectors. Namely, for every basis α^1,α^2 of K^2, there exist a basis $w_1,\dots,w_m$ of K^m and integers $s_1,\dots,s_r$ with $r=m-\dim L$, $s_1+\cdots+s_r=m$, and $s_1\geq s_2\geq\cdots\geq s_r\geq 1$ such that

$$
\begin{aligned}
L=\langle&\alpha^1\otimes w_1-\alpha^2\otimes w_2,\alpha^1\otimes w_2-\alpha^2\otimes w_3,\dots,\alpha^1\otimes w_{s_1-1}-\alpha^2\otimes w_{s_1},\\
&\alpha^1\otimes w_{s_1+1}-\alpha^2\otimes w_{s_1+2},\alpha^1\otimes w_{s_1+2}-\alpha^2\otimes w_{s_1+3},\dots,\\
&\alpha^1\otimes w_{s_2+s_1-1}-\alpha^2\otimes w_{s_2+s_1},\dots,\\
&\alpha^1\otimes w_{s_{r-1}+\cdots+s_1+1}-\alpha^2\otimes w_{s_{r-1}+\cdots+s_1+2},\dots,\alpha^1\otimes w_{m-1}-\alpha^2\otimes w_m\rangle. \quad (2)
\end{aligned}
$$

This normal form is a consequence of Kronecker's normal form for pencils of matrices (see e.g. [8, Chap. XII])—that is, elements of $K^2\otimes K^m\otimes K^\ell$ specialized to the constant-rank situation. Instead of considering the image of K^2 in $K^m\otimes K^\ell$, take the image of $K^\ell=L$ in $K^2\otimes K^m$. Note that the normal form gives a basis of L divided into r blocks of length $s_1-1,\dots,s_r-1$. In particular, if for some index j we have $s_j=1$, then the corresponding block is empty.

Applying this normal form, we obtain a normal form for T_EB. Note that in this case $r=m-\dim(T_EB/(E^*\otimes\Pi))=n-1-\dim T_EB+\dim T_E\mathcal{C}_x$. From now on we will assume $\dim B \geq n-2$, so $r \leq \dim \mathcal{C}_x + 1$, with equality holding generically if $\dim B = n-2$ and B is reduced.

LEMMA 3.1. *Let $X \subset \mathbb{P}W$ be as before and assume* $\deg(X) = d \geq 1 + s_1$. *Let E be a general point of B. Then there exist $p_j^E \in S^{d-s_j}E^*$, $1 \leq j \leq r$, such that*

$$\operatorname{Image} \sigma_{(X,E)} = S^{s_1}E^* \circ p_1^E + \cdots + S^{s_r}E^* \circ p_r^E.$$

We remark that here and in Lemma 3.3, one can drop the assumption that E is a general point of B. The only change at special points is that the normal form (2) will be different.

Proof. Choose a basis $w_1, \ldots, w_{n-1}$ of W/E such that $\Pi = \langle w_{m+1}, \ldots, w_{n-1} \rangle$ and $w_1, \ldots, w_m$ are adapted to the normal form (2). Apply the normal form to $\ker \sigma_{(X,E)}/(E^* \otimes \Pi)$. For $1 \leq j \leq s_1 - 1$, we have

$$\alpha^1 \circ (w_j \lrcorner P)|_E = \alpha^2 \circ (w_{j+1} \lrcorner P)|_E. \tag{3}$$

Since α^1, α^2 are linearly independent, for $j = 1$ this implies there exists a $\phi_1 \in S^{d-2}E^*$ such that $(w_1 \lrcorner P)|_E = \alpha^2 \circ \phi_1$ and $(w_2 \lrcorner P)|_E = \alpha^1 \circ \phi_1$. But for the same reason, when $j = 2$ we see there exists a $\phi_2 \in S^{d-3}E^*$ such that

$$\begin{aligned} (w_1 \lrcorner P)|_E &= (\alpha^2)^2 \circ \phi_2, \\ (w_2 \lrcorner P)|_E &= (\alpha^1 \circ \alpha^2) \circ \phi_2, \\ (w_3 \lrcorner P)|_E &= (\alpha^1)^2 \circ \phi_2, \end{aligned} \tag{4}$$

and so on until we arrive at $\phi_{s_1-1} =: p_1^E \in S^{d-s_1}E^*$ such that $(w_j \lrcorner P)|_E = (\alpha^1)^{j-1}(\alpha^2)^{s_1-j}p_1^E$ for $1 \leq j \leq s_1$. In particular, $S^{s_1}E^* \circ p_1^E \subset \operatorname{Image} \sigma_{(X,E)}$. Continuing in this way for the other chains in the normal form, we obtain polynomials $p_1^E, \ldots, p_r^E$ with $S^{s_k}E^* \circ p_k^E \subset \operatorname{Image} \sigma_{(X,E)}$. When $s_k = 1$ we set $p_k^E = (w_{s_{k-1}+\cdots+s_1+1} \lrcorner P)|_E$. $\square$

Note that without assumptions on the degree, the conclusion of Lemma 3.1 can fail. For example, if $d = 3$ and $s_1 = m = 3$, as in the case of a general cubic hypersurface, then (4) only says $(w_1 \lrcorner P)|_E = (\alpha^2)^2$, $(w_2 \lrcorner P)|_E = \alpha^1 \circ \alpha^2$, and $(w_3 \lrcorner P)|_E = (\alpha^1)^2$. This does imply that the image of $\mathbb{P}E$ under the Gauss map of X is a rational normal curve of degree 2 in $\mathbb{P}(E + \Pi)^\perp \subset \mathbb{P}W^*$, and one can obtain similar precise information about the Gauss image of $\mathbb{P}E$ in other cases.

When $s_1 = n - 1 - \dim \mathcal{C}_x$ there is a single polynomial on $\mathbb{P}E$ whose zero set corresponds to singular points of X.

THEOREM 3.2. *Let $X^{n-1} \subset \mathbb{P}^n$ be a hypersurface with $B, \tilde{\mathcal{C}}_x, \ldots$ as before. If* $\deg(X) \geq s_1 + 1$ *and* $s_1 = n - 1 - \dim \mathcal{C}_x$, *then for all $E \in B$, $\mathbb{P}E \cap X_{\mathrm{sing}} \neq \emptyset$.*

LEMMA 3.3. *Let X be as before and let E be a general point of B. Write* $\{\deg p_k^E : 1 \leq k \leq r\} = \{\delta_1 < \delta_2 < \cdots < \delta_c\}$ *and set* $i_j = \#\{p_k^E : \deg p_k^E \leq \delta_j\}$ *for all $j \leq c$. Note that if $s_r = 1$ in the normal form* (2), *then* $i_{c-1} = \#\{k : s_k > 1\}$ *and $i_c = r$. Consider the vector spaces*

$$\begin{aligned}
\hat{M}_1 = M_1 &:= \langle p_1^E, \dots, p_{i_1}^E \rangle \subset S^{\delta_1}E^*,\\
\hat{M}_2 &:= \langle p_{i_1+1}^E, \dots, p_{i_2}^E, \hat{M}_1 \circ S^{\delta_2-\delta_1}E^* \rangle \subset S^{\delta_2}E^*,\\
M_2 &:= \hat{M}_2/(\hat{M}_1 \circ S^{\delta_2-\delta_1}E^*) \subset S^{\delta_2}E^*/(\hat{M}_1 \circ S^{\delta_2-\delta_1}E^*),\\
&\ \vdots\\
\hat{M}_{c-1} &:= \langle p_{i_{c-2}+1}^E, \dots, p_{i_{c-1}}^E, \hat{M}_{c-2} \circ S^{\delta_{c-1}-\delta_{c-2}}E^* \rangle \subset S^{\delta_{c-1}}E^*,\\
M_{c-1} &:= \hat{M}_{c-1}/(\hat{M}_{c-2} \circ S^{\delta_{c-1}-\delta_{c-2}}E^*) \subset S^{\delta_{c-1}}E^*/(\hat{M}_{c-2} \circ S^{\delta_{c-1}-\delta_{c-2}}E^*),\\
\hat{M}_c &:= \langle p_{i_{c-1}+1}^E, \dots, p_{i_c}^E, \hat{M}_{c-1} \circ S^{\delta_c-\delta_{c-1}}E^* \rangle \subset S^{\delta_c}E^*,\\
M_c &:= \hat{M}_c/(\hat{M}_{c-1} \circ S^{\delta_c-\delta_{c-1}}E^*) \subset S^{\delta_c}E^*/(\hat{M}_{c-1} \circ S^{\delta_c-\delta_{c-1}}E^*).
\end{aligned}$$

These spaces are well-defined and depend only on X and E.

The lemma is an immediate consequence of the uniqueness of the normal form up to admissible changes of bases. Let $I_E \subset \operatorname{Sym}(E^*)$ denote the ideal generated by the $\hat{M}_j$. Note that the number of polynomials generating I_E is at most $\dim \mathcal{C}_x + 1$, independently of the normal form (and $\dim \mathcal{C}_x + 1$ is the expected number of generators). Let $B' \subset B$ denote the Zariski open subset where the normal form is the same as that of a general point.

PROPOSITION 3.4. *Let $E \in B'$ and let $[y] \in \mathbb{P}E$ be in the zero set of I_E. Then $[y] \in X_{\text{sing}}$.*

Proof. $[y] \in X_{\text{sing}}$ means that, for all $w \in W$, $(w \lrcorner P)(y) = 0$. Let $w_1, \dots, w_{n-1}$ be elements of W that descend to give a basis of W/E. Since $(u \lrcorner P)|_E = 0$ holds for all $u \in E$, the polynomial $(w \lrcorner P)|_E \in S^{d-1}E^*$ is a linear combination of the $(w_i \lrcorner P)|_E$. Because each $(w_i \lrcorner P)|_E$ contains one of the p_j^E as a factor, the hypothesis implies that $w \lrcorner P$ vanishes at y. □

We now allow E to vary. Let $\mathcal{S} \to G(2, W)$ denote the tautological rank-2 subspace bundle and note that the total space of $\mathbb{P}\mathcal{S}|_B$ is our incidence correspondence $\mathcal{I}_B$. Since all calculations are algebraic, M_1 gives rise to a rank-i_1 algebraic vector bundle $\mathcal{M}_1 \subset S^{\delta_1}\mathcal{S}^*|_{B'}$, M_2 gives rise to a rank-$(i_2 - i_1)$ algebraic vector bundle $\mathcal{M}_2 \subset ((S^{\delta_2}\mathcal{S}^*)/(\mathcal{M}_1 \circ S^{\delta_2-\delta_1}\mathcal{S}^*))|_{B'}$, and so forth, finally giving a bundle of ideals $\mathcal{I} \subset \operatorname{Sym}(\mathcal{S}^*)|_{B'}$.

Now, since Grassmannians are compact, along any curve E_t in B with $E_t \in B'$ for $t \neq 0$ we have well-defined limits as $t \to 0$, and thus we may define $\mathbf{I}_0^{E_t} \subset \operatorname{Sym}(E_0^*)$. Note that if we approach E_0 in different ways, we could obtain different limiting ideals; nevertheless, we have the following result.

PROPOSITION 3.5. *Let $E \in B$, let $\{E_t\} \subset B$ be a curve such that $E_0 = E$ and $E_t \in B'$ for $t \neq 0$, and let $[y] \in \mathbb{P}E$ be in the zero set of $\mathbf{I}_0^{E_t}$. Then $[y] \in X_{\text{sing}}$.*

Proof. Although this is a standard argument, we give details in a special case to show that, at points of $B \backslash B'$, the situation is even more favorable. We work locally in a coordinate patch. First note that we may choose a fixed $\alpha^1, \alpha^2 \in W^*$ that restrict to a basis of E^* for all E in our coordinate patch and still obtain the normal form by linear changes of bases in W/E. So along our curve E_t we consider α^1, α^2 and $w_1^t, \ldots, w_{n-1}^t$ such that, for $t \neq 0$ (and small), $\Pi = \langle w_{m+1}^t, \ldots, w_{n-1}^t \rangle$ and we have a fixed normal form for $w_1^t, \ldots, w_m^t$—say, for example, $\alpha^1 \otimes w_1^t - \alpha^2 \otimes w_2^t$ and $\alpha^1 \otimes w_3^t - \alpha^2 \otimes w_4^t \in \ker \sigma_{(X, E_t)}$ for all small t—giving rise to polynomials ϕ_t and ψ_t such that

$$w_1^t \lrcorner P|_{E_t} = \alpha^2 \circ \phi_t, \qquad w_2^t \lrcorner P|_{E_t} = \alpha^1 \circ \phi_t,$$
$$w_3^t \lrcorner P|_{E_t} = \alpha^2 \circ \psi_t, \qquad w_4^t \lrcorner P|_{E_t} = \alpha^1 \circ \psi_t.$$

In the limit, we may not assume that $w_1^0, \ldots, w_m^0$ are linearly independent.

First notice that if $\psi_0 = \mu \phi_0$ then, although we have a well-defined plane $\lim_{t \to 0} [\phi_t \wedge \psi_t]$ (which equals $[\phi_0 \wedge (\psi_0' - \mu \phi_0')]$ if $\phi_0 \wedge (\psi_0' - \mu \phi_0') \neq 0$), the vanishing of ϕ_0 already implies $[y] \in X_{\text{sing}}$ as long as $w_1^0, \ldots, w_4^0$ are linearly independent.

Now consider the case in which we have a relation $\lambda^1 w_1^0 + \cdots + \lambda^4 w_4^0 = 0$. This implies we have a relation

$$\begin{aligned} 0 &= \lambda_1 \alpha^2 \circ \phi_0 + \lambda^2 \alpha^1 \circ \phi_0 + \lambda^3 \alpha^2 \circ \psi_0 + \lambda^4 \alpha^1 \circ \psi_0 \\ &= \alpha^1 \circ (\lambda^2 \phi_0 + \lambda^4 \psi_0) + \alpha^2 \circ (\lambda^1 \phi_0 + \lambda^4 \psi_0), \end{aligned}$$

which implies (assuming all coefficients nonzero) $\psi_0 = \mu \phi_0$ with $\mu = -\lambda^2/\lambda^4 = -\lambda^1/\lambda^3$. In particular, the relation among the w_j^0 was not arbitrary. We also see that

$$(\lambda^1 w_1^{0\prime} + \cdots + \lambda^4 w_4^{0\prime}) \lrcorner P|_{E_0} = (\alpha^1 + \mu \alpha^2)(\lambda^2 \phi_0' + \lambda^4 \psi_0').$$

That is, assuming $z := (\lambda^1 w_1^{0\prime} + \cdots + \lambda^4 w_4^{0\prime})$ is linearly independent of $w_1^0, \ldots, w_4^0$, we obtain that $\mathbf{I}_0^{E_t\prime}$ is generated by ϕ_0 and $z \lrcorner P|_{E_0}$. Otherwise, just differentiate further. □

We would like to work with vector bundles over our entire space, which can be achieved by considering the product of Grassmann bundles $G(\operatorname{rank} \hat{M}_1, S^{\delta_1}\mathcal{S}^*) \times \cdots \times G(\operatorname{rank} \hat{M}_c, S^{\delta_c}\mathcal{S}^*) \to B$. Over $B' \subset B$ we have a well-defined section of this bundle. Using the compactness of the Grassmannian and the limiting procedure described previously, we extend this section to obtain a space $\tau : \mathcal{B} \to B$ with fiber over points of B' a single point. Here $\mathcal{B}$ is given by the section over points of B' and the union of the limit points over the points of $B \backslash B'$. Thus each M_j (resp. $\hat{M}_j$) gives rise to a well-defined vector bundle $\mathbf{M}_j \to \mathcal{B}$ (resp. $\hat{\mathbf{M}}_j \to \mathcal{B}$, where $\hat{\mathbf{M}}_j \subset \tau^*(S^{\delta_j}\mathcal{S}^*)$), and we have the corresponding bundle of ideals $\mathbf{I} \subset \tau^*(\operatorname{Sym}(\mathcal{S}^*))$.

Let $\mathbf{S} = \tau^*(\mathcal{S})$ and $\mathcal{O}_{\mathbb{P}(\mathbf{S})}(\delta) = \tilde{\tau}^*(\mathcal{O}_{\mathbb{P}\mathcal{S}}(\delta))$, where $\tilde{\tau} : \mathbf{S} \to \mathcal{S}$ is the lift of τ. Consider the projection $q : \mathbb{P}(\mathbf{S}) \to \mathcal{B}$ and the bundles

$$q^*(\mathbf{M}_j)^* \otimes \mathcal{O}_{\mathbb{P}(\mathbf{S})}(\delta_j).$$

Then $q^*(\mathbf{M}_1)^* \otimes \mathcal{O}_{\mathbb{P}(\mathbf{S})}(\delta_1) = q^*(\hat{\mathbf{M}}_1)^* \otimes \mathcal{O}_{\mathbb{P}(\mathbf{S})}(\delta_1)$ has a canonical section $\mathbf{s}_1$ whose zero set $Z_1 \subset \mathbb{P}(\mathbf{S})$ is the zero set of $(\mathbf{I})_{\delta_1}$. For each $2 \leq j \leq c$, the corresponding bundle $q^*(\hat{\mathbf{M}}_j)^* \otimes \mathcal{O}_{\mathbb{P}(\mathbf{S})}(\delta_j)$ has a canonical section $\hat{\mathbf{s}}_j$ whose zero set $Z_j \subset \mathbb{P}(\mathbf{S})$ is the zero set of $(\mathbf{I})_{\delta_j}$.

Fix a general point $x \in X_B$, and let $\mathbf{C}_x = \tau^{-1}(\mathcal{C}_x) \subset \mathcal{B}$. The essential observation is that $\dim \tilde{\mathcal{C}}_x \geq r = \sum_j \operatorname{rank} \mathbf{M}_j$, so we expect $Z_c \cap q^{-1}(\mathbf{C}_x)$ to be nonempty. This would imply the existence of singular points because the image of Z_c in X_B is contained in X_{sing}.

In more detail, we have a sequence of vector bundles $q^*(\mathbf{M}_1)^* \otimes \mathcal{O}_{\mathbb{P}(\mathbf{S})}(\delta_1), \dots, q^*(\mathbf{M}_c)^* \otimes \mathcal{O}_{\mathbb{P}(\mathbf{S})}(\delta_c)$ over $\mathbb{P}(\mathbf{S})$, whose ranks add up to r, such that $q^*(\mathbf{M}_1)^* \otimes \mathcal{O}_{\mathbb{P}(\mathbf{S})}(\delta_1)$ is equipped with a canonical section $\mathbf{s}_1$; and restricted to its zero set Z_1, $q^*(\mathbf{M}_2)^* \otimes \mathcal{O}_{\mathbb{P}(\mathbf{S})}(\delta_2)$ has a canonical section $\mathbf{s}_2$; ... such that if everything were to work out as expected then the zero set Z_c of $\mathbf{s}_c$, which is defined as a section of $q^*(\mathbf{M}_c)^* \otimes \mathcal{O}_{\mathbb{P}(\mathbf{S})}(\delta_c)$ over Z_{c-1}, would have codimension r, which is the dimension of $\mathbb{P}(\mathbf{S})|_{\mathbf{C}_x}$. Thus we expect $Z_c \cap \mathbb{P}(\mathbf{S})|_{\mathbf{C}_x} \neq \emptyset$, which would imply that $\tilde{\mathcal{C}}_x \cap X_{\text{sing}} \neq \emptyset$. Note that a sufficient condition for this is

$$c_{\text{top}}(q^*(\mathbf{M}_1)^* \otimes \mathcal{O}_{\mathbb{P}(\mathbf{S})|_{\mathbf{C}_x}}(\delta_1)) \cdot c_{\text{top}}(q^*(\mathbf{M}_2)^* \otimes \mathcal{O}_{\mathbb{P}(\mathbf{S})|_{\mathbf{C}_x}}(\delta_2)) \\ \cdots c_{\text{top}}(q^*(\mathbf{M}_c)^* \otimes \mathcal{O}_{\mathbb{P}(\mathbf{S})|_{\mathbf{C}_x}}(\delta_c)) \neq 0, \quad (5)$$

where the intersection takes place in the Chow group of codimension-r cycles on $\mathbb{P}(\mathbf{S})|_{\mathbf{C}_x}$.

We were not able to prove this in general, but we are able to show the following.

THEOREM 3.6. *The zero set of the canonical section of* $q^*\hat{\mathbf{M}}_1^* \otimes \tau^*(\mathcal{O}_{\mathbb{P}(W/\hat{x})}(\delta_1))|_{\mathbf{C}_x}$ *is always at least of the expected dimension.*

Another natural case to consider is the case where the $\mathbf{M}_j$ are all line bundles. For instance, consider the even further special case where there is just $\mathbf{M}_1, \mathbf{M}_2$ and both are line bundles. This case splits into two subcases based on whether or not the zero section of $\mathbf{s}_1$ surjects onto all of X_B. In Section 7 we show that, if $Z(\mathbf{s}_1)$ fails to surject onto X_B, then Conjecture 1.1 indeed holds.

Since $q^*(\mathbf{M}_j)^* \otimes \mathcal{O}_{\mathbb{P}(\mathbf{S})}(\delta_j)$ has only a section defined over Z_{j-1}, it will be more convenient to work with the bundles $q^*(\hat{\mathbf{M}}_j)^* \otimes \mathcal{O}_{\mathbb{P}(\mathbf{S})}(\delta_j)$, which have everywhere defined sections $\hat{\mathbf{s}}_j$.

The best situation for proving results about sections of bundles is when the bundles are ample, which fails here. However, in Section 5 we show that if x is sufficiently general then the bundles $\hat{\mathbf{M}}_j^* \otimes \tau^*(\mathcal{O}_{\mathbb{P}(W/\hat{x})}(\delta_j))$ are *generically ample* when restricted to $\mathbf{C}_x$. As we will prove in Lemma 6.1, this ensures that the zero locus $Z(\hat{\mathbf{s}}_j) \subset \mathbb{P}(\mathbf{S})|_{\mathbf{C}_x}$ of the canonical section of $q^*(\hat{\mathbf{M}}_j)^* \otimes \mathcal{O}_{\mathbb{P}(\mathbf{S})|_{\mathbf{C}_x}}(\delta_j)$ is nonempty if the rank of $\hat{\mathbf{M}}_j$ is smaller than r. This is what we will use in Section 6 to prove Theorem 3.6.

4. Examples

In this section, we illustrate how to construct local equations for the singular locus on $\mathcal{C}_x$ for some well-known cases of hypersurfaces containing large families of lines. In the first case we consider hypersurfaces containing a complete intersection of sufficiently low multidegree; in the second example, we consider hypersurfaces in $\mathbb{P}^5$ containing the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^2$. In both cases it is possible to prove directly that such hypersurfaces have to be singular. Instead of doing this, we explicitly write out $M_1, \dots, M_c$ in each case and use this construction to prove the existence of singular points.

Although our approach is based on the study of tangent spaces, it is important to stress that $T_E\mathbb{F}(X) \geq n-2$ at a general point $E \in B$ is not a sufficient condition for the singularity of a degree-d hypersurface X with $d \geq n$. For instance, it is well known that there are nonsingular hypersurfaces X for which $\mathbb{F}(X)$ can be nonreduced; a typical example is given by Fermat hypersurfaces [4, Sec. 2.5]. In Section 4.3 we revisit this example from our point of view.

4.1. Complete Intersections. Let $Y = \{g_1 = \cdots = g_r = 0\} \subset \mathbb{P}(W)$ be a complete intersection of multidegree $(d_1, \dots, d_r)$. Assume that $\sum_{i=1}^{r} d_i \leq n-r$, $r \geq 2$, and that Y is general. Then, by [5, Thm. 2.1], the Fano scheme $B := \mathbb{F}(Y)$ has dimension $2n-2-r-\sum_{i=1}^{r} d_i \geq n-2$.

Let X be a degree-$d \geq n$ hypersurface containing Y. Then there exist polynomials $h_1, \dots, h_r$, respectively of degrees $d-d_1, \dots, d-d_r$, such that X is defined by the equation $P = g_1h_1 + g_2h_2 + \cdots + g_rh_r = 0$.

By computing derivatives, one finds that the linear subspace

$$\{(w \lrcorner P)|_E : w \in W/E\} \subset S^{d-1}E^*$$

for $E \in B$ equals the degree-$(d-1)$ part of the ideal generated by the restrictions of $h_1, \dots, h_r$ to E. This yields

$$\operatorname{Image} \sigma_{(X,E)} = S^{d_1}E^* \circ h_1|_E + \cdots + S^{d_r}E^* \circ h_r|_E.$$

Assume that

$$d_1 = \cdots = d_{i_1} > d_{i_1+1} = \cdots = d_{i_1+i_2} > \cdots > d_{i_1+\cdots+i_{c-1}+1} = \cdots = d_r$$

and set $\delta_j = d - d_{i_j}$ for every $j = 1, \dots, c$. Then the vector bundles $M_1, \dots, M_c$ are given by

$$\begin{aligned} M_{1,E} &= \langle h_1|_E, \dots, h_{i_1}|_E \rangle \subset S^{\delta_1}E^*, \\ M_{2,E} &= \langle h_{i_1+1}|_E, \dots, h_{i_1+i_2}|_E \rangle \subset S^{\delta_2}E^*, \\ &\ \ \vdots \\ M_{c,E} &= \langle h_{i_1+\cdots+i_{c-1}+1}|_E, \dots, h_r|_E \rangle \subset S^{\delta_c}E^*. \end{aligned}$$

This implies that the zero locus of the canonical section of $q^*M_j^* \otimes \mathcal{O}_{\mathbb{P}(\mathcal{S})_B}(\delta_j)$ is the preimage of $X_B \cap \{h_1 = \cdots = h_{i_1+\cdots+i_j} = 0\}$ under the projection $\mathbb{P}(\mathcal{S})|_B \to X_B$. The cone $\tilde{\mathcal{C}}_x$ has dimension at least r for every point x of X_B. Since the conditions $h_1 = \cdots = h_r = 0$ define a subscheme of $\mathbb{P}^n$ of codimension at most r, the zero locus of the canonical section $q^*M_j^* \otimes \mathcal{O}_{\mathbb{P}(\mathcal{S})|_{\mathcal{C}_x}}(\delta_j)$ is nonempty for every $x \in X$. This implies $X_{\text{sing}} \cap \tilde{\mathcal{C}}_x \neq \emptyset$.

4.2. Hypersurfaces in $\mathbb{P}^5$ Containing a Segre Product. This example is due to J. Harris and was communicated to us by A. J. de Jong.

Consider the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^5$. Its image Y is a subvariety of codimension 2 in $\mathbb{P}^5$ defined by the condition

$$\operatorname{rank}\begin{pmatrix} x_0 & x_1 & x_2 \\ x_3 & x_4 & x_5 \end{pmatrix} = 1.$$

The Fano scheme of the variety Y has two irreducible components: a 2-dimensional family A of lines of the form $\mathbb{P}^1 \times \{p\}$ for $p \in \mathbb{P}^2$ and a 3-dimensional family B of lines of the form $\{p\} \times \ell$ with $p \in \mathbb{P}^1$ and $\ell \in \check{\mathbb{P}}^2$.

Let X be a hypersurface of degree $d \geq 5$ containing Y. Then X has an equation of the form

$$P = \begin{vmatrix} h_0 & h_1 & h_2 \\ x_0 & x_1 & x_2 \\ x_3 & x_4 & x_5 \end{vmatrix},$$

where the h_i are homogeneous polynomials of degree $d-2$. Here $\dim B = 3 = n-2$ and X_B is the Segre variety Y. Let x be a general point of X_B. Then the cone $\tilde{\mathcal{C}}_x$ of B-lines passing through x is the unique 2-plane contained in X_B and passing through x.

Without loss of generality, we may assume that $x = [1,0,0,0,0,0]$ and that $\tilde{\mathcal{C}}_x \subset \mathbb{P}^5$ is defined by equations $x_3 = x_4 = x_5 = 0$. Consider the line E_0: $x_2 = x_3 = x_4 = x_5 = 0$. Then the linear subspace

$$\Lambda_0 := \{(w \lrcorner P)|_{E_0} : w \in W/E_0\} \subset S^{d-1}E_0^*$$

equals the degree-$(d-1)$ part of the ideal generated by the polynomials $h_2|_{E_0}$ and $(h_0x_1 - h_1x_0)|_{E_0}$, which are respectively of degrees $d-2$ and $d-1$. This follows from the fact that Λ_0 is generated by $(e_3 \lrcorner P)|_{E_0} = -h_2x_1$, $(e_4 \lrcorner P)|_{E_0} = h_2x_0$, and $(e_5 \lrcorner P)|_{E_0} = h_0x_1 - h_1x_0$. Hence

$$\operatorname{Image} \sigma_{(X,E_0)} = (h_2|_{E_0}, (h_0x_1 - h_1x_0)|_{E_0})_{d-1}.$$

This description can be extended to every $E \in \mathcal{C}_x$ and can be used to define the line bundle $M_1 \subset S^{d-2}\mathcal{S}^*|_{\mathcal{C}_x}$. Note that in this case one has $\hat{M}_{2,E} = \{(w \lrcorner P)_E : w \in W/E\} \subset S^{d-1}E^*$ and $M_2 = \hat{M}_2/(\mathcal{S}^* \circ M_1)$. Furthermore, $(w \lrcorner P)|_E \in S^{d-1}E^*$ lies in the ideal generated by $M_{1,E} \subset S^{d-2}E^*$ if and only if the 2-plane $\mathbb{P}(\langle w, E\rangle)$ contains a line of the form $\mathbb{P}^1 \times \{p\}$ for some $p \in \mathbb{P}E$.

One can prove the existence of singular points on $\tilde{\mathcal{C}}_x$ as follows. In this case $\mathcal{C}_x$ is a projective line, the vector bundle $\mathcal{S}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$, and $\mathbb{P}(\mathcal{S})|_{\mathcal{C}_x}$ is the blow-up of $\tilde{\mathcal{C}}_x$ at x. In particular, its Weil group is generated by the class ξ_0 of the exceptional divisor and by the class F of a fiber of $\mathbb{P}(\mathcal{S})|_{\mathcal{C}_x} \to \mathcal{C}_x$ with intersections given by $\xi_0^2 = -1$, $\xi_0 \cdot F = 0$, and $F^2 = 0$. By computing the intersection numbers with ξ_0 and F, one obtains

$$c_1(q^*(M_1^*) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{S})|_{\mathcal{C}_x}}(d-1)) = (d-1)\xi_0 + (d-2)F$$

and

$$c_1(q^*(M_2^*) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{S})|_{\mathcal{C}_x}}(d-2)) = (d-2)\xi_0 + (d-1)F.$$

Therefore, the number of points in $X_{\text{sing}} \cap \tilde{\mathcal{C}}_x$ is

$$(d-1)^2 + (d-2)^2 - (d-1)(d-2) = d^2 - 3d + 3.$$

4.3. Fermat Hypersurfaces. In this section, we consider degree-d Fermat hypersurfaces in $\mathbb{P}^n$ with $d \geq n \geq 4$. This is the typical example of a nonsingular hypersurface with $\mathbb{F}(X)$ of dimension $n-3$ but nonreduced, so that $\dim T_E\mathbb{F}(X) \geq n-2$ holds for every $E \in \mathbb{F}(X)$ (see [4, Sec. 2.5]). It is interesting to see which vector bundles M_i arise in this case and to gain thereby intuition regarding why the degeneracy of $\mathbb{F}(X)$ does not imply singularity here.

Let $X \subset \mathbb{P}^n$ be defined by the vanishing of $P = x_0^d + x_1^d + \cdots + x_n^d$ with $d \geq n$. Let B be an irreducible component of $\mathbb{F}(X)$. Then, up to reordering the coordinates $x_0, \dots, x_n$, there is an index $2 \leq j \leq \lfloor n/2 \rfloor$ such that the lines in B are exactly the lines joining a point of

$$X_1 := X \cap \{x_0 = x_1 = \cdots = x_{j-1} = 0\}$$

with a point of

$$X_2 := X \cap \{x_j = x_{j+1} = \cdots = x_n = 0\}.$$

In other words, the variety X_B is the join of the varieties X_1 and X_2. Note that $\dim X_1 = n-j-1$ and $\dim X_2 = j-2$, so that $\dim B = (n-j-1)+j-2 = n-3$ and $\dim X_B = \dim B + 1 = n-2$. For every $x \in X_B \setminus (X_1 \cup X_2)$, there is exactly one line of B passing through x. Nevertheless, for every $E \in B$ the embedded tangent space to X_B at points on $\mathbb{P}E \setminus (X_1 \cup X_2)$ is constant, so that $T_E\mathcal{C}_x$ has dimension $\dim X_B - 1 = n-3$.

In view of Lemma 3.1, the points on $X_{\mathrm{sing}} \cap \mathbb{P}E$ are defined on $\mathbb{P}E$ by the vanishing of $r = n-1-\dim T_E B + \dim T_E\mathcal{C}_x$ equations on $\mathbb{P}E$. Let us compute these equations for a general line $E \in B$. Every line $E \in B$ is spanned by vectors of the form

$$v_1 = (\mu_0, \mu_1, \dots, \mu_{j-1}, 0, \dots, 0), \quad \sum_{i=0}^{j-1} \mu_i^d = 0,$$

$$v_2 = (0, \dots, 0, \mu_j, \mu_{j+1}, \dots, \mu_n), \quad \sum_{i=j}^{n} \mu_i^d = 0,$$

so that every point of $\mathbb{P}E$ is of the form $[sv_1 + tv_2]$.

Consider the basis $\{e_i\}$ of W dual to the coordinate system $\{x_i\}$. Then $(e_i \lrcorner P)_E = \mu_i^{d-1}\xi_i^{d-1}$, where $\xi_i = t$ if $i \leq j-1$ and $\xi_i = s$ if $i \geq j$. From this it follows that the image of $\sigma_{(X,E)}$ is the subspace of $S^dE^* \cong S^d\langle t, s\rangle$ generated by t^d, $t^{d-1}s$, ts^{d-1}, and s^d. This implies $\dim T_E B = \dim\ker\sigma_{(X,E)} = \dim(E^* \otimes W/E) - 4 = 2n-6$. Hence the number of equations for $X_{\mathrm{sing}} \cap \mathbb{P}E$ is $n-1-(2n-6)+(n-3) = 2$, and these equations are $s^{d-1} = t^{d-1} = 0$.

Note that the equation $s = 0$ defines the point $\mathbb{P}E \cap X_1$, whereas $t = 0$ defines $\mathbb{P}E \cap X_2$. Therefore, any putative singular points of X must lie on the intersection $X_1 \cap X_2$, which is empty.

In this case, if one lets E move in B, then the equations for $X_{\mathrm{sing}} \cap \mathbb{P}E$ give rise to a rank-2 vector bundle $M_1 \subset S^{d-1}\mathcal{S}^*$ that can be described as follows. Consider the line subbundle $N_1 \subset S^{d-1}\mathcal{S}^*$ defined by the condition of vanishing on X_1 with multiplicity $d-1$. Analogously, consider the line subbundle $N_2 \subset S^{d-1}\mathcal{S}^*$ of forms vanishing on X_2 with multiplicity $d-1$. Then, for every $E \in B$, the fiber $M_{1,E}$ is the linear span of $N_{1,E}$ and $N_{2,E}$ inside $S^{d-1}E^*$.

5. Generic Ampleness

Recall [7, Ex. 12.1.10] that a vector bundle $\mathcal{E}$ over a variety X is *generically ample* if it is generated by global sections and the canonical map $\mathbb{P}\mathcal{E}^* \to \mathbb{P}(H^0(X, \mathcal{E})^*)$ is generically finite. The locus where it is not finite is called the *disamplitude locus* $\mathrm{Damp}(\mathcal{E})$. In particular, if $Y \subset X$ is a subvariety such that $\mathcal{E}|_Y$ has a trivial quotient subbundle, then $Y \subset \mathrm{Damp}(\mathcal{E})$.

Generically ample bundles of rank $r \leq \dim X$ have the property that $c_1(\mathcal{E}), \ldots, c_r(\mathcal{E})$ are all positive in the sense that their classes in the Chow group of X are linear combinations of effective classes with nonnegative coefficients not all equal to 0.

To use a more compact notation, from now on we will write

$$\mathcal{O}_{\mathcal{C}_x}(k) := \mathcal{O}_{\mathbb{P}(W/\hat{x})}(k)|_{\mathcal{C}_x} \quad \text{and} \quad \mathcal{O}_{\mathbf{C}_x}(k) := \tau^*(\mathcal{O}_{\mathcal{C}_x}(k)) = \tau^*(\mathcal{O}_{\mathbb{P}(W/\hat{x})}(\delta_j))|_{\mathbf{C}_x}.$$

Analogous conventions will be used for subvarieties of $\mathcal{C}_x$ and $\mathbf{C}_x$.

LEMMA 5.1. *For general $x \in X_B$, the bundles $\hat{\mathbf{M}}_j^* \otimes \mathcal{O}_{\mathbf{C}_x}(\delta_j)$ are generically ample.*

Proof. First, global generation is clear, as for all the $\hat{\mathbf{M}}_j$ we have a surjective map

$$S^{\delta_j}\mathbf{S} \otimes \mathcal{O}_{\mathbf{C}_x}(\delta_j) \to \hat{\mathbf{M}}_j^* \otimes \mathcal{O}_{\mathbf{C}_x}(\delta_j).$$

Now take any choice of splitting $W = \hat{x} \oplus W'$ so that the left-hand side becomes $\tau^*(\mathcal{O}_{\mathcal{C}_x} \oplus \mathcal{O}_{\mathcal{C}_x}(1) \oplus \cdots \oplus \mathcal{O}_{\mathcal{C}_x}(\delta_j))$, which is a direct sum of finitely generated bundles.

The locus where the canonical map

$$\mathbb{P}\big(\textstyle\bigoplus_{i=0}^{\delta_j} \mathcal{O}_{\mathbb{P}(W/\hat{x})}(-i)\big) \to \mathbb{P}\big(H^0\big(\mathbb{P}(W/\hat{x}), \textstyle\bigoplus_{i=0}^{\delta_j} \mathcal{O}_{\mathbb{P}(W/\hat{x})}(i)\big)^*\big)$$

is not finite is the $\mathbb{P}\mathcal{O}_{\mathbb{P}(W/\hat{x})}$ factor. Hence, when we restrict to $\mathcal{C}_x \subset B$ and pull back to $\mathbf{C}_x$, $\mathrm{Damp}(\hat{\mathbf{M}}_j^* \otimes \mathcal{O}_{\mathbf{C}_x}(\delta_j))$ is contained in the union of the following two loci:

- the locus where the map $\tau : \mathbf{C}_x \to \mathcal{C}_x$ has positive-dimensional fibers;
- the projection to $\mathbf{C}_x$ of the locus where the image of

  $$\mathbb{P}(\hat{\mathbf{M}}_j \otimes \mathcal{O}_{\mathbf{C}_x}(-\delta_j)) \to \mathbb{P}\big(\textstyle\bigoplus_{i=0}^{\delta_j} \mathcal{O}_{\mathbf{C}_x}(-i)\big)$$

  intersects $\mathbb{P}(\mathcal{O}_{\mathbf{C}_x})$.

The lemma will follow from Lemma 5.2 and the fact that the general fiber of $\mathbf{C}_x \to \mathcal{C}_x$ is finite if x is a general point of X_B. Note that the image of $\mathbb{P}(\hat{\mathbf{M}}_j \otimes \mathcal{O}_{\mathbf{C}_x}(-\delta_j))$ inside $\mathbb{P}\big(\bigoplus_{i=0}^{\delta_j} \mathcal{O}_{\mathbf{C}_x}(-i)\big)$ intersects $\mathbb{P}(\mathcal{O}_{\mathbf{C}_x})$ precisely over the points $E \in \mathbf{C}_x$ such that the fiber $\hat{\mathbf{M}}_{j,E}$ contains a nonzero polynomial vanishing at x with multiplicity δ_j. □

LEMMA 5.2. *For general $x \in X_B$ and general $E \in \mathcal{C}_x$, all nonzero elements $P \in (\mathbf{I}_E)_k$ vanish at x with multiplicity at most $k - 1$ for any integer $k \leq \delta_c$.*

Proof. Fix $E \in B$. Then the locus

$$\{[P] \in \mathbb{P}((\mathbf{I}_E)_k) \mid P = f^k \text{ for some } f \in E^*\}$$

is the intersection of $\mathbb{P}((\mathbf{I}_E)_k)$ with a degree-k rational normal curve contained in $\mathbb{P}(S^k E^*)$. Hence, it consists of at most a finite number of points $[P_1], \dots, [P_R]$. Thus it suffices to choose a point $x \in \mathbb{P}E$ such that $P_j(x) \neq 0$ for all $j = 1, \dots, R$. □

6. Proof of Theorem 3.6

Theorem 3.6 is a consequence of Lemma 5.1 for $j = 1$ when combined with the following lemma for $M = \hat{\mathbf{M}}_1|_{\mathbf{C}_x}$.

LEMMA 6.1. *Let $M \subset S^p \mathbf{S}^*|_{\mathbf{C}_x}$ be a vector bundle such that $M^* \otimes \mathcal{O}_{\mathbf{C}_x}(p)$ is generically ample. Then the zero locus of the canonical section of the bundle $q^* M^* \otimes \mathcal{O}_{\mathbb{P}\mathbf{S}|_{\mathbf{C}_x}}(p)$ over $\mathbb{P}(\mathbf{S})|_{\mathbf{C}_x}$ is of dimension at least* $\dim \mathbf{C}_x + 1 - \operatorname{rank}(M)$.

The proof of Lemma 6.1 follows by several reductions that reduce the question to a basic fact about intersections on nontrivial $\mathbb{P}^1$-bundles over a curve.

LEMMA 6.2. *Let $p_\xi \colon S \to \xi$ be a $\mathbb{P}^1$-bundle over a curve ξ with a section $e \colon \xi \to S$ of negative self-intersection. If $\tilde{D}_1$ and $\tilde{D}_2$ are effective divisors of S not contained in the image of e such that the restriction of p_ξ to each of them is finite, then $\tilde{D}_1 \cap \tilde{D}_2 \neq \emptyset$.*

Proof. The group of Weil divisors of S (up to numerical equivalence) is generated by the class ξ_0 of the image of e and the class F of a fiber of p_ξ. Since S is not a product, one has $F^2 = 0$, $\xi_0 \cdot F = 1$, and $\xi_0^2 = -k$ with k a positive integer. Choose irreducible components D_1, D_2 of the divisors, different from the image of e. Then $D_i = a_i \xi_0 + b_i F$ with $a_i \geq 1$ (since a_i is the degree of $p_\xi|_{D_i}$) and $D_i \cdot \xi_0 = b_i - a_i k \geq 0$. Then $D_1 \cdot D_2 = -a_1 a_2 k + a_1 b_2 + a_2 b_1 \geq a_1 a_2 k > 0$. From this the claim follows. □

The proof of Lemma 6.1 relies on the following lemma.

LEMMA 6.3. *Let $M \subset S^p \mathbf{S}^*|_{\mathbf{C}_x}$ be a vector bundle such that $M^* \otimes \mathcal{O}_{\mathbf{C}_x}(p)$ is generically ample. Let $W' \subset W$ be any hyperplane not containing $\hat{x}$, set $H' = \mathcal{C}_x \cap \mathbb{P}W'$, and let $H \subset \mathbb{P}(\mathbf{S}|_{\mathbf{C}_x})$ be the preimage of H' under the map $\mathbb{P}(\mathbf{S}|_{\mathbf{C}_x}) \to \tilde{\mathcal{C}}_x$ induced by τ. Let $\mathbf{s}_M$ denote the canonical section of $q^* M^* \otimes \mathcal{O}_{\mathbb{P}\mathbf{S}|_{\mathbf{C}_x}}(p)$.*

Then the intersection $Z(\mathbf{s}_M) \cap H$ has dimension at least $\dim \mathbf{C}_x - \operatorname{rank}(M)$. *In particular, it is nonempty if* $\operatorname{rank} M \leq \dim \mathbf{C}_x$.

Proof. Consider the section $s_{M,W'} \in H^0(H', q^* \otimes \mathcal{O}_{\mathbb{P}(\mathbf{S}|_{\mathbf{C}_x})}(p))$ obtained by restricting $\mathbf{s}_M$ to H'. Then $Z(\mathbf{s}_M) \cap H = Z(s_{M,W'})$.

Observe that $\rho \colon \tilde{\mathcal{C}}_x \to \mathcal{C}_x$ and $q \colon \mathbb{P}(\mathbf{S}|_{\mathbf{C}_x}) \to \mathbf{C}_x$ become isomorphisms when restricted to, respectively, H' and H. In particular, since H' was a hyperplane section of $\mathcal{C}_x$, the isomorphism $H \cong \mathbf{C}_x$ so obtained induces an isomorphism

$\mathcal{O}_{\mathbb{P}(\mathbf{S}|_{\mathbf{C}_x})}(1)|_H \cong \mathcal{O}_{\mathbf{C}_x}(1)$. Since the isomorphism $H \cong \mathbf{C}_x$ also induces an isomorphism $(q^*M)|_H \cong M$, one can view $s_{M,W'}$ as a global section of $M^* \otimes \mathcal{O}_{\mathbf{C}_x}(p)$. Therefore, if $Z(s_{M,W'}) \subset H$ is nonempty, it has codimension at most rank M in H [7, Prop. 14.1b]. It remains to show $Z(s_{M,W'}) \neq \emptyset$ if rank $M \leq \dim \mathbf{C}_x$.

Recall from [7, Sec. 14.1] that there is a localized Chern class associated to the section $s_{M,W'}$, which is a class in the Chow group of $Z(s_{M,W'})$ whose pull-back under the inclusion $Z(s_{M,W'}) \to \mathcal{C}_x$ is the top Chern class of $M^* \otimes \mathcal{O}_{\mathbf{C}_x}(p)$. Since $M^* \otimes \mathcal{O}_{\mathbf{C}_x}(p)$ is generically ample and of rank $\leq \dim \mathbf{C}_x$, its top Chern class is positive. So the Chow group of $Z(s_{M,W'})$ contains a nontrivial class, and in particular $Z(s_{M,W'})$ cannot be empty. □

Proof of Lemma 6.1. For every $E \in \mathbf{C}_x$, consider $N_E := (S^{p-1}E^* \circ \hat{x}^{\perp}) \cap M_E$, the linear subspace of M_E of forms vanishing on the point x. Without loss of generality, when E varies N_E gives rise to a vector subbundle $N \subset M$ of codimension 1. Indeed, if it were not so, there would be a point $E \in \mathbf{C}_x$ such that $N_E = M_E$, and then (E, x) would be a point of the zero locus of the canonical section, thus implying the claim.

We have an exact sequence $0 \to N \to M \to L \to 0$, where L is the quotient line bundle. Since $q^*N^* \otimes \mathcal{O}_{\mathbf{C}_x}(p)$ is a corank-1 quotient of $q^*M^* \otimes \mathcal{O}_{\mathbf{C}_x}(p)$, we can apply Lemma 6.3 to it. Hence, the zero locus of the canonical section of $q^*N^* \otimes \mathcal{O}_{\mathbf{C}_x}(p)$ contains an irreducible component Z, which intersects all subvarieties $H \subset \mathbb{P}(\mathbf{S})|_{\mathbf{C}_x}$ that come from preimages of general hyperplane sections of $\tilde{\mathcal{C}}_x$.

Without loss of generality, we may assume that Z is of dimension 1 and that $q' := q|_Z \colon Z \to q(Z) =: \xi$ is a finite surjective map. Recall that the group of Weil divisors (up to numerical equivalence) of the ruled surface $\mathbb{P}(\mathbf{S})|_\xi$ is generated by the class ξ_0 of the tautological section of q' (i.e., $(\xi_0)_E = (E, x)$) and the class F of a fiber of q. From the effectivity of Z and from Lemma 6.3 we obtain $Z \cdot F \geq 1$ and $Z \cdot \xi_0 \geq 0$.

To prove the claim, it suffices to show $Z \cdot c_1(q^*L^*|_\xi \otimes \mathcal{O}_{\mathbb{P}\mathbf{S}|_\xi}(p)) > 0$. We have $c_1(q^*L^*|_\xi \otimes \mathcal{O}_{\mathbb{P}\mathbf{S}|_\xi}(p)) \cdot F = d$ because $c_1(q^*L^* \otimes \mathcal{O}_{\mathbb{P}\mathbf{S}|_\xi}(p)) \cdot F = c_1(q^*L^*) \cdot F + c_1(\mathcal{O}_{\mathbb{P}\mathbf{S}|_\xi}(p)) \cdot F = 0 + p = p$. Now recall that the canonical section of $q^*N^*|_\xi \otimes \mathcal{O}_{\mathbb{P}\mathbf{S}|_\xi}(p)$ vanishes on ξ_0 by construction. Therefore, the canonical section of $q^*M^*|_\xi \otimes \mathcal{O}_{\mathbb{P}\mathbf{S}|_\xi}(p)$ induces a section s_L of $q^*L^*|_\xi \otimes \mathcal{O}_{\mathbb{P}\mathbf{S}|_\xi}(p)$ on ξ_0. Since $N_E \subsetneq M_E$ for every $E \in \mathbf{C}_x$, we have that s_L cannot vanish identically on ξ_0. Hence $c_1(q^*L^*|_\xi \otimes \mathcal{O}_{\mathbb{P}\mathbf{S}|_\xi}(p)) \cdot \xi_0 \geq 0$, because it is the class of $Z(s_L)$ on ξ_0. Then the asserted inequality follows from Lemma 6.2 because $c_1(q^*L^*|_\xi \otimes \mathcal{O}_{\mathbb{P}\mathbf{S}|_\xi}(p))$ is numerically equivalent to an effective divisor satisfying the hypotheses of Lemma 6.2. □

7. Two Line Bundles

In this section we prove the following result, which was announced in Section 3.

LEMMA 7.1. *Assume that $\mathbf{M}_1$ and $\mathbf{M}_2$ are line bundles and that the projection $Z(\mathbf{s}_1) \to X_B$ is not surjective. Then, for every $x \in X_B$, the zero set of $\hat{\mathbf{s}}_2|_{\mathbf{C}_x}$ is of codimension ≤ 2 in $\mathbf{C}_x$.*

As in the previous arguments, it will be sufficient to work with a general point $x \in X_B$ and a sufficiently general irreducible curve $\xi \subseteq \mathbf{C}_x$ and show that the zero set of $\hat{\mathbf{s}}_2$ restricted to $\mathbb{P}(\mathbf{S})|_\xi$ is nonempty. The proof is based on showing that $Z(\hat{\mathbf{s}}_2) \cap \mathbb{P}(\mathbf{S})|_\xi$ coincides with the zero set of the canonical section of $(q|_\xi)^* N^* \otimes \mathcal{O}_{\mathbb{P}(\mathbf{S})|_\xi}(\delta_2)$, where $N \subset S^{\delta_2}\mathbf{S}|_\xi$ is a rank-2 vector bundle satisfying the hypotheses of Lemma 6.1. We construct N under the assumption that the zero set $Z(\mathbf{s}_1)$ does not intersect the tautological section of $\mathbb{P}(\mathbf{S})|_\xi \to \xi$.

Since $\mathbf{M}_1$ is a line bundle, we have that $Z(\mathbf{s}_1) \subset \mathbb{P}(\mathbf{S})$ intersects every fiber of $\mathbb{P}(\mathbf{S}) \to \mathcal{B}$ in δ_1 points, counted with multiplicity. This follows from the very construction of the canonical section $\mathbf{s}_1$.

Without loss of generality in the choice of x and ξ, we may assume that:

(i) $\mathcal{O}_{\mathbf{C}_x}(1)$ restricts to a generically ample line bundle $\mathcal{O}_\xi(1)$ on ξ;
(ii) $Z := Z(\mathbf{s}_1) \cap \mathbb{P}(\mathbf{S})|_\xi$ is not contained in the tautological section $\xi \to \mathbb{P}(\mathbf{S})|_\xi$;
(iii) $\hat{\mathbf{M}}_2^* \otimes \mathcal{O}_{\mathbf{C}_x}(\delta_2)$ is generically ample when restricted to ξ;
(iv) the map $q|_Z\colon Z \to \xi$ is finite.

The first assumption follows from the fact that $\mathbf{C}_x \to \mathcal{C}_x$ is generically finite; so $\mathbf{C}_x \not\subset \mathrm{Damp}(\mathcal{O}_{\mathbf{C}_x}(1))$, and the same holds for a generic $\xi \subset \mathbf{C}_x$. Assumption (ii) follows from the genericity of x, and (iii) follows from Lemma 5.1. Finally, if (iv) did not hold then $Z(\hat{\mathbf{s}}_2)$ would contain δ_2 points on every 1-dimensional fiber of $q|_Z$ (counted with multiplicity), thus showing $Z(\hat{\mathbf{s}}_2) \neq \emptyset$.

For the rest of this section, we will often omit the restriction to ξ from our notation. Recall that we have a short exact sequence

$$0 \to S^{\delta_2-\delta_1}\mathbf{S}^* \circ \mathbf{M}_1 \to \hat{\mathbf{M}}_2 \to \mathbf{M}_2 \to 0.$$

As a consequence, the section $\hat{\mathbf{s}}_2 \in H^0(\mathbb{P}(\mathbf{S}), q^*\hat{\mathbf{M}}_2^* \otimes \mathcal{O}_{\mathbb{P}(\mathbf{S})}(\delta_2))$ canonically induces a section $s \in H^0(Z, q^*\mathbf{M}_2^* \otimes \mathcal{O}_{\mathbb{P}(\mathbf{S})}(\delta_2))$. Assume that $Z(\hat{\mathbf{s}}_2) = \emptyset$; that is, $Z(s) = \emptyset$ on Z. Then s induces a trivialization $q^*\mathbf{M}_2^*|_Z \otimes \mathcal{O}_Z(\delta_2) \cong \mathcal{O}_Z$.

In this setup, Lemma 7.1 is equivalent to the following lemma.

LEMMA 7.2. *Assume Z does not intersect the image of the tautological section $s_0\colon \xi \to \mathbb{P}(\mathbf{S})|_\xi$. Then $Z(\hat{\mathbf{s}}_2) \cap \mathbb{P}(\mathbf{S})|_\xi \neq \emptyset$.*

Proof. Assume by way of contradiction that $Z(\hat{\mathbf{s}}_2) \cap \mathbb{P}(\mathbf{S})|_\xi$ is empty. Fix a line $E \in \xi$. The fiber $\hat{\mathbf{M}}_{2,E}$ is spanned by all degree-δ_2 multiples of polynomials in $\mathbf{M}_{1,E}$ and by an additional polynomial ϕ that does not vanish on any point of Z.

Recall that no nonzero polynomial in $\mathbf{M}_{1,E}$ vanishes at x. Therefore, the condition of vanishing at $x \in \mathbb{P}(E)$ with multiplicity $\delta_2 - \delta_1$ defines a 1-dimensional subspace of $S^{\delta_2-\delta_1}\mathbf{S}^* \circ \mathbf{M}_{1,E}$ and (for dimensional reasons) a 2-dimensional subspace N_E of $\hat{\mathbf{M}}_{2,E}$. Hence, without loss of generality we may assume that ϕ is a polynomial vanishing at x with multiplicity $\delta_2 - \delta_1$. If we let E vary, then N_E defines a rank-2 vector subbundle $N \subset \hat{\mathbf{M}}_2 \subset S^{\delta_2}\mathbf{S}^*$. Moreover, we have $N \otimes \mathcal{O}_\xi(-\delta_2 + \delta_1) \subset S^{\delta_1}\mathbf{S}^*$. This follows from the fact that the condition of vanishing at x with multiplicity at least k defines the subbundle

$$\mathcal{O}_\xi(k) \oplus \cdots \oplus \mathcal{O}_\xi(\delta_2) \subset \mathcal{O}_\xi \oplus \mathcal{O}_\xi(1) \oplus \cdots \oplus \mathcal{O}_\xi(\delta_2) \cong S^{\delta_2}\mathbf{S}^*.$$

For every $E \in \xi$, if we choose any $0 \neq \eta \in E^*$ that vanishes on $x \in \mathbb{P}(E)$, then the fiber of $N \otimes \mathcal{O}_\xi(-\delta_2 + \delta_1) \subset S^{\delta_1}\mathbf{S}^*$ over $E \in \xi$ is the locus of degree-δ_1 polynomials ψ over $\mathbb{P}(E)$ satisfying $\eta^{\delta_2-\delta_1} \circ \psi \in \hat{\mathbf{M}}_{2,E}$.

By the description of the fibers of $\hat{\mathbf{M}}_2$ just given, all points in the zero locus of the canonical section of $q^*(N \otimes \mathcal{O}_\xi(-\delta_2 + \delta_1)) \otimes \mathcal{O}_{\mathbb{P}(\mathbf{S})|_\xi}(\delta_1)$ belong to $Z(\hat{\mathbf{s}}_2)$. Hence, the canonical section of $q^*(N \otimes \mathcal{O}_\xi(-\delta_2 + \delta_1)) \otimes \mathcal{O}_{\mathbb{P}(\mathbf{S})|_\xi}(\delta_1)$ has empty zero locus.

On the other hand, $(N \otimes \mathcal{O}_\xi(-\delta_2+\delta_1))^* \otimes \mathcal{O}_\xi(\delta_1) = N^* \otimes \mathcal{O}_\xi(\delta_2)$ is a quotient of $\hat{\mathbf{M}}_2^* \otimes \mathcal{O}_\xi(\delta_2)$ and so, in particular, it is generically ample on ξ. Then Lemma 6.1 implies that the canonical section of $q^*(N \otimes \mathcal{O}_\xi(-\delta_2 + \delta_1)) \otimes \mathcal{O}_{\mathbb{P}(\mathbf{S})|_\xi}(\delta_1)$ is nonempty—a contradiction. □

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