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On the decomposition numbers of the Hecke algebra of $G(m, 1, n)$

By

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1. Introduction

It is recently found that many complex reflection groups have deformation of group algebras [A1], [AK], [BM]. For the group $G(m, 1, n)$ in the Shephard-Todd notation [C], [ST], the Hecke algebra $\mathcal{H}_A$ is the algebra over the polynomial ring $A=\boldsymbol{Z}[\boldsymbol{v}_1, \cdots, \boldsymbol{v}_m, \boldsymbol{q}, \boldsymbol{q}^{-1}]$ defined by generators $a_1, \cdots, a_n$ and relations

$$(a_1-\boldsymbol{v}_1)\cdots(a_1-\boldsymbol{v}_m)=0,\ (a_i-\boldsymbol{q})(a_i+\boldsymbol{q}^{-1})=0\ (2\leq i\leq n)$$
$$a_1a_2a_1a_2=a_2a_1a_2a_1,\quad a_ia_j=a_ja_i\ (j\geq i+2)$$
$$a_ia_{i+1}a_i=a_{i+1}a_ia_{i+1}\ (2\leq i\leq n-1)$$

This algebra is known to be A-free. If we specialize it to $\boldsymbol{v}_i=v_i$, $\boldsymbol{q}=q$, where $v_i\in\boldsymbol{C}$, $q\in\boldsymbol{C}^\times$, this algebra is denoted by $\mathcal{H}_C$.

We note here that the study of this algebra over a ring of integers is conjecturely related to the modular representation theory for the block algebras of the general linear group [BM].

One of the building blocks for the modular representation theory of $\mathcal{H}_C$ is the case that $v_1, \cdots, v_m$ are powers of $q^2\neq 1$, and we consider this case in this paper.

Let u_n be the Grothendieck group of the category of $\mathcal{H}_C$-modules. We set $u=\oplus u_n$. The purpose of this paper is to show that the graded dual of u is a highest weight module of $\mathfrak{g}(A_\infty)$ (resp. $\mathfrak{g}(A^{(1)}_{r-1})$) if q^2 is not root of unity (resp. a primitive r-th root of unity), and the dual basis of irreducible modules coincides with canonical basis. The proof heavily depends on Lusztig's theory of affine Hecke algebras and quantum groups, and Ginzburg's theory of affine Hecke algebras.

For $m=1$, our result verifies a conjecture of [LLT]. Hence their conjectural algorithm actually computes the decomposition numbers of the Hecke algebra of type A. We note here that there is an announcement of Grojnowski [Gr] on the decomposition numbers of the Hecke algebra of type A, but what we see here is that we can avoid the result at roots of unity to compute the de-

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composition numbers, although his result is interesting in its own right, since it focuses on the affine Hecke algebra of general type. The paper is organized as follows. In section 2, we review the semi-normal form representation of $\mathcal{H}_C$, and introduce Specht modules. In section 3, we transplant the induction theorem of Kazhdan-Lusztig to Ginzburg's theory. Since no literature is available, we add a proof using the properties of both K-theories explained in [KL], [CG] respectively, and some results from [KL] [CG]. Then Ginzburg's theory allows us to describe the decomposition numbers in terms of intersection cohomology complex. In section 4, we define the action on the graded dual of u and show that it is a highest weight module and the canonical basis corresponds to the dual basis of irreducible modules. Finally, we describe the module u^* in terms of Young diagrams.

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2. Hecke Algebra of $G(m, 1, n)$

(2.1) As was defined in the introduction, we denote by $\mathcal{H}_C$ the Hecke algebra of $G(m, 1, n)$ with specialized parameters. And we consider the case v_i are powers of $q^2 \neq 1$. In particular, $t_1, \cdots, t_n$ are invertible.

We recall that we set $t_1 = a_1$, $t_i = a_i t_{i-1} a_i$ $(2 \leq i \leq n)$, and we have $t_i t_j = t_j t_i$ and $t_i a_j = a_j t_i$ $(j \neq i-1, i)$. A consequence of this property is that this algebra is a quotient of the affine Hecke algebra for the general linear group, since we assume that $t_i (1 \leq i \leq n)$ are invertible. To see it, we use Bernstein presentation of the affine Hecke algebra H_{q^2}. Let $X = \oplus_{i=1}^{n} \mathbf{Z}\varepsilon_i$, $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, $S = \{s_i\}_{1 \leq i \leq n-1}$ be the simple reflections of S_n. Then H_{q^2} is generated by $\theta_x (x \in X)$ and $T_i (1 \leq i \leq n-1)$, and the defining relations are

$$\theta_x\theta_y = \theta_y\theta_x, \theta_0 = 1, \ (T_i - q)(T_i + q^{-1}) = 0 \ (1 \leq i \leq n-1)$$
$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \ (1 \leq i \leq n-2), \ T_i T_j = T_j T_i \ (j \geq i+2)$$
$$T_i \theta_x = \theta_x T_i \ (s_i x = x), \ T_i \theta_x T_i = \theta_{s_i x} \ (s_i x = x + \alpha_i)$$

Thus, by sending T_i to a_{i+1} $(1 \leq i \leq n-1)$ and θ_{ε_i} to t_i^{-1} $(1 \leq i \leq n)$, we have a surjective homomorphism $H_{q^2} \to \mathcal{H}_C$.

We sometimes call H_{q^2} the affine Hecke algebra of rank n. It is naturally embedded into the affine Hecke algebra of rank $n+m$ by the following homomorphism.

$$shift_m : \begin{cases} T_i \to T_{i+m} \\ \theta_{\varepsilon_i} \to \theta_{\varepsilon_{i+m}} \end{cases}$$

It is easy to see that it is an injective algebra homomorphism.

Remark. *Let $u_1, \cdots, u_r$ be the representative of distinct values of $v_1, \cdots, v_m$. We remark that there is a simple $\mathcal{H}_C$-module on which $(a_1 - u_1) \cdots (a_1 - u_r)$ acts as nonzero linear transformation. A simplest example is the Hecke algebra whose para-*

meters are $v_1=v_2=1$, $q^2\neq 1$, $n=2$, *and its representation*

$$a_1 \mapsto \begin{bmatrix} 1 & -q+q^{-1} \\ 0 & 1 \end{bmatrix}, \qquad a_2 \mapsto \begin{bmatrix} 0 & 1 \\ 1 & q-q^{-1} \end{bmatrix}.$$

It is irreducible, but t_1-1 *is not zero. Hence we can not assume* $v_1, \cdots, v_m$ *are distinct unless* $n=1$ *even when we consider irreducible modules.*

(2.2) We also recall that if we define the algebra $\mathcal{H}$ over the Laurent polynomial ring $\boldsymbol{C}[\boldsymbol{q}, \boldsymbol{q}^{-1}]$, then the specialization $f: \boldsymbol{q}\rightarrow q$ gives the isomorphism $\boldsymbol{C}\otimes_f\mathcal{H} \simeq \mathcal{H}_C$. By the semi-simplicity criterion [A2], $\boldsymbol{C}(\boldsymbol{q})\otimes\mathcal{H}$ is semi-simple if and only if $r=m$. In the general case $r<m$, the study of block structure of $\boldsymbol{C}(\boldsymbol{q})\otimes\mathcal{H}$ is closely related to the study of asymptotic Hecke algebra. The asymptotic Hecke algebra itself is deeply studied in [L1] [L2] [L3] [L4].

(2.3) We review the semi-normal form representation, which was obtained as a natural generalization of Hoefsmit's work [H].

Let F be a field. As is explained in [A2] (see also [AK]), as long as the specialized algebra $\mathcal{H}_F=F\otimes\mathcal{H}_A$ is a semi-simple algebra, we can associate a representation of $\mathcal{H}_F$ to each m-tuple of Young diagrams $\lambda=(\lambda^{(1)}, \cdots, \lambda^{(m)})$. They have a basis indexed by Young tableaux of shape λ, and these basis elements are simultaneous eigenvectors of $t_1, \cdots, t_n$. More precisely, let $T=(T^{(1)}, \cdots, T^{(m)})$ be a tableau of shape λ. If the number i is located at the (a,b)-th entry of $\lambda^{(c)}$, the eigenvalue of t_i on this vector is $v_cq^{2(b-a)}$. We call it the semi-normal representation of $\mathcal{H}_F$. Among tableaux of shape λ, we have a tableau of a special kind, which is called the canonical tableau of shape λ and is denoted by T_λ, defined as follows.

For each cell $x\in\lambda$, we attach a coordinate $(a(x), b(x), c(x))$ if it is located at the $(a(x), b(x))$-th entry of $\lambda^{(c(x))}$. Then we can introduce a lexicographic order on the set of cells by declaring $x<y$ if and only if $c(x)<c(y)$ or $c(x)=c(y)$, $a(x)<a(y)$ or $c(x)=c(y)$, $a(x)=a(y)$, $b(x)<b(y)$. Then the canonical tableau is the tableau cells are filled in with $1, \cdots, n$ according to this order.

Let R be $\boldsymbol{C}[\boldsymbol{t}]$ or its localization, and assume that $\boldsymbol{C}(\boldsymbol{t})\otimes\mathcal{H}_A$ is semi-simple. Then we denote by $S^\lambda{}_R$ the $\mathcal{H}_R$-lattice of the semi-normal representation generated by the basis vector correspoding to the canonical tableau. It is R-free. Then for specialization $R\rightarrow\boldsymbol{C}$, we have a $\mathcal{H}_C$-module $S^\lambda=\boldsymbol{C}\otimes S^\lambda{}_R$. We call these modules Specht modules.

We set $c_n=\theta_{-\varepsilon_1}+\cdots\theta_{-\varepsilon_n}$ and use the same c_n for $t_1+\cdots+t_n$. This element plays an important role in later sections.

Lemma 2.1. (1) *Let* R *be a principal ideal domain,* F *be its quotient field, such that* $\mathcal{H}_F$ *is semi-simple. We denote by* $S^\lambda{}_R$ *a Specht module for the*

Hecke algebra of rank n, and we restrict it to the Hecke algebra of rank n-1. Then there is a sequence of submodules $S^{\lambda}{}_R = V_1 \supset V_2 \supset \cdots$ *such that each quotient* V_i/V_{i+1} *is isomorphic to a full rank submodule of a Specht module* $S^{\mu}{}_R$ *for some Young diagram* μ *of size* $n-1$.

(2) *The element* c_n *acts on* $S^{\lambda}{}_R$ *as a scalar multiplication whose value is given by*

$$\sum_{x\in\lambda} v_{c(x)}q^{2(b(x)-a(x))}$$

Proof. We recall that $F\otimes S^{\lambda}{}_R$ decomposes into a direct sum of $F\otimes S^{\mu}{}_R$, each of which is spanned by basis vectors whose location of n are the same. Hence we have a natural sequence of submodules $F\otimes S^{\lambda}{}_R = W_1 \supset W_2 \supset \cdots$ such that each W_i/W_{i+1} is isomorphic to some $F\otimes S^{\mu}{}_R$. If we set $V_i = W_i \cap S^{\lambda}{}_R$, it is clear that it gives a sequence of submodules such that V_i/V_{i+1} is isomorphic to a submodule of some $F\otimes S^{\mu}{}_R$. Hence we have (1). (2) is obvious.

Note that if R is $\boldsymbol{C}[\boldsymbol{t}]$ or its localization, we can prove that the specialization $\boldsymbol{t}\to t$ defines a well-defined map from the Grothendieck group of $\boldsymbol{C}(\boldsymbol{t})\otimes \mathcal{H}_R$-modules to that of $\mathcal{H}_C$-modules. [CR, Proposition 16.16]

Further, if $\boldsymbol{C}(\boldsymbol{t})\otimes\mathcal{H}_R$ is semi-simple, then by virtue of Lemma 2.1, this so-called decomposition map sends $[\boldsymbol{C}(\boldsymbol{t})\otimes S^{\lambda}{}_R]$ to $[S^{\lambda}]$ such that the restriction rule between Specht modules described in Lemma 2.1 is also valid for these modules.

We also note here that symmetric functions of $t_1, \cdots, t_n$ belong to the center of $\mathcal{H}_A$, but the center does not coincide with the subalgebra consisting of these symmetric functions. Hence the situation is not the same as the affine Hecke algebra for which the center is the ring of symmetric functions with respect to $\theta_{\varepsilon_1}, \cdots, \theta_{\varepsilon_n}$ as determined by Bernstein. We will give an example below.

Example. *Let us consider the case* $n=2$, $m=3$. *By using the formula,*

$$a_i t_{i-1}{}^k = t_i{}^k a_i - (q-q^{-1})\sum_{j=1}^{k} t_{i-1}{}^{k-j}\, t_i{}^j$$

$$a_i t_i{}^k = t_{i-1}{}^k a_i + (q-q^{-1})\sum_{j=1}^{k} t_{i-1}{}^{k-j}\, t_i{}^j$$

and the fact that $\{t_1{}^{k_1}t_2{}^{k_2}a_w \mid k_1, k_2 = 0, 1, w\in S_2\}$ *is a basis although the algebra generated by* t_1, t_2 *is not* 3^2*-dimensional, we can explicitly compute the center. Then it turns out that the center is* A*-free, and its basis is given by*

$$1,\ t_1+t_2,\ t_1t_2,\ (t_1+t_2)^2,\ t_1t_2(t_1+t_2),\ (t_1t_2)^2,$$
$$(e_2(\boldsymbol{x}) - e_1(\boldsymbol{x})(t_1+t_2) + (t_1+t_2)^2 - t_1t_2)a_2,$$
$$(e_3(\boldsymbol{x}) + t_1t_2(t_1+t_2) - e_1(\boldsymbol{x})t_1t_2)a_2,$$
$$(e_3(\boldsymbol{x})(t_1+t_2) - e_2(\boldsymbol{x})t_1t_2 + (t_1t_2)^2)a_2,$$

where $e_i(\boldsymbol{x})$ is the i-th symmetric function of $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3$. If we specialize it to the group algebra of $G(3, 1, 2)$, this center specializes to 9-dimensional subalgebra. But it cannot be generated by t_1+t_2, t_1t_2, since $t_1^3=t_2^3=1$ and the subalgebra generated by these is 6-dimensional. Hence the center is not generated by t_1+t_2, t_1t_2 over A.

3. Connection to geometric theory

(3.1) Let $G=GL(n, \boldsymbol{C})$ be the general linear group, T be the maximal torus consisting of diagonal matrices, Z be the Steinberg variety, $\mathcal{B}$ be the flag variety, $\mathcal{N}$ be the nilpotent variety. We denote by $K^{G\times C^\times}(Z)$ (resp. $K^{G\times C^\times}(\mathcal{B})$) the Grothendieck group of $G\times \boldsymbol{C}^\times$-equivariant coherent sheaves on Z (resp. $\mathcal{B}$). Here the coefficient is extended to $\boldsymbol{C}$. The convolution product

$$[\mathcal{F}]\cdot[\mathcal{G}]=\sum(-1)^i[R^ip_{13*}(p_{12}{}^*\mathcal{F})\otimes(p_{23}{}^*\mathcal{G})]$$

makes $K^{G\times C^\times}(Z)$ into an algebra and $K^{G\times C^\times}(\mathcal{B})$ into a $K^{G\times C^\times}(Z)$-module.

A theorem of Ginzburg [CG, Theorem 7.6.10] says that $K^{G\times C^\times}(Z)$ is isomorphic to H_{q^2}, which is by definition the $\boldsymbol{C}[\boldsymbol{q}, \boldsymbol{q}^{-1}]$-algebra whose generators and relations are given by Bernstein presentation. The proof is given by explicit construction of the isomorphism. $\boldsymbol{q}$ will be specialized in (3.5).

Since $\mathcal{B}$ is a homogeneous space, we can identify $K^{G\times C^\times}(\mathcal{B})$ with the complexified representation ring $R_{T\times C^\times}=\boldsymbol{C}[\boldsymbol{x}_1, \boldsymbol{x}_1^{-1}, \cdots, \boldsymbol{x}_n, \boldsymbol{x}_n^{-1}, \boldsymbol{q}, \boldsymbol{q}^{-1}]$. The action of H_{q^2} is explicitly given by

$$T_i\bullet f=\boldsymbol{q}^{-1}\boldsymbol{x}_{i+1}\frac{f-s_if}{\boldsymbol{x}_i-\boldsymbol{x}_{i+1}}-\boldsymbol{q}\frac{\boldsymbol{x}_{i+1}f-s_i(\boldsymbol{x}_{i+1}f)}{\boldsymbol{x}_i-\boldsymbol{x}_{i+1}}$$

$$\theta_{\varepsilon_i}\bullet f=\boldsymbol{x}_i^{-1}f$$

where s_if is the exchange of variable $\boldsymbol{x}_i$ and $\boldsymbol{x}_{i+1}$ [CG, Theorem 7.2.16].

For any algebraic subgroup M of $T\times\boldsymbol{C}^\times$, we have [CG, Theorem 6.2.10]

$$R_M\underset{R_{G\times C^\times}}{\otimes}K^{G\times C^\times}(\mathcal{B})\simeq K^M(\mathcal{B}).$$

Hence the H_{q^2}-action on $K^M(\mathcal{B})$ is determined by $\bullet$-action.

(3.2) Let $\lambda=(\lambda_1, \cdots, \lambda_l)$ be a composition of size n. For $z_1, \cdots, z_l\in\boldsymbol{C}^\times$ and λ, we associate a pair of block diagonal matrices (x, s) by

$$x=\oplus_{i=1}^l J(\lambda_i),\ s=\oplus_{i=1}^l z_iq^{-\lambda_i+1}D(\lambda_i),$$

where

$$J(k)=\sum_{i=1}^{k-1}E_{i,i+1},\ D(k)=\sum_{i=1}^{k}q^{k+1-2i}E_{i,i}.$$

If we set $L=GL(\lambda_1)\times\cdots\times GL(\lambda_l)$, the collection of $(s, q)\in T\times\boldsymbol{C}^\times$ for various

$z_1, \cdots, z_l, q \in \boldsymbol{C}^\times$ form $(l+1)$-dimensional torus of $L \times \boldsymbol{C}^\times$. It contains the center of $L \times \boldsymbol{C}^\times$. For $u=e^x$, it is also a subgroup of

$$M(u) = \{(g, q) \in G \times \boldsymbol{C}^\times | gug^{-1} = u^{q^2}\}.$$

We take the parabolic subgroup of block upper triangular matrices whose Levi subgroup is L. Then for some $z_1, \cdots, z_l, q$,

$$d(z_1, \cdots, z_l\ ;\ q) = \prod_{i \neq j} \det(1 - q^{\lambda_i - \lambda_j + 2} z_j z_i^{-1} D(\lambda_j) D(\lambda_i^{-1}))$$

is not zero, and thus we can apply the induction theorem [KL,Theorem 6.2].

Let $\mathcal{B}_x$ be the subvariety of $\mathcal{B}$ consisting of x-stable (or equivalently u-stable) flags. In [KL], they defined H_{q^2}-action on $K_0^M(\mathcal{B}_x)$. We note here that this K-theory is different from the previous one.

Let $\widehat{\mathcal{B}}$ be the flag variety of L, $\widehat{\mathcal{B}}_x$ be the subvariety of x-stable flags.

We denote by $\widehat{H}_{q^2}$ the subalgebra of H_{q^2} generated by $T_i\,(i \neq \lambda_1, \lambda_1+\lambda_2, \cdots)$ and $\theta_y\,(y \in X)$. Then the induction theorem implies

$$H_{q^2} \underset{\widehat{H}_{q^2}}{\otimes} (R_M[d^{-1}] \underset{R_M}{\otimes} K_0^M(\widehat{\mathcal{B}}_x)) \simeq R_M[d^{-1}] \underset{R_M}{\otimes} K_0^M(\mathcal{B}_x)$$

where $R_M = \boldsymbol{C}[\boldsymbol{z}_1, \boldsymbol{z}_1^{-1}, \cdots, \boldsymbol{z}_l, \boldsymbol{z}_l^{-1}, \boldsymbol{q}, \boldsymbol{q}^{-1}]$ is the complexified representation ring of M, d is the polynomial in $\boldsymbol{z}_1, \cdots, \boldsymbol{z}_l, \boldsymbol{q}$ whose value at $z_1, \cdots, z_l, q$ is $d(z_1, \cdots, z_l, q)$.

Since $\widehat{\mathcal{B}}_x$ is a point, $K_0^M(\widehat{\mathcal{B}}_x)$ is a free R_M-module of rank 1. By [KL, 5.11 (a)], $K_0^M(\mathcal{B}_x)$ and $K_0^M(\mathcal{B})$ are projective R_M-modules.

Further, in the language of [KL], if we set $\Lambda_1 = \mathcal{B}_x$, $\Lambda_2 = \mathcal{B}$, then application of [KL,1.3(d)] for

$$\begin{array}{ccc} (\hat{\pi}^r)^{-1}\widehat{\Lambda}_1 & \subset & (\hat{\pi}^r)^{-1}\widehat{\Lambda}_2 \\ \downarrow & & \downarrow \\ \widehat{\Lambda}_1 & \subset & \widehat{\Lambda}_2 \end{array}$$

and application of [KL, 1.3(fl)] for $X = (\hat{\pi}^r)^{-1}\widehat{\Lambda}_2$, $X_0 = \Lambda_2$, $X' = (\hat{\pi}^r)^{-1}\widehat{\Lambda}_1$, $X'_0 = \Lambda_1$ lead to the following commutative diagram

$$\begin{array}{ccc} K_0^M(\mathcal{B}_x) & \xrightarrow{\tau^r} & K_0^M(\mathcal{B}_x) \\ \downarrow & & \downarrow \\ K_0^M(\mathcal{B}) & \xrightarrow{\tau^r} & K_0^M(\mathcal{B}). \end{array}$$

Hence by [KL, 5.11(b)], $K_0^M(\mathcal{B}_x) \to K_0^M(\mathcal{B})$ is a H_{q^2}-homomorphism. Since τ^r is R_M-linear, it descends to specialization.

To summarize, if we specialize $R_M[d^{-1}] \to R$, then $R \underset{R_M}{\otimes} K_0^M(\mathcal{B}_x) \to R \underset{R_M}{\otimes} K_0^M(\mathcal{B})$ and $H_{q^2} \underset{\widehat{H}_{q^2}}{\otimes} (R \underset{R_M}{\otimes} K_0^M(\widehat{\mathcal{B}}_x)) \simeq R \underset{R_M}{\otimes} K_0^M(\mathcal{B}_x)$ are H_{q^2}-homomorphisms, $R \underset{R_M}{\otimes} K_0^M(\mathcal{B}_x)$ and $R \underset{R_M}{\otimes} K_0^M(\mathcal{B})$ are projective R-modules, $R \underset{R_M}{\otimes} K_0^M(\widehat{\mathcal{B}}_x) \simeq R$.

(3.3) Take $z'_1, \cdots, z'_l, q'$ sufficiently generic such that $d(z'_1, \cdots, z'_l, q') \neq 0$, $|q'| \neq 1$, the corresponding diagonal matrix $s' \in M$ has distinct eigenvalues. We specialize $\boldsymbol{z}_i = (1-\boldsymbol{t})z'_i + \boldsymbol{t}z_i$, $\boldsymbol{q} = (1-\boldsymbol{t})q' + \boldsymbol{t}q$. We localize $\boldsymbol{C}[\boldsymbol{t}]$ at points where $\boldsymbol{z}_i$ or $\boldsymbol{q}$ is zero, and denote it by R.

For this R, we show that $R \underset{R_M}{\otimes} K_0{}^M(\mathscr{B}_x) \to R \underset{R_M}{\otimes} K_0{}^M(\mathscr{B})$ is injective. In fact, if it is not injective, then the kernel is a direct summand of $R \underset{R_M}{\otimes} K_0{}^M(\mathscr{B}_x)$, since R is a principal ideal domain and these modules are projective (or equivalently free) R-modules. Hence if we specialize $\boldsymbol{t}$ to 0, $\boldsymbol{C} \underset{R_M}{\otimes} K_0{}^M(\mathscr{B}) \to \boldsymbol{C} \underset{R_M}{\otimes} K_0{}^M(\mathscr{B})$ is not injective. By [KL, 1.3(k)], vertical arrows in the following diagram are isomorphisms.

$$\begin{array}{ccc} \boldsymbol{C} \underset{R_M}{\otimes} K_0{}^M(\mathscr{B}_x^{s'}) & \to & \boldsymbol{C} \underset{R_M}{\otimes} K_0{}^M(\mathscr{B}^{s'}) \\ \downarrow & & \downarrow \\ \boldsymbol{C} \underset{R_M}{\otimes} K_0{}^M(\mathscr{B}_x) & \to & \boldsymbol{C} \underset{R_M}{\otimes} K_0{}^M(\mathscr{B}) \end{array}$$

Hence $\boldsymbol{C} \underset{R_M}{\otimes} K_0{}^M(\mathscr{B}_x^M) \to \boldsymbol{C} \underset{R_M}{\otimes} K_0{}^M(\mathscr{B}^M)$ should be not injective. But it is contradiction since $\mathscr{B}_x^M \subset \mathscr{B}^M$ are discrete sets.

Therefore, $R \underset{R_M}{\otimes} K_0{}^M(\mathscr{B}_x)$ is a submodule of $R \underset{R_M}{\otimes} K_0{}^M(\mathscr{B})$.

(3.4) [KL, 1.3(p3)], [KL, 1.3(p2)] and [KL, 1.3(n3)] deduce the isomorphism

$$R_M \underset{R_{G\times C^\times}}{\otimes} K_0^{G\times C^\times}(\mathscr{B}) \simeq R_M \underset{R_{T\times C^\times}}{\otimes} K_0^{T\times C^\times}(\mathscr{B}).$$

By [KL, 1.3(p)], we have a natural map $R \underset{R_{T\times C^\times}}{\otimes} K_0^{T\times C^\times}(\mathscr{B}) \to R \underset{R_M}{\otimes} K_0{}^M(\mathscr{B})$. If it is not injective, similar argument as above shows that $\boldsymbol{C} \underset{R_{T\times C^\times}}{\otimes} K_0^{T\times C^\times}(\mathscr{B}) \to \boldsymbol{C} \underset{R_M}{\otimes} K_0{}^M(\mathscr{B})$ is not injective. But since $\mathscr{B}^M = \mathscr{B}^{T\times C^\times}$, it is impossible.

Moreover, since $R \underset{R_M}{\otimes} K_0{}^M(\mathscr{B})$ is R-free, and $\boldsymbol{C} \underset{R_M}{\otimes} K_0{}^M(\mathscr{B}) \simeq \boldsymbol{C} \underset{R_M}{\otimes} K_0{}^M(\mathscr{B}^M)$ is n!-dimensional, we know that the left hand side and the right hand side of this injective map have the same rank.

Hence, we conclude that $R \underset{R_M}{\otimes} K_0{}^M(\mathscr{B}) \subset \boldsymbol{C}(\boldsymbol{t}) \underset{R_{G\times C^\times}}{\otimes} K_0^{G\times C^\times}(\mathscr{B})$. In particular, the $H_{\boldsymbol{q}^2}$-action on $R \underset{R_M}{\otimes} K_0{}^M(\mathscr{B})$ is determined by that on $K_0^{G\times C^\times}(\mathscr{B})$.

(3.5) We now identify $K_0^{G\times C^\times}(\mathscr{B})$ with $\boldsymbol{C}[\boldsymbol{x}_1, \boldsymbol{x}_1^{-1}, \cdots, \boldsymbol{x}_n, \boldsymbol{x}_n^{-1}, \boldsymbol{q}, \boldsymbol{q}^{-1}]$. The action of $H_{\boldsymbol{q}^2}$ on this module is given by

$$T_i \circ f = (\phi^+)^{-1}(\boldsymbol{q} - \boldsymbol{q}^{-1} - T_i) \cdot (\phi^+ f)$$
$$\theta_{\varepsilon_i} \circ f = (\phi^+)^{-1}(\theta_{-\varepsilon_i}) \cdot (\phi^+ f)$$

where $\phi^+ = \prod_{i<j}(1 - \boldsymbol{q}^2 \boldsymbol{x}_i \boldsymbol{x}_j^{-1})$. This follows from [KL, 3.2(c), 3.6(a)], [KL, 1.3(o2)]. (Hence it is the action given at [KL, Lemma 3.10].)

If we set $\phi^- = \prod_{i>j}(1-\boldsymbol{q}^2\boldsymbol{x}_i\boldsymbol{x}_j^{-1})$, $\phi^+\phi^- \in R_{G\times C^\times}$ specializes to a non-zero element in R. Hence we have the commutative diagram

$$\begin{array}{ccc} R[d^{-1}]\underset{R_M}{\otimes}K_0^M(\mathcal{B}_x) & \xrightarrow{\phi^+\cdot} & R[d^{-1}]\underset{R_M}{\otimes}K_0^M(\mathcal{B}_x) \\ \downarrow & & \downarrow \\ R[d^{-1}]\underset{R_M}{\otimes}K_0^M(\mathcal{B}) & \xrightarrow{\phi^+\cdot} & R[d^{-1}]\underset{R_M}{\otimes}K_0^M(\mathcal{B}) \end{array}$$

where we use the same ϕ^+ for the specialization to $R[d^{-1}]\underset{R_M}{\otimes}K_0^M(\mathcal{B})$. The vertical arrows are injective by (3.3).

If we equip the left hand side with $\circ$-action and the right hand side with $\bullet$-action twisted by the automorphism $*$ of H_{q^2} defined by $T_i^* = \boldsymbol{q}-\boldsymbol{q}^{-1}-T_i$, $\theta_x^* = \theta_{-x}$, this diagram is commutative as H_{q^2}-modules. Because of the induction theorem, we also know that the vertical arrows in the following diagram are injective.

$$\begin{array}{ccc} R[d^{-1}]\underset{R_M}{\otimes}K_0^M(\hat{\mathcal{B}}_x) & \rightarrow & R[d^{-1}]\underset{R_M}{\otimes}K_0^M(\hat{\mathcal{B}}_x) \\ \downarrow & & \downarrow \\ R[d^{-1}]\underset{R_M}{\otimes}K_0^M(\mathcal{B}_x) & \rightarrow & R[d^{-1}]\underset{R_M}{\otimes}K_0^M(\mathcal{B}_x) \end{array}$$

The specialization we are interested in is $\boldsymbol{z}_i = z_i$, $\boldsymbol{q} = q$. Thus we take R-lattice $K_0^M(\mathcal{B}_x)$ of $R[d^{-1}]\underset{R_M}{\otimes}K_0^M(\mathcal{B}_x)$ on the right hand side and specialize $\boldsymbol{t}$ to 1. Then we have a diagram of H_{q^2}-modules

$$\begin{array}{ccc} \boldsymbol{C}\underset{R_M}{\otimes}K_0^M(\mathcal{B}_x) & \simeq & \boldsymbol{C}\underset{R_M}{\otimes}K_0^M(\mathcal{B}_x) \\ \downarrow & & \downarrow \\ \boldsymbol{C}\underset{R_M}{\otimes}K_0^M(\mathcal{B}) & \simeq & \boldsymbol{C}\underset{R_M}{\otimes}K_0^M(\mathcal{B}) \end{array}$$

where the H_{q^2}-action on the left hand side and the right hand side are as before, and in the Grothendieck group of the category of finite dimensional H_{q^2}-modules, we have

$$[\boldsymbol{C}\underset{R_M}{\otimes}K_0^M(\mathcal{B}_x)] = [H_{q^2}\underset{\hat{H}_{q^2}}{\otimes}(\boldsymbol{C}\underset{R_M}{\otimes}K_0^M(\hat{\mathcal{B}}_x))],$$

since it holds on the left hand side.

(3.6) We now consider

$$\begin{array}{ccc} \boldsymbol{C}\underset{R_M}{\otimes}K_0^M(\hat{\mathcal{B}}_x) & \leftarrow & \boldsymbol{C}\underset{R_M}{\otimes}K^M(\hat{\mathcal{B}}_x) \\ \downarrow & & \downarrow \\ \boldsymbol{C}\underset{R_M}{\otimes}K_0^M(\mathcal{B}_x) & \leftarrow & \boldsymbol{C}\underset{R_M}{\otimes}K^M(\mathcal{B}_x) \\ \downarrow & & \downarrow \\ \boldsymbol{C}\underset{R_M}{\otimes}K_0^M(\mathcal{B}) & \leftarrow & \boldsymbol{C}\underset{R_M}{\otimes}K^M(\mathcal{B}) \end{array}$$

As is pointed out in [L5, section 6], horizontal arrows are isomorphisms. And if we equip these modules with • -action twisted by $*$, it is commutative as H_{q^2}-modules. Thus by (3.5), we have the following proposition.

Proposition 3.1. *Let* $\boldsymbol{C}\underset{R_M}{\otimes}K^M(\mathscr{B})$ *be the* H_{q^2}*-module with convolution product action. Then* $\boldsymbol{C}\underset{R_M}{\otimes}K^M(\mathscr{B}_x)$ *is a* H_{q^2}*-submodule of* $\boldsymbol{C}\underset{R_M}{\otimes}K^M(\mathscr{B})$, *which is equal to* $[H_{q^2}\underset{\hat{H}_{q^2}}{\otimes}(\boldsymbol{C}\underset{R_M}{\otimes}K^M(\hat{\mathscr{B}}_x))]$ *in the Grothendieck group of finite dimensional* H_{q^2}*-modules.*

(3.7) Let $M(s,q)$ be the smallest algebraic subgroup of M containing (s,q). Then we have the diagram

$$\begin{array}{ccccc}\boldsymbol{C}\underset{R_M}{\otimes}K_0{}^M(\mathscr{B}_x) & \simeq & \boldsymbol{C}\underset{R_M}{\otimes}K^M(\mathscr{B}_x) & \subset & \boldsymbol{C}\underset{R_M}{\otimes}K^M(\mathscr{B}) \\ \downarrow\alpha_0 & & \downarrow\alpha & & \downarrow \\ \boldsymbol{C}\underset{R_{M(s,q)}}{\otimes}K_0^{M(s,q)}(\mathscr{B}_x) & \simeq & \boldsymbol{C}\underset{R_{M(s,q)}}{\otimes}K^{M(s,q)}(\mathscr{B}_x) & \subset & \boldsymbol{C}\underset{R_{M(s,q)}}{\otimes}K^{M(s,q)}(\mathscr{B})\end{array}$$

Since the rightmost vertical arrow is an isomorphism [CG, Theorem 6.2.10], α_0 is injective. On the other hand, [KL, 1.3(k)] and [KL, 1.3(m2)] imply

$$\boldsymbol{C}\underset{R_{M(s,q)}}{\otimes}K_0^{M(s,q)}(\mathscr{B}_x)\simeq\boldsymbol{C}\otimes K_0{}^M(\mathscr{B}_x^s)\simeq H_{even}(\mathscr{B}_x^s)$$

As is explained in [X, 5.8], the main result of [DLP] and a result of Steinberg [S] calculate the dimension of $H_*(\mathscr{B}_x^s)$, which coincides with that of $\boldsymbol{C}\underset{R_M}{\otimes}K_0{}^M(\mathscr{B}_x)$ by Proposition 3.1. Hence α is an isomorphism.

(3.8) Finally, we consider the following diagram in order to descend to K groups without group action.

$$\begin{array}{ccccc}\boldsymbol{C}\underset{R_M}{\otimes}K^M(\hat{\mathscr{B}}_x) & \simeq & \boldsymbol{C}\underset{R_{M(s,q)}}{\otimes}K^{M(s,q)}(\hat{\mathscr{B}}_x) & \xrightarrow{r_a} & K(\hat{\mathscr{B}}_x^s) \\ \downarrow & & \downarrow & & \downarrow \\ \boldsymbol{C}\underset{R_M}{\otimes}K^M(\mathscr{B}_x) & \simeq & \boldsymbol{C}\underset{R_{M(s,q)}}{\otimes}K^{M(s,q)}(\mathscr{B}_x) & \xrightarrow{r_a} & K(\mathscr{B}_x^s) \\ \downarrow & & \downarrow & & \downarrow \\ \boldsymbol{C}\underset{R_M}{\otimes}K^M(\mathscr{B}) & \simeq & \boldsymbol{C}\underset{R_{M(s,q)}}{\otimes}K^{M(s,q)}(\mathscr{B}) & \xrightarrow{r_a} & K(\mathscr{B}^s)\end{array}$$

By [CG, Lemma 5.11.9, Theorem 6.2.4], the bottom r_a is an isomorphism. Further, [CG, Theorem 5.11.14] says it commutes with convolution. Then bivariant Riemann-Roch theorem [CG, Theorem 5.11.15] says

$$\begin{array}{ccc}K(\hat{\mathscr{B}}_x^s) & \xrightarrow{RR} & H_*(\hat{\mathscr{B}}_x^s) \\ \downarrow & & \downarrow \\ K(\mathscr{B}_x^s) & \xrightarrow{RR} & H_*(\mathscr{B}_x^s) \\ \downarrow & & \downarrow \\ K(\mathscr{B}^s) & \xrightarrow{RR} & H_*(\mathscr{B}^s)\end{array}$$

also commutes with convolution. It is an isomorphism by [CG, Theorem 6.2.4] which proves [CG, 6.2.1(2)]. Hence the injective H_{q^2}-homomorphism

$$\boldsymbol{C} \underset{R_{M(s,q)}}{\otimes} K^{M(s,q)}(\mathcal{B}_x) \to H_*(\mathcal{B}_x^s)$$

is an isomorphism, since both have the same dimension.

Let $\boldsymbol{C} \otimes H_{q^2}$ be the specialized affine Hecke algebra, i.e. whose center is specialized to $\boldsymbol{C}$ by the evaluation at s. We define a subvariety Z^a of $\mathcal{B} \times \mathcal{B} \times \mathcal{N}$ by

$$Z^a = \{(b_1, b_2, N) | Ad(s) b_1 = b_1,\ Ad(s) b_2 = b_2,\ N \in b_1 \cap b_2,\ Ad(s) N = q^2 N\}$$

then $\boldsymbol{C} \otimes H_{q^2} \simeq \boldsymbol{C} \underset{R_{G\times C^\times}}{\otimes} K^{G\times C^\times}(Z) \simeq H_*(Z^a)$ as algebra and H_{q^2}-action is transported to the convolution action of $H_*(Z^a)$. [CG, Proposition 8.1.5]

Since $\boldsymbol{C} \underset{R_M}{\otimes} K^M(\widehat{\mathcal{B}}_x^s)$ corresponds to $H_*(\widehat{\mathcal{B}}_x^s)$, we can determine $\widehat{H}_{q^2}$-module structure on $H_*(\widehat{\mathcal{B}}_x^s)$ by the character formula [CG, Theorem 8.2.1]. In fact, we know that $K^M(\widehat{\mathcal{B}}_x)$ is free of rank 1, thus it is enough to consider it on $H_*(\widehat{\mathcal{B}}_x^{s'})$. Then since $\widehat{\mathcal{B}}_x^{s'}$ is a point, the formula reduces to $Tr(\theta_y, H_*(\widehat{\mathcal{B}}_x^{s'})) = y(s')$. Here we identify X with $Hom(T, \boldsymbol{C}^\times)$ as usual. Therefore $T_i \theta_{-\varepsilon_i} T_i = \theta_{-\varepsilon_{i+1}}$ shows T_i maps to q.

Hence we have transplanted the induction theorem into Ginzburg's theory.

Theorem 3.2. *For each composition λ and $z_1, \cdots, z_l \in \boldsymbol{C}^\times$, we have*

$$[H_*(\mathcal{B}_x^s)] = [H_{q^2} \underset{\widehat{H}_{q^2}}{\otimes} \boldsymbol{C}_\lambda]$$

where $\lambda_1, \cdots, \lambda_l$ are the size of Jordan blocks of x, and $\boldsymbol{C}_\lambda$ is the $\widehat{H}_{q^2}$-module defined by sending θ_y to $y(s)$, T_i to q.

Recall that λ, z determines s, x as was introduced in (3.2). Hence we denote $H_{q^2} \underset{\widehat{H}_{q^2}}{\otimes} \boldsymbol{C}_\lambda$ by $M_{\lambda,z}$.

(3.9) Let

$$\widetilde{\mathcal{N}}^a = \{(b, N) | Ad(s) b = b,\ N \in b,\ Ad(s) N = q^2 N\} \subset \mathcal{B} \times \mathcal{N}$$

and $\mu : \widetilde{\mathcal{N}}^a \to \mathcal{N}^a = \{N | Ad(s) N = q^2 N\} \subset \mathcal{N}$. We have $\mu^{-1}(x) = \mathcal{B}_x^s$. Since the correspondence from $\mathcal{B}$ to the variety of Borel subalgebras is by taking the stabilizer, $N \in b$ is that $\mathcal{F}$ is N-stable. We note that $\widetilde{\mathcal{N}}^a$ is smooth and μ is proper. To see it, we write $\phi_{\mathcal{F}}(i) = j$ for s-stable flag $\mathcal{F}$ if the i-dimensinal subspace F_i is obtained from F_{i-1} by adding an eigenvector of s whose eigenvalue is q^{2j}. Then the set of flags with a same $\phi_{\mathcal{F}}$ bijectively correspond to r-tuples of flags where r is the number of distinct eigenvalues of s. This correspondence respects $C_G(s)$-action. Hence $C_G(s)$-orbits give connected components of the set $\mathcal{B}^s$. We now take the nilpotent element N into the consideration. For each

$C_G(s)$-orbit in $\mathcal{N}$, its inverse image by the first projection $p_1 : \widetilde{\mathcal{N}}^a \to \mathcal{B}^s$ is also a connected component. It is obvious since it is a vector bundle over the orbit. Hence, $\widetilde{\mathcal{N}}^a$ is smooth.

This explicit description is helpful to find an elementary proof of non-vanishing result for irreducible modules in our case. For example, to find a connected component $\widehat{\mathcal{O}}$ of $\widetilde{\mathcal{N}}^a$ such that $\mu(\widehat{\mathcal{O}})$ coincides with an orbit closure, and its intersection with $\mathcal{B}_x^s$ is non-empty. But, here in the next subsection, we appeal to the general result of Grojnowski, which can be found in [CG,Theorem 8.7.1], for the case q^2 is not roots of unity.

(3.10) We are in a position to state a result of Ginzburg-Vasserot. We state it for our special case. As is explained in [CG, Theorem 8.6.6] [GV, p.76],

$$H_*(Z^a) \simeq Ext_{D^b(\mathcal{N}^a)}(\mu_* \boldsymbol{C}_{\widetilde{\mathcal{N}}^a}, \mu_* \boldsymbol{C}_{\widetilde{\mathcal{N}}^a})$$

such that convolution product corresponds to Yoneda product. Further, as is explained in [CG, p.361], $H_*(Z^a)$-action on $H_*(\mathcal{B}_x^s)$ corresponds to the action on $H^*(i_x^! \mu_* \boldsymbol{C}_{\mathcal{N}^a})$ through the restriction map

$$Ext_{D^b(\mathcal{N}^a)}(\mu_* \boldsymbol{C}_{\widetilde{\mathcal{N}}^a}, \mu_* \boldsymbol{C}_{\widetilde{\mathcal{N}}^a}) \to Ext_{D^b(\mathcal{N}^a)}(i_x^! \mu_* \boldsymbol{C}_{\widetilde{\mathcal{N}}^a}, i_x^! \mu_* \boldsymbol{C}_{\widetilde{\mathcal{N}}^a}).$$

Since the decomposition theorem [CG, Theorem 8.4.5] is applicable, we have

$$\mu_* \boldsymbol{C}_{\widetilde{\mathcal{N}}^a} = \bigoplus_{i,\mathcal{O}} L_{\mathcal{O}}(i) \otimes IC(\mathcal{O}, \boldsymbol{C}_{\mathcal{O}}),$$

where $i \in \boldsymbol{Z}$ and $\mathcal{O}$ are $C_G(s)$-orbits in $\mathcal{N}^a$. Then the argument [CG,p.359] tells that $L_{\mathcal{O}} = \oplus_{i \in Z} L_{\mathcal{O}}(i)$ is irreducible or 0 as H_{q^2}-module, and non-zero ones are inequivalent modules. Thus we have,

Theorem 3.3 ([CG, Theorem 8.6.15] [GV, Theorem 6.6]). *If $L_{\mathcal{O}}$ is non-zero, we have,*

$$[H_*(\mathcal{B}_x^s) : L_{\mathcal{O}}] = P_{\mathcal{O},x}(1)$$

where $P_{\mathcal{O},x}(\mathbf{v}) = \sum \dim H^k(i_x^! IC(\mathcal{O}, \boldsymbol{C}_{\mathcal{O}}))\mathbf{v}^k$.

For the case that q^2 has infinite order, [CG, Theorem 8.7.1] and the argument which follows the theorem proves that they are actually non-zero. In fact, $\mu_* \boldsymbol{C}_{\widehat{\mathcal{O}}}$ is a direct summand of $\mu_* \boldsymbol{C}_{\widetilde{\mathcal{N}}^a}$ and the multiplicity of $IC(\mathcal{O}, \boldsymbol{C}_{\mathcal{O}})$ in $\mu_* \boldsymbol{C}_{\widehat{\mathcal{O}}}$ is $H^*(\mathcal{B}_x^s \cap \widehat{\mathcal{O}})$. Hence, we can state a special case of non-vanishing result of Grojnowski, due to Lusztig.

Proposition 3.4 ([CG, Proposition 8.4.12]). *If q^2 is not roots of unity, then $L_{\mathcal{O}}$ are all non-zero.*

We remark that if we order $C_G(s)$-orbits such a way that it is compatible with the closure relation, we have an unitriangular transition matrix and hence we can conclude that $L_{\mathcal{O}}$ can be expressed as alternating sum of $M_{\lambda,z}$.

4. Littlewood-Richardson ring and canonical basis

(4.1) Let $V(i, h)$ be an indecomposable representation $(V_j, f_{j,j+1})$ of the quiver A_∞ given by

$V_j = \begin{cases} \boldsymbol{C} & (i \leq j \leq i+h-1) \\ 0 & (else) \end{cases}$, $f_{j,j+1} = \begin{cases} 1 & (i \leq j \leq i+h-2) \\ 0 & (else) \end{cases}$. For the cyclic quiver $A^{(1)}_{r-1}$, we denote by $\overline{V}(i,h) = (\overline{V}_j, \overline{f}_{j,j+1})$ the folded representation, i.e.

$$\overline{V}_j = \oplus_{k \equiv j(r)} V_k, \ \overline{f}_{j,j+1} = \oplus_{k \equiv j(r)} f_{k,k+1}$$

For λ and $z_1 = q^{2i_1}, \cdots, z_l = q^{2i_l}$, the corresponding s and x (see (3.2)) define a representation of $\Gamma = A_\infty$ (resp. $A^{(1)}_{r-1}$) if q^2 is not roots of unity (resp. a primitive r-th root of unity) by putting the eigenspace of s^{-1} of the eigenvalue q^{2j} on the j-th node and considering x^t as the set of linear maps between them. It then becomes isomorphic to $V(\lambda, z) = V(i_1, \lambda_1) \oplus \cdots \oplus V(i_l, \lambda_l)$. They exaust all finite dimensional representations of Γ whose composition of arrows are nilpotent. In fact, any such representation gives a pair (x, s) such that $sxs^{-1} = q^2 x$, which is extendable to a $SL(2, \boldsymbol{C})$-module by [KL, 2.4 (g)]. Hence we have the exaustion.

Since $C_G(s)$-orbits of $\mathcal{N}^a$ are nothing but representations of Γ, we write $L_{\lambda,z}$ instead of $L_{\mathcal{O}}$, and $P_{\lambda,z\,;\,\mu,w}(\mathbf{v})$ instead of $P_{\mathcal{O},x}(\mathbf{v})$. In this notation, Theorem 3.3 says that $[M_{\mu,w} : L_{\lambda,z}] = P_{\lambda,z\,;\,\mu,w}(1)$ (if $L_{\lambda,z} \neq 0$).

(4.2) We introduce the graded dual of the Littlewood-Richardson ring of H_{q^2}. Let U_n be the subgroup of the Grothendieck group of the affine Hecke algebra of rank n spanned by $\{[M_{\mu,w}]\}$, and consider the direct sum $U = \oplus U_n$. Here the coefficient is extended to $\boldsymbol{C}$. We give it an algebra structure by using the shift homomorphism introduced in (2.1). We note that non-zero elements of $\{[L_{\lambda,z}]\}$ give a basis of U. The graded dual $\oplus\, Hom\,(U_n, \boldsymbol{C})$ is denoted by U^*. We also denote by $\{[L_{\lambda,z}]^*\}$ the dual basis of non-zero irreducible modules $\{[L_{\lambda,z}]\}$.

Since $c_n = \theta_{-\varepsilon_1} + \cdots + \theta_{-\varepsilon_n}$ commutes with $T_1 \cdots, T_{n-1}$, any H_{q^2}-module is a direct summand of the generalized eigenspace of c_n. For each module M and eigenvalue c, we denote this summand by $P_{c_n,c}(M)$. The induced up (resp.restricted) module of M is denoted by $Ind\,(M)$ (resp. $Res\,(M)$). Here we consider the affine Hecke algebra of rank n (resp. $n-1$) as a subalgebra of the affine Hecke algebra of rank $n+1$ (resp. n) in the natural way.

Let M be such that $P_{c_n,c}(M) = M$ (resp. $P_{c_{n+1},c}(M) = M$). Then we define i-$Ind\,(M)$ (resp. i-$Res\,(M)$) by

$$i\text{-}Ind\,(M) = P_{c_n, c+q^{2i}}(Ind\,(M)) \quad (\text{resp. } i\text{-}Res\,(M) = P_{c_n, c-q^{2i}}(Res\,(M)).)$$

We now introduce operators $e_i, f_i : U^* \to U^*$ as follows.

$$e_i\phi([M]) = \phi([i\text{-}Ind\,(M)]),\ f_i\phi([M]) = \phi([i\text{-}Res\,(M)])$$

where $\phi \in U^*, [M] \in U$.

Let u_n be the Grothendieck group of the category of $\mathscr{H}_C$-modules and we set $u = \oplus u_n$ as in the introduction. Its graded dual is denoted by u^*.

Then we define *i-Ind*, *i-Res*, e_i, f_i in the same way as before. By exploiting the fact that

$$\{t_1^{k_1} \cdots t_n^{k_n} a_w | 0 \leq k_i \leq m-1, w \in S_n\}$$

is a basis of $\mathscr{H}_C$ and $t_n a_w = a_w t_n$ if $w \in S_{n-1}$, we know that the Hecke algebra of $G(m, 1, n)$ is a free module as a right module over the Hecke algebra of $G(m, 1, n-1)$.
Hence these definitions are well-defined.

Lemma 4.1. *The following diagram is commutative.*

$$\begin{array}{ccc} U^* & \longrightarrow & u^* \\ f_i \downarrow & & \downarrow f_i \\ U^* & \longrightarrow & u^* \end{array}$$

Proof. For any irreducible $\mathscr{H}_C$-module D, $i\text{-}Res(D)$ is an $\mathscr{H}_C$-module. Hence the lemma is clear.

(4.3) According to the cases q^2 is not roots of unity or a primitive r-th root of unity, we let $\mathfrak{g}$ be the Kac-Moody Lie algebra of type A_∞ or $A_{r-1}^{(1)}$. The triangular decomposition of the universal enveloping algebra is denoted by $U(\mathfrak{g}) = U^- \otimes U^0 \otimes U^+$, where U^- is the algebra generated by generators f_i's. When we have a need to specify $\mathfrak{g}$, we write $U^-(\mathfrak{g})$, etc.

We now recall some result about PBW-type basis and canonical basis of Lusztig. Let $\boldsymbol{k}$ be an algebraically closed field. We fix $l \in \boldsymbol{k}^\times$. Let I be the set of nodes of the quiver Γ. For each I-graded vector space $V = \oplus V_i$ over $\boldsymbol{k}$, we set $G = \prod_{i \in I} Aut(V_i)$, $E = \oplus_{i \to j} Hom(V_i, V_j) \subset Hom(V, V)$. Then the isomorphism classes of representations of the quiver Γ with a fixed dimension type is nothing but G-orbits supported on E. Let $\mathscr{Q}_V$ be the subcategory of the bounded derived category of complexes of $\overline{\mathbf{Q}}_l$-sheaves on E consisting of finite direct sum of $IC(\mathscr{O}, \overline{\mathbf{Q}}_l)[i]$ for G-orbits $\mathscr{O}$ whose singular support is nilpotent and $i \in \boldsymbol{Z}$. By running through all isomorphism classes of I-graded spaces V, we have the category $\mathscr{Q} = \oplus \mathscr{Q}_V$. This is nothing but the category introduced in [L7, 2.1]. It is because that the perverse sheaves of [L7, 2.1] must satisfy the condition that the singular support is nilpotent [L7, Corollary 13.6], thus the orbits with trivial local system with non-nilpotent conormal bundle cannot appear. Then in the language of [L7], only aperiodic orbits may appear [L7, Proposition 15.5], and because of the orientation we choose, we have the same number of orbits as the dimension of U^-. Hence by the isomorphism mentioned below, we have that the set of aperiodic orbits give a basis of U^-.

Let $\mathscr{K}$ be $\boldsymbol{Z}[\mathbf{v}, \mathbf{v}^{-1}]$-free module whose basis (L) are indexed by objects in

$\mathcal{Q}$. Addition is given by $(L)+(L')=(L\oplus L')$ [L7, 10.1]. We can introduce a standard multiplication on $\mathcal{K}$ via geometric method [L7, 3.1] and we modify it to define a new algebra structure on $\mathcal{K}$ [L7, 10.2]. Then [L7, Theorem 10.7] states that we have an isomorphism

$$\lambda_{\Omega}:\mathbf{Q}(\mathbf{v})\otimes U_{\mathbf{v}}^{-}\simeq\mathbf{Q}(\mathbf{v})\otimes\mathcal{K}$$

such that if we denote by $L_i^{(d)}$ the unique orbit in $V=V_i=\boldsymbol{k}^d$, the divided power $f_i^{(d)}$ maps to $(L_i^{(d)})$.

The elements correponding to $IC(\mathcal{O},\overline{\mathbf{Q}}_l)$ are called canonical basis. We also call $(i_{\mathcal{O}})_!\overline{\mathbf{Q}}_l$ PBW-type basis. Here, $i_{\mathcal{O}}$ is the inclusion map for $\mathcal{O}$. Note that PBW-type basis may not live in $\mathcal{K}$.

Through the isomorphism λ_{Ω}, we often identify the canonical basis with elements in $U_{\mathbf{v}}^{-}$. By the fact that $(L_i^{(d)})$ generate $\mathcal{K}$ as $\boldsymbol{Z}[\mathbf{v},\mathbf{v}^{-1}]$-algebra [L7, Proposition 10.13], this lattice is the usual $\boldsymbol{Z}[\mathbf{v},\mathbf{v}^{-1}]$-form of $U_{\mathbf{v}}^{-}$. Hence if we specialize it to $\mathbf{v}=1$, we have canonical basis of $U^{-}(\mathfrak{g})$.

Since orbits are indexed by $V_{\lambda,z}$, we denote by $\mathcal{O}_{\lambda,z}$ (resp. $\overline{\mathcal{O}}_{\lambda,z}$) the corresponding PBW-type basis (resp. canonical basis).
The gradings we are interested in are $I=\boldsymbol{Z}$ and $I=\boldsymbol{Z}/r\boldsymbol{Z}$. The corresponding group $G=\prod_{i\in I}\mathrm{Aut}(V_i)$ are denoted by G_∞ or G_r. Then, G_r-orbits are unions of G_∞-orbits. More precisely, the union of G_∞-orbits corresponding to representations of the quiver A_∞ which fold to a same representation of the quiver $A_{r-1}^{(1)}$. It gives the relation between PBW-type bases of A_∞ and $A_{r-1}^{(1)}$.

We note that $\overline{\mathcal{O}}_{\lambda,z}=\sum P_{\lambda,z\,;\,\mu,w}(\mathbf{v})\mathcal{O}_{\mu,w}$.
The orbit closures we are concerned with are locally isomorphic to affine Schubert varieties [L6, Theorem 11.3] and $P_{\lambda,z\,;\,\mu,w}(\mathbf{v})$ are known to be Kazhdan-Lusztig polynomials. The defining field $\boldsymbol{k}$ of these varieties does not matter as long as l is invertible in $\boldsymbol{k}$. That is, we have the same transition matrix between PBW-type basis and canonical basis of $U_{\mathbf{v}}^{-}$ if we move to a field of positive characteristic. For these general principles, we refer [BBD, Lemma 6.1.9, Lemma 6.2.6].

Lemma 4.2. *Let μ be a composition and $w_1,\cdots,w_l$ be powers of q^2. Then, the PBW-type basis of U^- satisfies the following multiplication formula.*

$$\mathcal{O}_{\mu,w}f_i=\sum c_{\mu,w\,;\,\mu',w'}\mathcal{O}_{\mu',w'}$$

where $c_{\mu,w\,;\,\mu',w'}$ is the number of ways to go from (μ',w') to (μ,w) by changing μ'_k to μ'_k-1 if $w'_kq^{2\mu'_k-2}=q^{2i}$.

Proof. By the definition of the multiplication rule, the verification of the above formula reduces to the proof of $\mathcal{O}_{\mu,w}f_2=\sum c_{\mu,w\,;\,\mu',w'}\mathcal{O}_{\mu',w'}$ for the quiver of type A_3. As was remarked above, we can take $\boldsymbol{k}$ to be positive characteristic. Then [L6, Proposition 9.8 (d)] gives the isomorphism $\lambda'_{\Omega}:\mathcal{K}\simeq U_{\mathbf{v}}^{-}$ such that PBW-type basis actually correspond to PBW basis $E^c{}_i$ which is determined at $\mathbf{v}=1$. We have $\lambda_\Omega=\lambda'_\Omega$, since $\lambda'_\Omega((L_i^{(d)}))=f_i^{(d)}$. Then we have the result by the use of the algebra isomorphism from the Hall algebra R_Ω of the representations of the quiver to $U_\mathbf{v}^-$, which sends $[V_c]$ to $E^c{}_i$ at $\mathbf{v}=1$ [L6, 5.1o (b)]. Recall that the multiplication in the Hall algebra is by definition,

$$[V'][V'']=\sum g_{V,V',V''}(\mathbf{v})\,[V]$$

such that if $\mathbf{v}$ is specialized to a power of a prime, i.e. the cardinality of a finite field $\boldsymbol{k}$, then $g_{V,V',V''}$ is the cardinality of the following set for almost all $\boldsymbol{k}$.

$$\{W\subset V\,|\,W\simeq V'', V/W\simeq V'\}$$

Here, the representations of the quiver are realized over the field $\boldsymbol{k}$. Thus we see that the only freedom of choosing linear maps comes from those from $0\to\boldsymbol{k}\to 0$ to a sum of indecomposable modules of type $\boldsymbol{k}\to\boldsymbol{k}\to 0$ or $0\to\boldsymbol{k}\to 0$. Let l_1 (resp. l_2) be the multiplicity of $\boldsymbol{k}\to\boldsymbol{k}\to 0$ (resp. $0\to\boldsymbol{k}\to 0$). Take a submodule $0\to\boldsymbol{k}v\to 0$ of

$$0\to\boldsymbol{k}^{l_1}\to\boldsymbol{k}^{l_1}\oplus\boldsymbol{k}^{l_2}\to 0.$$

We write $v=v_1+v_2\in\boldsymbol{k}^{l_1}\oplus\boldsymbol{k}^{l_2}$. If $v_1\neq 0$, we take $C\subset\boldsymbol{k}^{l_1}$ such that $C\oplus\boldsymbol{k}v_1=\boldsymbol{k}^{l_1}$. Then the direct sum of $C\to C\to 0$ and $0\to\boldsymbol{k}^{l_2}\to 0$ is a submodule of

$$0\to\boldsymbol{k}^{l_1}\to\boldsymbol{k}^{l_1}\oplus\boldsymbol{k}^{l_2}/\boldsymbol{k}v\to 0.$$

Hence the above module is isomorphic to

$$(\boldsymbol{k}\to\boldsymbol{k}\to 0)^{\oplus(l_1-1)}\oplus(\boldsymbol{k}\to 0\to 0)\oplus(0\to\boldsymbol{k}\to 0)^{\oplus l_2}.$$

If $v_1=0$, it is isomorphic to

$$(\boldsymbol{k}\to\boldsymbol{k}\to 0)^{\oplus l_1}\oplus(0\to\boldsymbol{k}\to 0)^{\oplus(l_2-1)}.$$

Hence the coefficients in question are $\dfrac{\mathbf{v}^{l_1+l_2}-\mathbf{v}^{l_2}}{\mathbf{v}-1}$ and $\dfrac{\mathbf{v}^{l_2}-1}{\mathbf{v}-1}$ respectively, which specialize to l_1,l_2 at $\mathbf{v}=1$. We are done.

(4.4) We return to the graded dual of Littlewood-Richardson ring. The following proposition is a key to prove Theorem 4.4. In the theorem, the universal enveloping algebra is a little fatter than finite linear span of basis, since infinite sum is allowed in each degree.

We define an antiautomorphism σ of U^- by $\sigma(f_i)=f_i$. We consider U^- as a U^--module via $xu=u\sigma(x)$. Then we have the following result.

Proposition 4.3. (1) *There is an isomorphism of vector spaces $U^-\simeq U^*$ such that the following diagram commutes.*

$$\begin{array}{ccc} U^- & \simeq & U^* \\ f_i\downarrow & & \downarrow f_i \\ U^- & \simeq & U^* \end{array}$$

In particular, U^* becomes a free U^--module.

(2) *Through the isomorphism given above, the canonical basis corresponds to the dual basis of irreducible H_{q^2}-modules* $\{[L_{\lambda,z}]^*\}$.

Proof. (1) We first prove it for the case that q^2 is not roots of unity. We can show that i-$Res\,(M_{\mu,w})$ is the direct sum of $M_{\mu',w'}$ with multiplicity where (μ', w') run through

$$\{(\mu', w')\,|\,\mu'=(\mu_1, \cdots, \mu_j-1, \cdots, \mu_l),\, w'=w,\, w_j q^{2\mu_j-2}=q^{2i}\}$$

In fact, $M_{\mu,w}$ is an induced up module from the algebra generated by $T_i\,(i\neq\mu_1, \mu_1+\mu_2, \cdots)$ and $\theta_y\,(y\in X)$. We denote its one-dimensional representation by φ. Let $v_1', \cdots, v_{m'}'$ be powers of q^2 appearing in $\varphi(\theta_{-\varepsilon_i})$. Then $M_{\mu,w}$ can be viewed as a $\mathcal{H}_C$-module with this different set of parameters, which is nothing but Specht module introduced in (2.3). Hence by Lemma 2.1, we have the restriction formula.

Since $\{[M_{\mu,w}]\}$ are basis by Proposition 3.4, we can consider its dual basis which we denote by $\{[M_{\mu,w}]^*\}$. Then

$$f_i[M_{\mu,w}]^*([M_{\mu',w'}])=[M_{\mu,w}]^*([i\text{-}Res\,(M_{\mu',w'})])$$

and we have that $f_i[M_{\mu,w}]^*=\sum c_{\mu,w\,;\,\mu',w'}[M_{\mu',w'}]^*$ where $c_{\mu,w\,;\,\mu',w'}$ is the coefficient defined in Lemma 4.2.

Let $U^-\Rightarrow U^*$ be an isomorphism sending $\mathcal{O}_{\mu,w}$ to $[M_{\mu,w}]^*$. By Lemma 4.2, we have the desired commutative diagram.

We now set U_r^* be the graded dual of Littlewood-Richardson ring of affine Hecke algebras whose q^2 is a primitive r-th root of unity. It is naturally a subspace of U^*, since $\{[M_{\mu,w}]\}$ spans U and U_r. Since i-Res for U_r^* is nothing but $\oplus_{i\equiv i'(r)}i'$-Res for U^*, we have the following commutative diagram

$$\begin{array}{ccccc} U_r^* & \rightarrow & U^* & \Leftarrow & U^- \\ f_i\downarrow & & \downarrow & & \downarrow f_i \\ U_r^* & \rightarrow & U^* & \Leftarrow & U^- \end{array}$$

where $f_i=\sum_{i'\equiv i(r)} f_{i'}$. In U^-, they generate $U^-(A_{r-1}^{(1)})$ [DJKM]. Thus if we identify U^* with $U^-(A_{r-1}^{(1)})$, we have $U^-(A_{r-1}^{(1)})\subset U_r^*$ since $1\in U_r^*$. Further, $U^-(A_{r-1}^{(1)})$ is isomorphic to the algebra $\mathcal{K}$ of Lusztig for $\mathfrak{g}=\mathfrak{g}(A_{r-1}^{(1)})$. Thus, by comparing dimension, we know $U^-(A_{r-1}^{(1)})=U_r^*$.

(2) If the order of q^2 is r, we have $\bar{\mathcal{O}}_{\lambda,z}=\sum P_{\lambda,z\,;\,\mu,w}(1)\,\mathcal{O}_{\mu,w}$ at $\mathbf{v}=1$ and $[M_{\mu,w}:L_{\lambda,z}]=P_{\lambda,z\,;\,\mu,w}(1)$. Since $[M_{\mu,w}]^*$ corresponds to PBW-type basis for the folded reprepentation, it is the infinite sum of $[M_{\mu',w'}]^*$ with respect to the quiver A_∞ which are folded to the same representation. Hence, for the case q^2 is a root of unity, the proof is as follows.

$$\begin{aligned}\bar{\mathcal{O}}_{\lambda,z}([M_{\mu,w}])&=\sum P_{\lambda,z\,;\,\mu',w'}(1)\,[M_{\mu',w'}]^*([M_{\mu,w}])\\&=P_{\lambda,z\,:\,\mu,w}(1)=[L_{\lambda,z}]^*([M_{\mu,w}]).\end{aligned}$$

Here, we consider $[M_{\mu,w}]^*$ as an infinite sum in U as is stated above. Hence the isomorphism maps $\bar{\mathcal{O}}_{\lambda,z}$ to $[L_{\lambda,z}]^*$. In view of [L7, 10.27], we have the result.

(4.5) We describe the module u^* as a submodule of Fock space, whose basis is given by Young diagrams.

Let $v_k = q^{2i_k}$ be the parameters of $\mathcal{H}_C$, r be the order of q^2. We consider the Hecke algebra over $\boldsymbol{C}(\mathbf{t})$ with parameters $\boldsymbol{v}_k = v_k \boldsymbol{t}^{k-1} (1 \leq k \leq m)$, $\boldsymbol{q} = q\boldsymbol{t}^m$. Then it is a semi-simple algebra [A2] and the graded dual of the Grothendieck group has a dual basis of Specht modules. Let $\mathcal{F}$ be the vector space with basis indexed by Young diagrams. Then, this graded dual can be identified with the m-fold tensor $\mathcal{F} \otimes \cdots \otimes \mathcal{F}$. Thus u^* is naturally embeded in $\mathcal{F} \otimes \cdots \otimes \mathcal{F}$ through this identification.

Let $\lambda = (\lambda^{(1)}, \cdots, \lambda^{(m)})$ be an m-tuple of Young diagrams. For each cell in λ, we define its r-residue, $r(x)$ by $r(x) \equiv i_{c(x)} - a(x) + b(x) \pmod r$, where the cell is located at $(a(x), b(x))$-th entry of $\lambda^{(c(x))}$. If $\lambda' = \lambda \cup \{x\}$, we write $r(\lambda'/\lambda) \equiv i \pmod r$.

We introduce operators e_i, f_i on $\mathcal{F} \otimes \cdots \otimes \mathcal{F}$ by

$$e_i\lambda = \sum_{r(\lambda/\mu) \equiv i} \mu, \quad f_i\lambda = \sum_{r(\lambda'/\lambda) \equiv i} \lambda'.$$

As is referred in [LLT, Theorem 4.1], a theorem of Hayashi states that $\mathcal{F}$ becomes $\mathfrak{g}$-module by these operators [Ha] [MM]. In fact, we introduce h_i by $h_i\lambda = (a_+^{(i)}(\lambda) - a_-^{(i)}(\lambda))\lambda$, where $a_+^{(i)}(\lambda)$ (resp. $a_-^{(i)}(\lambda)$) is the number of Young diagrams appearing in $f_i\lambda$ (resp. $e_i\lambda$). We let ν be a Young diagram appearing in $e_if_j\lambda$ or $f_je_i\lambda$. Then $\nu = (\lambda \setminus \{x\}) \cup \{y\}$ such that $r(x) \equiv i, r(y) \equiv j$. If x,y are not adjacent, ν appears in both $e_if_j\lambda$ and $f_je_i\lambda$. Hence we consider the rest of the case. If $i \not\equiv j$, x,y cannot be adjacent. If $i \equiv j$, x,y must coincide. Thus we have $[e_i, f_j] = \delta_{ij}h_i$. Nextly, we let ν be a Young diagram appearing in $e_j\lambda$. Then $\nu = \lambda \setminus \{x\}$ such that $r(x) \equiv j$. If the cell on the right hand side of x has residue i which is counted in $a_+^{(i)}(\lambda)$, then $a_+^{(i)}(\lambda) - a_-^{(i)}(\lambda)$ decreases by one if x is removed. If this right hand side cell is not counted in $a_+^{(i)}(\lambda)$, then the cell just above x becomes a cell counted in $a_-^{(i)}(e_i\lambda)$ and thus $a_+^{(i)}(\lambda) - a_-^{(i)}(\lambda)$ decreases by one. The same holds for the cell just below x. Hence we have $[h_i, e_j] = -e_j$ if $i = j \pm 1$, and $[h_i, e_i] = 2e_i$. Similar argument shows $ad(e_i)^2 e_j = 0, (r \geq 3, j = i \pm 1)$, $ad(e_i)^3 e_j = 0 (r = 2, j = i \pm 1)$, and the remaining relations.

If we shift the numbering of generators, the same is true for $\mathcal{F}$ which has a shifted r-residue, which appears as the components of $\mathcal{F} \otimes \cdots \otimes \mathcal{F}$. Hence $\mathcal{F} \otimes \cdots \otimes \mathcal{F}$ becomes a $\mathfrak{g}$-module via above defined operators.

By Lemma 2.1(2), it coincides with $U(\mathfrak{g})$-action on u^* if restricted to u^*.

Further, the m-tuple of the empty diagram ϕ is a weight vector. In particular, if we denote the weight by Λ, then $\Lambda(h_i)$ is the cardinality of the set $\{k | i_k \equiv i \pmod r\}$. (we set $h_i = [e_i, f_i]$ as usual.)

(4.6) We now state the theorem.

Theorem 4.4. *Let $\mathfrak{g}$ be the Kac-Moody Lie algebra of type A_∞ (resp. $A^{(1)}_{r-1}$) if q^2 is of infinite order (resp. of order r), and u^* be the graded dual of the Grothendieck groups of $\mathcal{H}_C$.*

We denote by n_i the multiplicity of q^{2i} in $\{v_1,\cdots,v_m\}$. Then we have,

(1) u^* is a highest weight $\mathfrak{g}$-module with highest weight $\Lambda=\sum n_i\Lambda_i$ where Λ_i are fundamental weights.

(2) Let $\{b\}=\{\bar{\mathcal{O}}_{\lambda,z}\}$ be canonical basis of $U^-(\mathfrak{g})$, v_Λ be the highest weight vector of u^*. Then the set of non-zero elements in $\{bv_\Lambda\}$ coincides with the dual basis of irreducible $\mathcal{H}_C$-modules.

Proof. We prove (2) first. By Lemma 4.1, u^* is again a cyclic U^--module, and the dual basis of irreducible H_{q^2}-modules coincides with canonical basis by Proposition 4.3 (2). Take an irreducible H_{q^2}-module D. Then $[D]^*$ maps to 0 if D is not a $\mathcal{H}_C$-module, and maps to a dual basis element of irreductible $\mathcal{H}_C$-modules otherwise. Hence non-zero elements of the image of canonical basis are linearly independent, and are nothing but the dual basis of irreducible $\mathcal{H}_C$-modules. Hence we have (2).

Since we have already seen that u^* is a $\mathfrak{g}$-submodule of $\mathcal{F}\otimes\cdots\otimes\mathcal{F}$, and (2) tells that it is a cyclic module generated by v_Λ, we have (1).

We have also proved the following proposition.

Proposition 4.5. *Let $\mathcal{F}\otimes\cdots\otimes\mathcal{F}$ be the m-fold tensor of the space of Young diagrams $\mathcal{F}$, and f_i be the operator $f_i\lambda=\sum_{r(\lambda'/\lambda)\equiv i}\lambda'$.*

Then u^* is the subspace of $\mathcal{F}\otimes\cdots\otimes\mathcal{F}$ spanned by $\{f_{i_1}\cdots f_{i_N}\phi\}$, where ϕ is the empty Young diagram.

As we have seen above, we regard u^* as a submodule of $\mathcal{F}\otimes\cdots\otimes\mathcal{F}$. To make it $U_{\mathbf{v}}^-$-module, we have to choose a Hopf-algebra structure. It should be Δ_- defined in [K, 1.4.2], since [K, Lemma 2.5.1] states that the tensor product of lower crystal lattices is a lower crystal lattice. We note that the module is irreducible, since it is a submodule of an integrable module. Hence the number of irreducible modules is computable by the q-dimension formula.

Another important remark about the module structure of u^* is for the Hecke algebra of type A. In this case, the number of irreducible modules is also known as the number of r-regular partitions [DJ], and its generating function coincides with the q-dimension. Hence we again know that the $\mathfrak{g}$-module u^* for the Hecke algebra of type A is irreducible and realized as $L(\Lambda_0)$. The canonical basis coincides with the dual basis of irreducible modules by the above theorem.

(4.7) Innerproduct on negative part of the quantized enveloping algebra $U_{\mathbf{v}}(\mathfrak{g})$ are introduced in two ways. That in [GL] satisfies $(uf_i, v)=(1-\mathbf{v}^2)^{-1}(u, d_i(v))$ where $d_i(v)\otimes f_i$ is the term appearing in $\Delta_{\tau,w}(v)$ defined in [GL].

On the other hand, that in [K] satisfies $(uf_i, v) = (u, Ad(t_i)e_i''(v))$ [K, 5.2.3]. It is easy to see that $d_i(f_j u) = f_j d_i(u) + \delta_{ij} u$. If we replace d_i by $Ad(t_i)e_i''$, we have the same formula [K, 3.4.4]. Hence these innerproduct coincide modulo $\mathbf{v}\boldsymbol{Z}[[\mathbf{v}]]$. By [K, Proposition 5.1.2], the global basis of Kashiwara satisfies $(b, b') = \delta_{ij} + \mathbf{v}\boldsymbol{Z}[[\mathbf{v}]]$.

Using these properties, Grojnowski and Lusztig have proved that the global basis of Kashiwara coincides with the canonical basis of Lusztig[GL].

When mapped to an irreducible integrable module, non-zero elements coincide with the lower crystal basis [K,Theorem 5], which is unique up to isomorphism [K,Theorem 3]. Hence, for the Hecke algebra of type A, Theorem 4.4 verfies a conjecture of Lascoux-Leclerc-Thibon if we recall the remark stated at the end of the previous subsection.

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