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ON THE DECOMPOSITION OF GENERALIZED K-CURVATURE TENSOR FIELDS

Dedicated to Professor Shigeo Sasaki on his 60th birthday

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In a work [2] K. Nomizu studied a decomposition of generalized curvature tensor fields on a Riemannian manifold. In this paper, we study its Kählerian analogous. For notations and the fundamental facts, we refer to [1]. The present author gratefully acknowledges his debt to his teacher S. Sasaki, who stimulated and guided his interest in differential geometry. The author wishes to express his hearty thanks to Dr. S. Tanno and T. Sakai for their kind advices. The author also thanks these people who gave encouragement during his stay at Tôhoku and Hirosaki Universities.

1. Algebraic preliminaries. Let V be a $2n$-dimensional real vector space with a complex structure J and hermitian inner product g. A tensor L of type (1, 3) over V, that is, an element of $V \otimes V^* \otimes V^* \otimes V^*$, where V^* is the dual space of V, can be considered as a bilinear mapping

$$(x, y) \in V \times V \rightarrow L(x, y) \in \mathrm{Hom}\,(V, V)\ .$$

Such a tensor L is called a *K-curvature tensor* on V if it has the following properties:

(1) $L(y, x) = -L(x, y)$;

(2) $L(x, y)$ is a skew-symmetric endomorphism of V, i.e.

$$g(L(x, y)u, v) + g(u, L(x, y)v) = 0\ ;$$

(3) $\sigma L(x, y)z = 0$ (the first Bianchi identity), where σ denotes the cyclic sum over x, y and z;

(4) $L(x, y)\circ J = J\circ L(x, y)$.

We denote by $\mathscr{L}(V)$ the vector space of all K-curvature tensors over V. It is a subspace of the tensor space of type (1, 3) over V and has a natural inner product induced from that in V. For $L \in \mathscr{L}(V)$, the *Ricci tensor* $K = K_L$ of type (1, 1) is a symmetric endomorphism of V defined by

$$K(x) = \text{trace of the bilinear map: } (y, z) \in V \times V \rightarrow L(x, y)z \in V\ .$$

If $\{e_1, \cdots, e_{2n}\}$ is an orthonormal basis of V with respect to hermitian inner product g, then we have

$$K(x) = \sum_{A=1}^{2n} L(x, e_A)e_A .$$

The trace of the Ricci tensor K_L is called the *scalar curvature* of L. Let P be a plane, that is, a 2-dimensional subspace in V and let x and y be an orthonormal basis for P. We set

$$k(P) = g(L(x, y)y, x) .$$

We see that $k(P)$ is independent of the choice of an orthonormal basis for P. In particular, if a 2-dimensional plane P is invariant by the complex structure J and x is a unit vector in P, then $\{x, Jx\}$ is an orthonormal basis for P and

$$k(P) = g(L(x, Jx)Jx, x) .$$

We call that $k(P)$ is the holomorphic sectional curvature for J-invariant P.

We now discuss some examples of K-curvature tensors. For $x, y \in V$, we denote by $x \wedge y$ the skew-symmetric endomorphism of V defined by

$$(x \wedge y)z = g(z, y)x - g(z, x)y .$$

Let A and B be two symmetric endomorphisms of V which commute with J. We define $L = L_{A,B}$ by

$$(\alpha) \qquad L(x, y) = Ax \wedge By + Bx \wedge Ay + JAx \wedge JBy + JBx \wedge JAy + 2g(Ax, Jy)JB - 2g(Jx, By)JA .$$

L is a K-curvature tensor (properties of A and B are used for (3) and (4)). The Ricci tensor K is given by

$$K = (\mathrm{tr}\, B)A + (\mathrm{tr}\, A)B + 2(AB + BA) ,$$

and the scalar curvature of L is given by

$$\mathrm{tr}\, K = 2(\mathrm{tr}\, A)(\mathrm{tr}\, B) + 4\mathrm{tr}\,(AB) .$$

As special cases we obtain following examples:

EXAMPLE 1. Take $A = cI/8$, $B = I$, where I is the identity transformation and c is a constant. Then L is given by

$$L(x, y) = c/4 \{x \wedge y + Jx \wedge Jy + 2g(x, Jy)J\} .$$

The Ricci tensor and the scalar curvature are

$$K = (n + 1)cI/2 , \qquad \mathrm{tr}\, K = n(n + 1)c .$$

The holomorphic sectional curvatures $k(P)$ for all planes P in V invariant by J is identically equal to c. It is well-known that if $L \in \mathscr{L}(V)$ has constant holomorphic sectional curvature, say c, then it is of the above form.

EXAMPLE 2. Take $B = cI/4$ and a symmetric endomorphism A which commutes with J. Then L is given by

$$L(x, y) = c/4\,\{Ax \wedge y + x \wedge Ay + JAx \wedge Jy + Jx \wedge JAy + 2g(Ax, Jy)J - 2g(Jx, y)JA\}\,.$$

The Ricci tensor K and the scalar curvature are

$$K = (n + 2)cA/2 + c(\mathrm{tr}\,A)I/4\,, \qquad \mathrm{tr}\,K = (n + 1)c(\mathrm{tr}\,A)\,.$$

The significance of this example is clear from the Bochner tensor of a Kählerian manifold.

In order to get an orthogonal decomposition of $\mathscr{L}(V)$ in [2], we define $\mathscr{L}_1(V)$ to be the subspace of $\mathscr{L}(V)$ consisting of all K-curvature tensors

$$L(x, y) = c/4\,\{x \wedge y + Jx \wedge Jy + 2g(x, Jy)J\}\,,$$

where c is an arbitrary constant, i.e.,

$$\mathscr{L}_1(V) = \{L \in \mathscr{L}(V) \text{ with constant holomorphic sectional curvature}\}\,.$$

Let $\mathscr{L}_1^{\perp}(V)$ be the orthogonal complement of $\mathscr{L}_1(V)$ in $\mathscr{L}(V)$. Then, we have the following:

PROPOSITION 1.

$$\mathscr{L}_1^{\perp}(V) = \{L \in \mathscr{L}(V) \textit{ with vanishing scalar curvature}\}$$

and

$$\mathscr{L}(V) = \mathscr{L}_1(V) \oplus \mathscr{L}_B(V) \oplus \mathscr{L}_2(V) \ (\textit{orthogonal})$$

where

$$\mathscr{L}_B(V) = \{L \in \mathscr{L}(V) \textit{ with vanishing Ricci tensor}\}\,,$$
$$\mathscr{L}_2(V) = \textit{orthogonal complement of } \mathscr{L}_B(V) \textit{ in } \mathscr{L}_1^{\perp}(V)\,.$$

PROOF. It is sufficient to show that $\mathscr{L}_1^{\perp}(V)$ consists of all $L \in \mathscr{L}(V)$ whose scalar curvature is 0. Since g is an hermitian inner product, we have vectors $e_1, \cdots, e_n$ of V such that $\{e_1, \cdots, e_n, Je_1, \cdots, Je_n\}$ is an orthonormal basis for V. Unless otherwise stated, Latin small indices $i, j, k, \cdots$ run from 1 to n, while Latin capitals $A, B, C, \cdots$ run through the range $1, \cdots, n;\ \bar{1}, \cdots, \bar{n}\ (\bar{i} = i + n)$. We set $Je_i = e_{\bar{i}}$. With respect to the above orthonormal basis $\{e_A\}$ of V, let L_{ABCD} be the components of L:

$$L(e_C, e_D)e_B = \sum_A L_{ABCD} e_A \,.$$

Since L satisfies the properties (1), (2), (3) and (4), we have the following identities:

(a) $L_{ABDC} = -L_{ABCD}$,
(b) $L_{BACD} = -L_{ABCD}$,
(c) $L_{ABCD} + L_{ACDB} + L_{ADBC} = 0$,
(d) $L_{CDAB} = L_{ABCD}$,
(e) $L_{\bar{i}\bar{j}CD} = L_{ijCD}$, $L_{\bar{i}jCD} = -L_{i\bar{j}CD}$, $L_{AB\bar{k}\bar{l}} = L_{ABkl}$, $L_{AB\bar{k}l} = -L_{ABk\bar{l}}$.

For $L' \in \mathscr{L}(V)$, the inner product $\langle L, L' \rangle$ is equal to

$$\sum_{A,B,C,D} L_{ABCD} L'_{ABCD} \,.$$

Now if $L' \in \mathscr{L}_1(V)$, then for some (unique) c we have,

$$\begin{aligned} L'_{ijkl} &= L'_{ij\bar{k}\bar{l}} = L'_{\bar{i}\bar{j}kl} = L'_{\bar{i}\bar{j}\bar{k}\bar{l}} \\ &= c/4\,(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) \,, \\ L'_{i\bar{j}k\bar{l}} &= L'_{\bar{i}j\bar{k}l} = -\,L'_{i\bar{j}\bar{k}l} = -L'_{\bar{i}jk\bar{l}} \\ &= c/4\,(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} + 2\delta_{ij}\delta_{kl}) \,, \end{aligned}$$

other components being 0. So, we see that

$$\begin{aligned} \langle L, L' \rangle &= 4 \sum_{i,j,k,l} (L_{ijkl}L'_{ijkl} + L_{i\bar{j}k\bar{l}}L'_{i\bar{j}k\bar{l}}) \\ &= c \sum_{i,j,k,l} L_{ijkl}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) \\ &\quad + c \sum_{i,j,k,l} L_{i\bar{j}k\bar{l}}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} + 2\delta_{ij}\delta_{kl}) \\ &= c \sum_{i,j} (L_{ijij} - L_{ijji}) + c \sum_{i,j} (L_{i\bar{j}i\bar{j}} + L_{i\bar{j}j\bar{i}} + 2L_{i\bar{i}j\bar{j}}) \\ &= 2c \sum_{i,j} (L_{ijij} + L_{i\bar{j}i\bar{j}} + L_{i\bar{i}j\bar{j}}) \\ &= 2c \sum_{i,j} (L_{ijij} + L_{i\bar{j}i\bar{j}} + L_{ijij} + L_{i\bar{j}i\bar{j}}) \\ &= 2c \sum_{A,B} L_{ABAB} \\ &= 2c \text{ (scalar curvature of } L), \end{aligned}$$

because

$$\begin{aligned} L_{i\bar{i}j\bar{j}} &= -\,L_{ij\bar{j}\bar{i}} - L_{i\bar{j}\bar{i}j} \\ &= -\,L_{ijji} + L_{i\bar{j}\bar{i}\cdot} \\ &= L_{ijij} + L_{i\bar{j}i\bar{j}} \,. \end{aligned}$$

This proves our assertion.

Proposition 2. *For* $L \in \mathscr{L}(V)$, *let*

$$L = L_1 + L_B + L_2 ,$$

where

$$L_1 \in \mathscr{L}_1(V),\ L_B \in \mathscr{L}_B(V),\ L_2 \in \mathscr{L}_2(V) .$$

Then we have

$$\begin{aligned} L_1(x, y) &= \frac{\operatorname{tr} K}{8n(n+1)} L_{I,I}(x, y) , \\ L_2(x, y) &= \frac{1}{2(n+2)}\left\{L_{K,I}(x, y) - \frac{\operatorname{tr} K}{2n} L_{I,I}(x, y)\right\} , \\ L_B(x, y) &= L(x, y) - \frac{1}{2(n+2)}\left\{L_{K,I}(x, y) - \frac{\operatorname{tr} K}{4(n+1)} L_{I,I}(x, y)\right\} , \end{aligned}$$

where K is the Ricci tensor L and $L_{A,B}$ is the tensor defined by the equation (α).

PROOF. Since we can show easily that tensors L_1 and L_B belong to $\mathscr{L}_1(V)$ and $\mathscr{L}_B(V)$ respectively, it is sufficient to show that tensor L_2 belongs to $\mathscr{L}_2(V)$. Since the Ricci tensor K of L is a symmetric endomorphism which commutes with the complex structure J of V, if a non-zero vector x is an eigen-vector of K with respect to an eigen-value λ, then so is Jx. Therefore we can assume that an orthonormal basis $\{e_1, \cdots, e_n, Je_1, \cdots, Je_n\}$ of V possesses the following properties:

e_i and Je_i are eigen-vectors of K with respect to the same eigen-value λ_i.

With respect to the above orthonormal basis $\{e_A\}$ of V, let $\tilde{L}_{ABCD}$ and $\bar{L}_{ABCD}$ be the components of $\tilde{L} = 2(n+2)L_2$ and $\bar{L} \in \mathscr{L}_B(V)$ respectively. Then we have

$$\begin{aligned} \tilde{L}_{ijkl} &= \tilde{L}_{ij\bar{k}\bar{l}} = \tilde{L}_{\bar{i}\bar{j}kl} = \tilde{L}_{\bar{i}\bar{j}\bar{k}\bar{l}} \\ &= \left(\lambda_k + \lambda_l - \frac{\operatorname{tr} K}{n}\right)(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) , \\ \tilde{L}_{i\bar{j}k\bar{l}} &= -\tilde{L}_{i\bar{j}\bar{k}l} = -\tilde{L}_{\bar{i}jk\bar{l}} = \tilde{L}_{\bar{i}j\bar{k}l} \\ &= \left(\lambda_k + \lambda_l - \frac{\operatorname{tr} K}{n}\right)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \\ &\quad + 2\left(\lambda_k + \lambda_j - \frac{\operatorname{tr} K}{n}\right)\delta_{ij}\delta_{kl} , \end{aligned}$$

other components being 0. So, we see that

$$
\begin{aligned}
&\langle \tilde{L}, \bar{L} \rangle \\
&= \sum_{i,j,k,l} (\tilde{L}_{ijkl}\bar{L}_{ijkl} + \tilde{L}_{ij\bar{k}\bar{l}}\bar{L}_{ij\bar{k}\bar{l}} + \tilde{L}_{\bar{i}\bar{j}kl}\bar{L}_{\bar{i}\bar{j}kl} \\
&\quad + \tilde{L}_{\bar{i}\bar{j}\bar{k}\bar{l}}\bar{L}_{\bar{i}\bar{j}\bar{k}\bar{l}} + \tilde{L}_{i\bar{j}k\bar{l}}\bar{L}_{i\bar{j}k\bar{l}} + \tilde{L}_{i\bar{j}\bar{k}l}\bar{L}_{i\bar{j}\bar{k}l} \\
&\quad + \tilde{L}_{\bar{i}jk\bar{l}}\bar{L}_{\bar{i}jk\bar{l}} + \tilde{L}_{\bar{i}j\bar{k}l}\bar{L}_{\bar{i}j\bar{k}l}) \\
&= 4 \sum_{i,j,k,l} \left(\lambda_k + \lambda_l - \frac{\operatorname{tr} K}{n}\right)(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})\bar{L}_{ijkl} \\
&\quad + 4 \sum_{i,j,k,l} \left\{\left(\lambda_k + \lambda_l - \frac{\operatorname{tr} K}{n}\right)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \right. \\
&\quad \left. + 2\left(\lambda_k + \lambda_j - \frac{\operatorname{tr} K}{n}\right)\delta_{ij}\delta_{kl}\right\}\bar{L}_{i\bar{j}k\bar{l}} \\
&= 8 \sum_{i,j} (\lambda_i + \lambda_j)(\bar{L}_{ijij} + \bar{L}_{i\bar{j}i\bar{j}} + \bar{L}_{i\bar{i}j\bar{j}}) \\
&\quad - \frac{8 \operatorname{tr} K}{n} \sum_{i,j} (\bar{L}_{ijij} + \bar{L}_{i\bar{j}i\bar{j}} + \bar{L}_{i\bar{i}j\bar{j}}) \\
&= 32 \sum_i \lambda_i \left(\sum_A \bar{L}_{iAiA}\right) - \frac{16 \operatorname{tr} K}{n} \sum_i \left(\sum_A \bar{L}_{iAiA}\right) \\
&= 0 ,
\end{aligned}
$$

because the Ricci tensor $\bar{K}$ of $\bar{L}$ is identically equal to 0. This proves our assertion.

For each $L \in \mathscr{L}(V)$, the $\mathscr{L}_B(V)$-component L_B is called the Bochner tensor associated to L.

REMARK. If the *Bochner tensor* associated to a K-curvature tensor L is 0, then

$$L(x, y) = \frac{1}{2(n+2)}\left\{L_{K,I} - \frac{\operatorname{tr} K}{4(n+1)} L_{I,I}\right\} .$$

By setting

$$A = \frac{1}{2(n+2)} K - \frac{\operatorname{tr} K}{8(n+1)(n+2)} I ,$$

we may write

$$L(x, y) = L_{A,I}(x, y) ,$$

as in Example 2. Conversely, if A is an arbitrary symmetric endomorphism of V which commutes with the complex structure J of V, then L given above is a K-curvature tensor whose associated Bochner tensor is 0 .

2. Generalized K-curvature tensor fields. Let M be a Kähler manifold with Kähler metric g and complex structure J. For each point p of

M we may consider K-curvature tensors in the sense of section 1 over the tangent space $T_p(M)$ with inner product g_p and complex structure J_p. A (differentiable) tensor field L of type (1, 3) on M will be called a *generalized K-curvature tensor field* if for each point p the tensor L_p is a K-curvature tensor over $T_p(M)$. We shall say that L is *proper* if it satisfies the second Bianchi identity, that is,

$$\sigma(\nabla_X L)(Y, Z) = 0 ,$$

where X, Y and Z are arbitrary vector fields on M.

Let L be a proper generalized K-curvature tensor field on M and let K be its Ricci tensor field. We prepare a few formulas (cf. [2])

$$(1) \qquad (\nabla_X K)Y - (\nabla_Y K)X = -\sum_{B=1}^{2n} (\nabla_{E_B} L)(X, Y)E_B ,$$

$$(2) \qquad (\nabla_{JX} K)JY - (\nabla_{JY} K)JX = (\nabla_X K)Y - (\nabla_Y K)X ,$$

$$(3) \qquad \text{trace of } \{X \to \sum_{B=1}^{2n} (\nabla_{E_B} L)(X, Y)E_B\} = \text{trace of } \{X \to (\nabla_X K)Y\} ,$$

$$(4) \qquad \text{trace of } \{X \to (\nabla_X K)Y\} = \frac{1}{2} Y(\mathrm{tr}\, K) ,$$

where X and Y are any vector fields and $\{E_B\}$ is an orthonormal frame field around p, for each point p of M. We can easily prove (1), (3) and (4) analogously as in [2]. To prove (2), we use the property of L:

$$L(JX, JY) = L(X, Y)$$

and its immediate consequence

$$(5) \qquad (\nabla_Z L)(JX, JY) = (\nabla_Z L)(X, Y) .$$

By virtue of (3) and (5), we can easily prove (2). We shall now prove two propositions.

PROPOSITION 3. *Let L be a proper generalized K-curvature tensor field on a connected Kähler manifold M. If the Ricci tensor field K of L satisfies Codazzi's equation*

$$(\nabla_X K)Y = (\nabla_Y K)X$$

then $\nabla K = 0$ on M.

PROOF. K satisfies $JK = KJ$. By a proposition and the proof of a theorem in S. Tanno [4, p. 502], we have $\nabla K = 0$.

PROPOSITION 4. *On a Kähler manifold M let A be a tensor field of* *type* (1, 1) *which is symmetric at each point and commutes with the complex structure J. Let* $L_{A,I}$ *be a generalized K-curvature tensor field defined by*

$$L_{A,I}(X, Y) = AX \wedge Y + X \wedge AY + JAX \wedge JY + JX \wedge JAY + 2g(AX, JY)J + 2g(X, JY)JA \, .$$

If $L_{A,I}$ *is proper and if* $\operatorname{tr} A$ *is constant, then* A *satisfies Codazzi's equation.*

PROOF. Since the Ricci tensor field K of $L \equiv L_{A,I}$ is of the form

$$K = 2(n + 2)A + (\operatorname{tr} A)I \, ,$$

then we have $\operatorname{tr} K = 4(n + 1) \operatorname{tr} A$. So, by virtue of (4), we have

$$\begin{aligned} &\text{trace of } \{X \to (\nabla_X A) Y\} \\ &= \text{trace of } \left\{X \to \left[\frac{1}{2(n + 2)}(\nabla_X K) Y - \frac{1}{2(n + 2)} X(\operatorname{tr} A)I\right]\right\} \\ &= \frac{1}{2(n + 2)} \frac{1}{2} Y(\operatorname{tr} K) \\ &= \frac{(n + 1)}{(n + 2)} Y(\operatorname{tr} A) \\ &= 0 \, . \end{aligned}$$

Therefore we get

$$\begin{aligned} (6) \qquad & g(Z, \sum_{B=1}^{2n} (\nabla_{E_B} L)(X, Y)E_B) \\ &= g(Z, (\nabla_Y A)X - (\nabla_X A) Y + (\nabla_{JY} A)JX - (\nabla_{JX} A)JY) \\ &\quad - 2g((\nabla_{JZ} A)X, JY) \, , \end{aligned}$$

$$(7) \qquad \sum_{B=1}^{2n} (\nabla_X L)(Y, E_B)E_B = 2(n + 2)(\nabla_X A) Y \, .$$

On the other hand, the second Bianchi identity implies that the left hand side of (6) is equal to

$$g(Z, - \sum_{B=1}^{2n} \{\nabla_X L)(Y, E_B)E_B + (\nabla_Y L)(E_B, X)E_B\}) \, .$$

Therefore by virtue of (2), (6) and (7), we get

$$(8) \qquad g((\nabla_{JZ} A)X, JY) = (n + 1)g(Z, (\nabla_X A) Y - (\nabla_Y A)X) \, .$$

By taking the cyclic sum over X, Y and Z, we get

$$\begin{aligned} (9) \qquad & g((\nabla_{JZ} A)X, JY) + g((\nabla_{JX} A) Y, JZ) \\ &+ g((\nabla_{JY} A)Z, JX) = 0 \, . \end{aligned}$$

Using (2) and $(\nabla_X A)J = J(\nabla_X A)$, we have

(10) $$g((\nabla_{JZ}A)X, JY) = g((\nabla_X A)Y - (\nabla_Y A)X, Z) .$$

From (8) and (10), we get

$$ng((\nabla_X A)Y - (\nabla_Y A)X, Z) = 0 .$$

Therefore A satisfies Codazzi's equation. This proves our assertion.

3. Main results. If L is a generalized K-curvature tensor field on a Kähler manifold M, then applying the decomposition in Proposition 2 at each point p of M we obtain

$$L = L_1 + L_B + L_2 ,$$

where L_1, L_B and L_2 are generalized K-curvature tensor fields, which at each point p, belong to $\mathscr{L}_1$, $\mathscr{L}_B$ and $\mathscr{L}_2$ over $T_p(M)$, respectively. Then we have similar results to [2]:

THEOREM I. *On a connected Kähler manifold M, let*

$$L = L_1 + L_B + L_2$$

be the natural decomposition of a proper generalized K-curvature tensor field L. If the Ricci tensor field K of L satisfies Codazzi's equation, then L_1, L_B and L_2 are proper. Conversely, if L_1, L_B and L_2 are proper and if $n \geqq 2$, then K satisfies Codazzi's equation.

PROOF. The first assertion is easy to prove. We prove the converse. The Ricci tensor K_1 of proper generalized K-curvature tensor field L_1 is of the form:

$$K_1 = \frac{\mathrm{tr}\, K}{2n} I .$$

If $n \geqq 2$, by a generalization of Schur's theorem we have $\mathrm{tr}\, K$ is constant on M. Since L_2 is proper, we see that $L_{K,I}$ defined by the equation in Proposition 4 is also proper. Hence we conclude that K satisfies Codazzi's equation. This completes the proof of Theorem I.

COROLLARY. *On a Kähler manifold M of complex dimension $n \geqq 2$ let L be a proper generalized K-curvature tensor field whose scalar curvature is constant. Then the associated Bochner curvature tensor field L_B is proper if and only if the Ricci tensor field K of L satisfies Codazzi's equation.*

Now let $\mathscr{U}(M)$ be the vector space of all tensor fields A of type (1, 1) on a Kähler manifold M which satisfy the following conditions:

1) A is symmetric at each point;
2) A commutes with the complex structure J of M;
3) A satisfies Codazzi's equation: $(\nabla_X A)Y = (\nabla_Y A)X$.

Let $\mathscr{L}(M)$ denote the vector space of all proper generalized K-curvature tensor fields whose Ricci tensor fields satisfy Codazzi's equation. Let n be the complex dimension of M. We have a linear mapping $A \in \mathscr{U}(M) \to L_A \in \mathscr{L}(M)$ given by

$$L_A = \frac{1}{2(n+2)} L_{A,I} - \frac{\operatorname{tr} A}{8(n+1)(n+2)} L_{I,I} .$$

Since the Ricci tensor field of L_A is precisely A, the mapping is one-to-one. We shall state the following result.

THEOREM II. *If $n = 1$, $A \to L_A$ is a linear isomorphism of $\mathscr{U}(M)$ onto $\mathscr{L}(M)$. If $n \geqq 2$, it is a linear isomorphism onto the subspace*

$$\{L \in \mathscr{L}(M);\, L_B = 0\} .$$

PROOF. If $n = 1$, then the fact that the Ricci tensor field K of $L \in \mathscr{L}(M)$ commutes with the complex structure J implies

$$K = \lambda I ,$$

where λ is a function. From the assumption on K we see that K is parallel by Proposition 3. Therefore λ is a constant. By Proposition 2, we have

$$L_2 = L_B = 0 ,$$

$$L = L_1 = \frac{\lambda}{8} L_{I,I} = L_{\lambda I} .$$

Thus the mapping is onto. Next, if $n \geqq 2$, then for $L \in \mathscr{L}(M)$ with $L_B = 0$ we have

$$L = \frac{1}{2(n+2)} L_{K,I} - \frac{\operatorname{tr} K}{8(n+1)(n+2)} L_{I,I}$$

by Proposition 3, where K is the Ricci tensor field of L. Therefore the mapping is onto. This completes the proof of Theorem II.

REFERENCES

[1] S. KOBAYASHI AND K. NOMIZU, Foundations of Differential Geometry, Volumes I and II, Interscience, 1963, 1969.
[2] K. NOMIZU, On the decomposition of generalized curvature tensor fields, Differential Geometry, in honor of K. Yano, Kinokuniya, Tokyo, 1972, 335–345.
[3] S. TACHIBANA, On the Bochner curvature tensor, Nat. Sci., Rep. Ochanomizu Univ., 18 (1967), 15–19.

[4] S. TANNO, 4-dimensional conformally flat Kähler manifolds, Tôhoku Math. J., 24 (1972), 501-504.
[5] S. TANNO, Curvature tensors and covariant derivatives, to appear.
[6] K. YANO AND S. BOCHNER, Curvature and Betti numbers, Ann. Math. Studies No. 32, 1953.

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