Canad. Math. Bull. Vol. **39** (3), 1996 pp. 257–265

ON THE DECOMPOSITION OF NONSINGULAR CS-MODULES

JOHN CLARK AND NGUYEN VIET DUNG

ABSTRACT. It is shown that if M is a nonsingular CS-module with an indecomposable decomposition $M = \bigoplus_{i\in I} M_i$, then the family $\{M_i \mid i \in I\}$ is locally semi-T-nilpotent. This fact is used to prove that any nonsingular self-generator Σ-CS module is a direct sum of uniserial Noetherian quasi-injective submodules. As an application, we provide a new proof of Goodearl's characterization of non-singular rings over which all nonsingular right modules are projective.

Introduction. A module M is called a CS-*module* if every submodule of M is essential in a direct summand of M. CS-modules present a generalization of quasi-continuous and continuous modules (see Mohamed and Müller [MM]), which in turn generalize quasi-injective and injective modules. It has recently been shown in [D1] (see also [DHSW, 8.13]) that if $M = \bigoplus_{i\in I} M_i$ is a decomposition of a CS-module M into summands M_i's with local endomorphism rings, then the family $\{M_i \mid i \in I\}$ is locally semi-T-nilpotent. It is natural to ask whether this result can be extended to the general case, without the local endomorphism ring hypothesis.

In this note, by using different techniques, we show that if $M = \bigoplus_{i\in I} M_i$ is any indecomposable decomposition of a nonsingular CS-module M, then the family $\{M_i \mid i \in I\}$ is locally semi-T-nilpotent. This result is used to show that any nonsingular self-generator Σ-CS module is a direct sum of uniserial Noetherian quasi-injective submodules. As an application, we give a new proof of a well-known theorem, due to Goodearl [G1], [G2], that every nonsingular right module over a right nonsingular ring R is projective if and only if R is (left and right) hereditary serial Artinian. We also show that, if R is a right nonsingular ring such that every $\aleph_1$-generated nonsingular right R-module is projective and either R is right quasi-continuous or R is commutative, then R is semisimple Artinian.

Definitions. Throughout this note all rings are associative with identity and all modules are unitary right modules. A submodule N of a module M is said to be essential in M if $N \cap K \neq 0$ for every nonzero submodule K of M. A submodule C of M is called *closed* in M provided C has no proper essential extensions in M. A module M is a CS-module (or, as in [DHSW] and [O], an *extending* module) if and only if every closed submodule of M is a direct summand. M is called Σ-CS (or Σ-*extending* in [CW]) if every direct sum of copies of M is a CS-module. Recall that M is said to be *continuous* if it is CS and every

Received by the editors February 6, 1995.
AMS subject classification: Primary: 16D70; secondary: 16D40, 16D50, 16D80.
Key words and phrases: nonsingular module, CS-module, indecomposable module, quasi-injective module, uniserial module, hereditary ring.

https://doi.org/10.4153/CMB-1996-033-4 Published online by Cambridge University Press

submodule of M which is isomorphic to a direct summand of M is also a direct summand of M. Also a module M is called *quasi-continuous* if it is CS and whenever M_1 and M_2 are direct summands of M with $M_1 \cap M_2 = 0$ then $M_1 \oplus M_2$ is also a direct summand of M.

A family of modules $\{M_i \mid i \in I\}$ is called *locally semi-T-nilpotent* if, for any countable set of non-isomorphisms $\{f_n : M_{i_n} \longrightarrow M_{i_{n+1}}\}$ with all i_n distinct in I, and for any $x \in M_{i_1}$, there exists k (depending on x) such that $f_k \cdots f_1(x) = 0$.

Results. We begin with the following result.

THEOREM 1. *Let M be a nonsingular* CS-*module such that $M = \bigoplus_{i \in I} M_i$, where M_i is indecomposable for all $i \in I$. Then the family $\{M_i \mid i \in I\}$ is locally semi-T-nilpotent.*

PROOF. Consider any infinite sequence of non-isomorphisms f_n

$$M_{i_1} \xrightarrow{f_1} M_{i_2} \xrightarrow{f_2} \cdots \longrightarrow M_{i_n} \xrightarrow{f_n} \cdots.$$

For simplicity we may write $i_n = n$. Clearly M_n is uniform, hence if $\operatorname{Ker} f_n \neq 0$ it would imply that $f_n(M_n)$ is a singular submodule of M_{n+1}, a contradiction because M is nonsingular. Thus f_n is a monomorphism for all $n \geq 1$. Let $E(M_1)$ be the injective hull of M_1. Then $f_1^{-1} : f_1(M_1) \longrightarrow M_1$ can be extended to a monomorphism $g_2 : M_2 \longrightarrow E(M_1)$. Setting $N_2 = g_2(M_2)$, we clearly have $M_1 \subset N_2$ and $M_1 \neq N_2$. Repeating this argument on the non-isomorphic monomorphism $f_2 g_2^{-1} : N_2 \longrightarrow M_3$ produces a submodule N_3 of $E(M_1)$ such that N_3 properly contains N_2 and $N_3 \simeq M_3$. Continuing this procedure we get a strictly ascending sequence

$$M_1 \subset N_2 \subset N_3 \subset \cdots \subset N_n \subset \cdots \subseteq E(M_1)$$

where $N_n \simeq M_n$ for all $n \geq 2$.

Next set $N_1 = M_1$ and let $\varphi : \bigoplus_{n=1}^{\infty} N_n \longrightarrow E(M_1)$ be the homomorphism defined by $\varphi(x) = x_1 + x_2 + \cdots + x_k$ for each $x = (x_1, \ldots, x_k, 0, 0, \ldots) \in \bigoplus_{n=1}^{\infty} N_n$. Since $E(M_1)$ is nonsingular, it follows that $A = \operatorname{Ker} \varphi$ is a closed submodule of $\bigoplus_{n=1}^{\infty} N_n$. But $\bigoplus_{n=1}^{\infty} N_n$ is isomorphic to $\bigoplus_{n=1}^{\infty} M_n$ which is CS, hence A is a direct summand of $\bigoplus_{n=1}^{\infty} N_n$. Therefore we have $\bigoplus_{n=1}^{\infty} N_n = A \oplus B$ for some submodule B. Then B is isomorphic to a submodule of $E(M_1)$, hence uniform, so B contains a finitely generated essential submodule C. There exists a positive integer m such that $C \subseteq \bigoplus_{n=1}^{m} N_n$, thus C is essential in a direct summand D of $\bigoplus_{n=1}^{m} N_n$. But $\bigoplus_{n=1}^{\infty} N_n$ is nonsingular, hence every submodule of $\bigoplus_{n=1}^{\infty} N_n$ has a unique maximal essential extension in $\bigoplus_{n=1}^{\infty} N_n$ (see *e.g.* [O1, Lemma 2]). In particular, it follows that $B = D$, thus $B \subseteq \bigoplus_{n=1}^{m} N_n$.

Now, for any $x_{m+1} \in N_{m+1}$, consider the element $y = (0, \ldots, 0, x_{m+1}, 0, 0, \ldots) \in \bigoplus_{n=1}^{\infty} N_n$. There are elements $a \in A$ and $b \in B$ such that $y = a + b$. Clearly a and b must have the forms $a = (x_1, \ldots, x_m, x_{m+1}, 0, 0, \ldots)$ and $b = (-x_1, \ldots, -x_m, 0, 0, \ldots)$, for some $x_k \in N_k$, $k = 1, \ldots, m$. But $a \in A = \operatorname{Ker} \varphi$, so this implies that $x_1 + \cdots + x_m + x_{m+1} = 0$, hence $x_{m+1} \in N_1 + \cdots + N_m = N_m$. Therefore we get $N_m = N_{m+1}$, a contradiction which completes the proof. ∎

https://doi.org/10.4153/CMB-1996-033-4 Published online by Cambridge University Press

It is well-known that over a right Noetherian ring any right CS-module has an indecomposable decomposition (see, *e.g.*, [MM, Theorem 2.19] or [GJ, Theorem 1.10]). However, by [GD, Corollary 1.7], if the ring R has merely finite right uniform dimension, then any nonsingular CS right R-module is a direct sum of uniform submodules.

We now show by example that we can not remove from Theorem 1 the hypothesis that the summands M_i are indecomposable.

EXAMPLE 2. Let V be an infinite-dimensional right vector space over a division ring D and let R be the ring $\mathrm{End}_D(V)$ of linear transformations on V. It is well-known that R is a von Neumann regular right self-injective ring (see [G2, Proposition 2.23]) and so, by [DS, Proposition 3], $R_R^{(\mathbb{N})}$ is CS. Moreover, for each $n \in \mathbb{N}$, $R_R \simeq R_R^{(n)}$, the direct sum of n copies of R, (see Anderson and Fuller [AF, Exercise 8.14]). Thus, if we take $M_n = R_R^{(n)}$ for each $n \in \mathbb{N}$, the module $\bigoplus_{n\in\mathbb{N}} M_n$ is CS. Now define $f_n: M_n \longrightarrow M_{n+1}$ to be the monomorphism from $R_R^{(n)}$ to $R_R^{(n+1)} = R \oplus R_R^{(n)}$ induced by the isomorphism from $R_R^{(n)}$ to the first summand R_R. Then, since each f_n is not an isomorphism, we get that $\{M_n \mid n \in \mathbb{N}\}$ is not locally semi-T-nilpotent.

If R is a right nonsingular ring such that the module R_R is Σ-CS then R is (left and right) Artinian (see, *e.g.*, [DS, Corollary 2]). Following [DS], a module M is called *countably* Σ-CS if $M^{(\mathbb{N})}$ is a CS-module. By [DS, Proposition 3], every (von Neumann) regular right self-injective ring R is countably Σ-CS as a right module over itself. However, since R need not be Artinian, it follows from above that R_R need not be Σ-CS. In view of this, it seems natural to ask if there exists a regular right self-injective ring R such that $R_R^{(I)}$ is CS for some index set I of uncountable cardinality but R_R is not Σ-CS. Rather surprisingly, such a ring does not exist. The authors are very grateful to Prof. Ken Goodearl for pointing out this fact and showing them how it could be deduced from results in Osofsky's paper [O2]. The following proposition extends this observation to quasi-continuous rings and is proved using Osofsky's ideas.

PROPOSITION 3. *Let R be a right nonsingular right quasi-continuous ring such that $R_R^{(I)}$ is* CS *for some index set I of uncountable cardinality. Then R is semisimple Artinian.*

PROOF. We show first that R does not contain an infinite set of (nonzero) orthogonal idempotents. Assume to the contrary that $\{e_n \mid n \in \mathbb{N}\}$ is a countably infinite set of orthogonal idempotents in R. Note that, since R_R is nonsingular CS, every right ideal K of R has a unique maximal essential extension K' which is a direct summand of R (see, *e.g.*, [O1, Lemma 2]). Let $\sum_{n\in\mathbb{N}} e_nR$ be essential in eR, where e is an idempotent in R.

Now let A be any subset of $\mathbb{N}$. Denote the unique maximal essential extensions of $\sum_{n\in A} e_nR$ and $\sum_{n\notin A} e_nR$ by F_A and G_A respectively. Then $F_A \cap G_A = 0$ and $\sum_{n\in\mathbb{N}} e_nR$ is essential in $F_A \oplus G_A$. Since R is right quasi-continuous, $F_A \oplus G_A$ is a direct summand of R_R, hence of eR. From this it follows that $eR = F_A \oplus G_A$. With respect to this decomposition let $e(A)$ denote the projection of e in F_A. In the particular case where A is the singleton subset $\{k\}$ for some $k \in \mathbb{N}$ we denote $e(A)$ by $\epsilon(k)$. We will show that the sets $\{\epsilon(k) \mid k \in \mathbb{N}\}$ and $\{e(A) \mid A \subseteq \mathbb{N}\}$ satisfy the hypotheses of [O2, Proposition 5], namely that

$\{\epsilon(k) \mid k \in \mathbb{N}\}$ is an infinite set of orthogonal idempotents such that, for all $k \in \mathbb{N}$ and all subsets A, B of $\mathbb{N}$,

$$(1) \qquad \epsilon(k)e(A) = e(A)\epsilon(k) = \epsilon(k)\chi_A(k)$$

where χ_A denotes the characteristic function of A and

$$(2) \qquad e(A)e(B) = e(B)e(A).$$

Firstly let A and B be disjoint subsets of $\mathbb{N}$. Since $\sum_{n\in A} e_nR \cap \sum_{n\in B} e_nR = 0$ it follows that $e(A)R \cap e(B)R = 0$. Now, since R is right quasi-continuous, we get that $e(A)R \oplus e(B)R$ is also a direct summand of R_R and hence of eR. This readily implies that $e(A)e(B) = e(B)e(A) = 0$ and $e(A \cup B) = e(A) + e(B)$. Now let A and B be arbitrary subsets of $\mathbb{N}$. Then, from above, we have $e(A) = e(A \setminus B) + e(A \cap B)$ and $e(B) = e(B \setminus A) + e(A \cap B)$. From this it follows that $e(A)e(B) = e(B)e(A) = e(A \cap B)$, verifying (2) and the first equation of (1). Moreover, if k and l are distinct elements of $\mathbb{N}$ then, since $\{k\} \cap \{l\} = \emptyset$ we also get from above that $\epsilon(k)\epsilon(l) = 0$. Thus the infinite set of idempotents $\{\epsilon(k) \mid k \in \mathbb{N}\}$ is orthogonal.

Next let $k \in \mathbb{N}$ and A be a subset of $\mathbb{N}$. If $k \in A$ then $\epsilon(k)R \subseteq e(A)R$ and this gives $e(A)\epsilon(k) = \epsilon(k)$. On the other hand, if $k \notin A$ then $\{k\} \cap A = \emptyset$ and so $e(A)\epsilon(k) = 0$. Thus $e(A)\epsilon(k) = \epsilon(k)\chi_A(k)$ for all possible k and A, completing the verification of (1).

We may now apply [O2, Proposition 5] to get a nice set of idempotents U in R such that the cardinality of U is $2^{\aleph_0}$ (see [O2, p. 641] for the definition of "nice"). Choose a subset $V = \{g_\alpha \mid \alpha \in \Omega\}$ of U where Ω is a set of cardinality $\aleph_1$. Then V is also a nice set of idempotents and it follows from [O2, Theorem A] that the right ideal of R generated by V is not projective (in fact of projective dimension one).

On the other hand, if we denote the cardinality of the index set I by $\aleph$, then, since $R_R^{(I)}$ is CS, it follows from the proof of [DS, Proposition 1] that every $\aleph$-generated nonsingular right R-module is projective. In particular, since $\aleph_1 \leq \aleph$, V must be projective. This contradiction shows that R does not contain an infinite set of orthogonal idempotents.

Since R is right nonsingular with no infinite set of orthogonal idempotents and $R_R^{(\mathbb{N})}$ is CS, it follows from [DS, Theorem 4] that R_R is Σ-CS and so, by Goodearl's Theorem [G1, Theorem 2.15] (see also [DS, Corollary 2]), R is (left and right) serial Artinian. Then it is well-known that the indecomposable summands of R_R are quasi-injective (see, *e.g.*, [W, 55.16]) and, since R is right quasi-continuous, these summands are relatively injective (see [MM, Proposition 2.19]). It follows that R is right self-injective and so, since R is right nonsingular, the Jacobson radical of R is zero. Hence R is semisimple Artinian, as required. ∎

REMARK 4. It would be interesting to know if the "right quasi-continuous" condition in Proposition 3 can be removed. In view of (the proof of) [DS, Proposition 1], this would mean that if R is a right nonsingular ring for which every $\aleph_1$-generated nonsingular right R-module is projective then every nonsingular right R-module is projective, thereby giving a stronger form of Goodearl's Theorem [G1, Theorem 2.15]. This will be true if

R is commutative, since in this case the proof of Proposition 3 can be modified (and simplified) as follows. Suppose that R contains an infinite set $\{e_k \mid k \in \mathbb{N}\}$ of orthogonal idempotents. For each subset A of $\mathbb{N}$, the ideal $\sum_{n\in A} e_n R$ is essential in a direct summand $e(A)R$, where $e(A)$ is an idempotent of R. Clearly $e_n = e(A)e_n$ for each $n \in \mathbb{N}$. Also, if $n \notin A$, then $e_n(\sum_{m\in A} e_m R) = 0$. By [CH, Lemma 1.1], $e(A)H \subseteq \sum_{m\in A} e_m R$ for some essential ideal H of R. Then $e_n e(A)H = 0$ and so, since R is nonsingular, $e_n e(A) = 0$. It now follows that the sets $\{e_n \mid n \in \mathbb{N}\}$ and $\{e(A) \mid A \subseteq \mathbb{N}\}$ satisfy conditions (1) and (2) above. We now complete the argument as in the proof of Proposition 3. Thus we have shown that if R is a commutative nonsingular ring for which the R-module $R_R^{(I)}$ is CS for some index set I of uncountable cardinality then R is a finite direct sum of fields.

Our next objective is to give necessary and sufficient conditions for a nonsingular countably Σ-CS module to be Σ-CS, which may be considered as a module-theoretic generalization of [DS, Theorem 4]. For this we need a result which has recently been proved in [D2].

LEMMA 5. *Let $\{M_i \mid i \in I\}$ be a family of uniform modules with local endomorphism rings. Then $M = \bigoplus_{i\in I} M_i$ is a* CS*-module if and only if $\bigoplus_{i\in K} M_i$ is a* CS*-module for each countable subset K of I.*

PROOF. See [D2, Theorem 2.4]. ∎

PROPOSITION 6. *Let M be a nonsingular module. Then the following conditions are equivalent:*

(a) M is Σ-CS;

(b) M is countably Σ-CS *and M is a direct sum of uniform submodules.*

PROOF. (a) $\Rightarrow$ (b). Suppose that M is nonsingular and Σ-CS. Let $\hat{M}$ be the M-injective hull of M, *i.e.* the injective hull of M in the Grothendieck category $\sigma[M]$ (see Wisbauer [W]). Then M generates injective objects of $\sigma[M]$ (see, *e.g.*, [W, 16.3]), hence it follows from [GD, Proposition 1.13] that $\hat{M}$ is Σ-quasi-injective. By [GD, Corollary 1.6], we get that M is a direct sum of uniform submodules. (See also [CW, Proposition 2.2] for a more general result.)

(b) $\Rightarrow$ (a). Assume that M is nonsingular countably Σ-CS and $M = \bigoplus_{i\in I} M_i$, where each M_i is uniform. Since $M_i^{(\mathbb{N})}$ is nonsingular and CS, it follows by Theorem 1 that every monomorphism $f\colon M_i \longrightarrow M_i$ must be an isomorphism. Hence each M_i is continuous, therefore M_i has a local endomorphism ring by [MM, Proposition 3.5]. Now the result follows by Lemma 5. ∎

A module M is called a *self-generator* provided M generates all submodules of M, *i.e.* every submodule of M is a quotient of a direct sum of copies of M. For basic properties, characterisations and examples of self-generators, we refer to Zimmermann-Huisgen [Z-H]. We note in particular that a self-generator need not be a generator (see [Z-H, Example 1.2]).

We now describe the structure of a nonsingular Σ-CS module which is also a self-generator. Recall that a module U is said to be *uniserial* if for any submodules A and B of U, we have either $A \subseteq B$ or $B \subseteq A$.

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THEOREM 7. *Let M be any nonsingular self-generator* Σ-CS *module. Then M has a direct decomposition $M = \bigoplus_{i \in I} M_i$, where each M_i is uniserial Noetherian quasi-injective. Furthermore, if the family $\{M_i \mid i \in I\}$ has only a finite number of non-isomorphic modules, then each M_i is of finite (composition) length.*

PROOF. By Proposition 6 ((a) $\Rightarrow$ (b)), there is a direct decomposition $M = \bigoplus_{i \in I} M_i$, where each M_i is uniform. As in the proof of Proposition 6 ((b) $\Rightarrow$ (a)) we can show that each M_i is continuous, hence $\operatorname{End} M_i$ is local. Now $M_i \oplus M_i$ is CS and M_i is not isomorphic to any proper submodule of M_i, so it follows that M_i is M_i-injective, *i.e.* M_i is quasi-injective (see *e.g.* [DHSW, 7.3]).

Next we show that M_i is Noetherian for each $i \in I$. Let A be any submodule of M_i. Then, since M is a self-generator, there is an exact sequence

$$M^{(J)} \xrightarrow{f} A \longrightarrow 0$$

for some index set J. But A is nonsingular, so it follows that $\operatorname{Ker} f$ is a closed submodule of $M^{(J)}$, hence $\operatorname{Ker} f$ is a direct summand of $M^{(J)}$. Therefore A is isomorphic to a direct summand of $M^{(J)} = (\bigoplus_{i \in I} M_i)^{(J)}$, and since $\operatorname{End} M_i$ is local for all $i \in I$, we get by the Krull-Schmidt-Azumaya theorem ([AF, Theorem 12.6]) that $A \simeq M_j$ for some $j \in I$. Now suppose that there exists a strictly ascending chain of submodules of M_i

$$A_1 \subset A_2 \subset \cdots \subset A_n \subset \cdots \subseteq M_i.$$

Then by the above argument, each A_n is isomorphic to M_{i_n} for some $i_n \in I$. Hence the external direct sum $D = \bigoplus_{n=1}^{\infty} A_n$ is isomorphic to $\bigoplus_{n=1}^{\infty} M_{i_n}$ which is a direct summand of $M^{(\mathbb{N})}$, so D is a CS-module. Obviously there is an infinite sequence of non-isomorphic monomorphisms

$$A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \longrightarrow A_n \xrightarrow{f_n} \cdots$$

where f_n is the identity map on A_n. By Theorem 1 we get a contradiction which proves that M_i is Noetherian for each $i \in I$.

Now we show that M_i is uniserial for each $i \in I$. Let A_1 and A_2 be any nonzero submodules of M_i. Then $A_1 \simeq M_{i_1}$ and $A_2 \simeq M_{i_2}$ for some $i_1, i_2 \in I$. Hence $\operatorname{End} A_1$ and $\operatorname{End} A_2$ are local, and the external direct sum $A_1 \oplus A_2$ is CS. By [KM, Corollary 14] either A_1 can be embedded in A_2 or A_2 can be embedded in A_1. Thus, without loss of generality, we may assume that there is a monomorphism $\varphi: A_1 \longrightarrow A_2$. Since M_i is quasi-injective, φ can be extended to a homomorphism $\psi: M_i \longrightarrow M_i$, and clearly ψ is an isomorphism. But A_1 is quasi-injective, so it is fully invariant in M_i, hence $A_1 = \psi(A_1) = \varphi(A_1) \subseteq A_2$. This proves that M_i is uniserial.

Finally, assume that the family $\{M_i \mid i \in I\}$ contains only a finite number of non-isomorphic members. Suppose that there exists a strictly descending chain of submodules of M_i

$$B_1 \supset B_2 \supset \cdots \supset B_n \supset \cdots.$$

As we have shown above, each B_n is isomorphic to M_{i_n} for some $i_n \in I$. Hence there are positive integers k and l with $k < l$ such that $B_k \simeq B_l$. But this is impossible because $B_k \simeq M_{i_k}$ is quasi-injective. This contradiction shows that M_i must be Artinian. Therefore, we conclude that M_i is of finite length for each i. ■

We are able now to derive Goodearl's characterization of right nonsingular rings for which all nonsingular right modules are projective. Recall that a ring R is called *right (left) serial* if R_R (respectively $_RR$) is a direct sum of uniserial submodules.

COROLLARY 8 (GOODEARL [G1], [G2]). *The following conditions are equivalent for a right nonsingular ring R:*

(a) Every nonsingular right R-module is projective;

(b) R is a (left and right) hereditary serial Artinian ring.

PROOF. (a) $\Rightarrow$ (b). Assume that (a) is satisfied. Then it is easy to see that this condition is equivalent to the property that R_R is a Σ-CS module (see, *e.g.*, [DS, Proposition 1]). By Theorem 7, R is right serial right Artinian. Every right ideal of R is nonsingular, hence projective as a right R-module, thus R is right hereditary. Now we essentially follow Goodearl [G2, Theorem 5.21] to show that every nonsingular left R-module is projective. Let Q be the maximal right quotient ring of R. Then clearly Q is semisimple, and since Q_R is projective, it follows by [C, Theorem 2.3] that Q is also the maximal left quotient ring of R. Let M be any nonsingular left R-module and consider the injective hull $E(M)$ of M. Then $E(M)$ is a flat left Q-module (*cf.* [G2, Theorem 2.2]), and since $_RQ$ is flat (see [G1, Lemma 2.2]), this implies that $E(M)$ is a flat left R-module. But R is right hereditary, so it follows that M is a flat left R-module (see, for example, [W, 39.12]). Because R is right Artinian, R is perfect, hence $_RM$ is projective (see [AF, Theorem 28.4]). Therefore $_RR$ is Σ-CS, and so by Theorem 7, R is left serial left Artinian. Obviously, R is left hereditary.

(b) $\Rightarrow$ (a). Assume that R is (left and right) hereditary serial Artinian, and let M be any nonsingular right R-module. By using the same argument as in (a) $\Rightarrow$ (b), we can show that M_R is flat, hence M_R is projective because R is perfect. ■

We finish with an example to show that the self-generator condition in Theorem 7 can not be replaced by "finitely generated projective". More precisely, we give an example of a uniform nonsingular projective Σ-injective module M which is neither uniserial nor Noetherian nor Artinian.

EXAMPLE 9. Let S denote the ring of 3×3 matrices over $\mathbb{Q}$, the field of rationals, and let R denote the subring of S generated by Se_{11}, $e_{33}S$ and the identity, where e_{11} and e_{33} denote the usual matrix units. Thus R is the lower triangular matrix ring

$$\begin{bmatrix} \mathbb{Q} & 0 & 0 \\ \mathbb{Q} & \mathbb{Z} & 0 \\ \mathbb{Q} & \mathbb{Q} & \mathbb{Q} \end{bmatrix},$$

where the $(2,2)$ entries are integers. The ring R is featured on p. 70 of Tachikawa's monograph [T]. It follows from [T, Theorem 5.4] that R is a left and right QF-3 ring with

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$M = e_{33}S$ as a minimal faithful injective right ideal. Moreover the simple Artinian ring S is the maximal quotient ring of R and so R is a right nonsingular ring of finite right uniform dimension (see, *e.g.*, [S, Chapter XII, Section 2]). Then, by [CH, Lemma 1.14], R satisfies the ascending chain condition on right annihilators. From this it follows by a result of Faith (see, *e.g.*, [AF, Theorem 25.1]) that the injective right R-module M is Σ-injective. Now the proper non-trivial submodules of M are precisely of the form

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \mathbb{Q} & H & 0 \end{bmatrix},$$

where H is a subgroup of the abelian group $\mathbb{Q}$. From this we get that M is uniform but is neither Noetherian nor Artinian nor uniserial.

ACKNOWLEDGMENTS. This paper was written while the second author was visiting the University of Otago. He wishes to thank the University for its generous financial support and the Department of Mathematics and Statistics for its kind hospitality.

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Department of Mathematics and
Statistics
University of Otago
PO Box 56
Dunedin
New Zealand
e-mail: jclark@maths.otago.ac.nz

Institute of Mathematics
P.O. Box 631 Bo Ho
Hanoi
Vietnam
e-mail: nvdung@thevinh.ac.vn