ON THE DEFINITION OF CESÀRO-PERRON INTEGRALS

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(Received February 11, 1959)

1. Introduction. The Cesàro-Perron integral was defined by J. C. Burkill [1]*) using the Cesàro-continuous upper and lower functions.

G. Sunouchi and M. Utagawa [3] proved that the Cesàro-Perron scale of integration can be defined without assuming the Cesàro-continuity of upper and lower functions and that the indefinite integral is Cesàro-continuous.

We denote by CP_0 and CP the Burkill's Cesàro-Perron integral and the generalized Cesàro-Perron integral defined by G. Sunouchi and M. Utagawa respectively. It is clear that CP-integral includes CP_0-integral. But, in this paper, we will prove the equivalence of these integrals by using the Cesàro-Denjoy integral introduced by W. L. C. Sargent [2].

I must express my best thanks to Dr. G. Sunouchi for his suggestions and criticisms.

2. CP_0-integral and CP-integral.

DEFINITION 2. 1. We put

$$C(f, a, b) = \frac{1}{b-a}\int_a^b f(t)\,dt,$$

where the integral is taken in the restricted Denjoy sense.

If $\lim_{h\to 0} C(f, x_0, x_0+h) = f(x_0)$, then $f(x)$ is termed *Cesàro-continuous at* x_0.

If $\overline{CD}\, f(x_0) = \underline{CD}\, f(x_0)$, where

$$\overline{\lim_{h\to 0}} \left\{C(f, x_0, x_0+h) - f(x_0)\right\} \Big/ \frac{1}{2}h = \overline{CD}\, f(x_0)$$

and

$$\underline{\lim_{h\to 0}} \left\{C(f, x_0, x_0+h) - f(x_0)\right\} \Big/ \frac{1}{2}h = \underline{CD}\, f(x_0),$$

then $f(x)$ is called *Cesàro differentiable at* x_0 and we denote the common value by $CD\, f(x_0)$.

DEFINITION 2. 2. $U(x)$ $[L(x)]$ is termed *upper* [*lower*] *function of a measurable* $f(x)$ *in* $[a, b]$, provided that

*) Numbers in brackets refer to the bibliography at the end.

(i) $U(a) = 0 \qquad [L(a) = 0]$,

(ii) $U(x)$ $[L(x)]$ is Cesàro-continuous on $[a, b]$,

(iii) $\underline{CD}\ U(x) > -\infty$ $[\overline{CD}\ L(x) < +\infty]$ at each point x,

(iv) $\underline{CD}\ U(x) \geqq f(x)$ $[\overline{CD}\ L(x) \leqq f(x)]$ at each point x.

DEFINITION 2.3. If $f(x)$ has upper and lower functions in $[a, b]$ and l. u. b. $U(b)$ = g. l. b. $L(b)$, then $f(x)$ is termed *integrable in Cesàro-Perron sense* or *CP_0-integrable.* The common value of the two bounds is called the definite CP_0-integral of $f(x)$ and denote by $(CP_0)\int_a^b f(t)\, dt$.

DEFINITION 2.4. If in the definition 2.2, the condition (ii) is omitted, then the Perron-scale of integration constructed by the Definition 2.3 is called *CP-integral* and its definite on $[a, b]$ is denoted by $(CP)\int_a^b f(t)\, dt$.

The CP-integral has the following properties, cf. [3].

THEOREM 2.1. *The function $U(x) - L(x)$ is increasing and non-negative.*

THEOREM 2.2. *If $f(x)$ is CP-integrable in $[a, b]$, then $f(x)$ is so also in any subinterval.*

THEOREM 2.3. *The indefinite integral $F(x) = (CP)\int_a^x f(t)dt$ is Cesàro-continuous.*

THEOREM 2.4. *The function $F(x)$ is Cesàro differentiable almost everywhere and $CD\,F(x) = f(x)$, a.e.*

3. Cesàro-Denjoy integral.

DEFINITION 3.1. The function $f(x)$ is said to be AC^* on a set E if it is Denjoy-integrable in the restricted sense in an interval containing E, and if to each positive number ε, there corresponds a number δ such that

$$\sum_{r=1}^{n} \sup_{x\in(a_r, b_r)} |\ C(f, a_r, x) - f(a_r)| < \varepsilon, \tag{1}$$

$$\sum_{r=1}^{n} \sup_{x\in(a_r, b_r)} |\ C(f, b_r, x) - f(b_r)| < \varepsilon, \tag{2}$$

for all finite non-overlapping sequence of intervals

$$(a_1, b_1),\ (a_2, b_2),\ \ldots\ldots\ldots\ldots\ldots,\ (a_n, b_n)$$

with end points on E and such that

$$\sum_{r=1}^{n} (b_r - a_r) < \delta. \tag{3}$$

If the inequalities (1) and (2) are replaced by the following conditions respectively

$$\sum_{r=1}^{n} \inf_{x\in(a_r, b_r)} \{C(f,a_r, x) - f(a_r)\} > - \varepsilon, \tag{4}$$

$$\sum_{\gamma=1}^{n} \inf_{x\in(a_r, b_r)} \{f(b_r) - C(f, b_r, x)\} > - \varepsilon, \tag{5}$$

then $f(x)$ is called AC^* *below on* E. There is a corresponding definition of AC^* *above on* E. If the set E is the sum of a countable number of sets E_n on each of which f is AC^* and if f is Cesàro-continuous on E, then f is termed ACG^* on E. cf. [2]

The function $f(x)$ is AC^* on E if and only if $f(x)$ is both AC^* below and AC^* above on E.

DEFINITION 3. 2. The function $f(x)$ defined on $[a, b]$ is called *integrable in the Cesàro-Denjoy sense or CD-integrable* provided that there exsists a function $F(x)$ ACG^* on $[a, b]$ and such that

$$CD\ F(x) = f(x), \text{ a. e.}$$

We call the function $F(x)$ the *indefinite CD-integral* and define the definite CD-integral as $F(b) - F(a)$, cf. [2].

The following results have been proved by Sargent, cf. [2].

THEOREM 3. 1. *If* $\underline{CD}\, f(x) > -\infty$ *at each point of* E, *then* E *is the sum of a countable number of sets on each of which* $f(x)$ *is* AC^* *below.*

THEOREM 3. 2. The CD-*integral is a descriptive definition of the* CP_0-*integral.*

4. Theorem

THEOREM. *The CP-integral is equivalent to the* CP_0*-integral.*

PROOF. Since the CD-integral is equivalent to the CP_0-integral, it is sufficient to prove that the CD-integral includes the CP-integral and that the following equality holds,

$$(CD)\int_a^b f(t)\ dt = (CP)\int_a^b f(t)\, dt. \tag{6}$$

Let $F(x) = (CP)\int_a^x f(t)\, dt$. Then, by Theorems 2. 3 and 2. 4, the function $F(x)$ is Cesàro-continuous on $[a, b]$ and $CD\ F(x) = f(x)$ a. e.

We shall prove that $F(x)$ is ACG^* on $[a, b]$.

For a given $\varepsilon > 0$, we can select the upper and lower functions $U(x)$, $L(x)$ such that

$$U(b) - L(b) \leqq \frac{1}{2}\varepsilon \tag{7}$$

and

$$CD\,U(x) > -\infty \quad (a \leqq x \leqq b). \tag{8}$$

It follows from (8) and Theorem 3. 1 that $[a, b]$ is the sum of a countable number of sets E_n on each of which $U(x)$ is AC^* below. Consequently, for any finite non-overlapping intervals (a_1, b_1), (a_2, b_2),......(a_m, b_m) with end point on E_n and such that

$$\sum_{r=1}^{m} (b_r - a_r) < \delta_n,$$

we have

$$\sum_{r=1}^{m} \inf \{C(U, a_r, x)\} > -\frac{\varepsilon}{2} \tag{9}$$

and

$$\sum_{r=1}^{m} \inf \{U(b_r) - C(U, b_r, x)\} > -\frac{\varepsilon}{2} \tag{10}$$

Suppose that $a_r < x < b_r$. Then it follows that

$$\begin{aligned} C(F, a_r, x) - F(a_r) &= C(U, a_r, x) - U(a_r) - \frac{1}{x - a_r}\int_{a_r}^{x}[U(t) - F(t)]\,dt \\ &\quad + \{U(a_r) - F(a_r)\} \\ &\geqq C(U, a_r, x) - U(a_r) - \{U(b_r) - F(b_r)\} \\ &\quad + \{U(a_r) - F(a_r)\}, \end{aligned}$$

since $U(x) - F(x)$ is increasing and non-negative by Theorem 2. 1. Therefore, we obtain from (7) and (9)

$$\sum_{r=1}^{m} \inf \{C(F, a_r, x) - F(a_r)\} \geqq$$

$$\sum_{r=1}^{m} \inf \{C(U, a_r, x) - U(a_r)\} - \{U(b) - F(b)\} > -\varepsilon.$$

Similarly, we have from (7) and (10)

$$\sum_{r=1}^{m} \inf \{F(b_r) - C(F, b_r, x)\} > -\varepsilon.$$

Hence the function $F(x)$ is AC^* below on E_n.

Since $-f(x)$ is CP-integrable and its indefinite integral is $-F$, the interval $[a, b]$ is the sum of a countable number of sets E'_m on each of which $-F$ is AC^* below. Therefore F is AC above on E'_m and is AC^* on $E_n \cap E'_m$.

Since F is Cesàro-continuos on $[a, b]$ and $[a, b] = \sum_m \sum_n E_n \cap E'_m$, the function $F(x)$ is ACG^* on $[a, b]$. Thus, f is CD-integrable on $[a, b]$ and

$$(CD)\int_a^b f(t)dt = F(b) - F(a) = (CP)\int_a^b f(t)\ dt.$$

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