ON THE DEGREE OF VARIATION IN CONFORMAL MAPPING OF VARIABLE REGIONS(1)

BY

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1. **Introduction.** Suppose C is a closed Jordan curve and the function $w=f(z)$ maps the circle $|z|<1$ conformally onto the interior R of C; $f(z)$ is analytic in $|z|<1$ and continuous in $|z|\leqq 1$. It is well known that the function $f(z)$ varies continuously in the closed circle $|z|\leqq 1$ under a suitable continuous deformation(2) of C. It is of interest, however, to go beyond this merely "qualitative" statement and to estimate the *degree of variation* of $f(z)$ in this dependence upon the change in C. If C_1 is a "neighboring" closed Jordan curve and if $w=f_1(z)$, normalized in the same manner as $f(z)$, maps $|z|<1$ conformally onto the interior of C, it is desired to find an upper bound for $|f(z)-f_1(z)|$ for $|z|\leqq 1$ which measures the effect of the deformation of C.

This problem has been treated with some degree of completeness in the case of "nearly circular" regions, that is, in the special case in which C is a circle. The principal contributions here were made by L. Bieberbach [1], 1924, A. R. Marchenko [10], 1935, and Jacqueline Ferrand [4], 1945. Marchenko's and Ferrand's estimates are, in a certain sense, best possible results.

The general problem of two arbitrary regions presents a more diversified aspect because of the various degrees of smoothness which may be imposed upon one or both of the boundary curves. Under suitable differentiability assumptions regarding the curve C one can reduce the problem to the "nearly circular" case by a conformal transformation. In this way A. Markoushevitch [11], 1936, extended Marchenko's result to the more general configuration of two Jordan curves. A different approach is used by J. Ferrand in [4], where a theorem on "nearly polygonal" curves(3) is presented. The method indicated here is based on appraisals of the change of the harmonic measure

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(2) See, for example, Gattegno and Ostrowski [5, section 14]. Numbers in brackets refer to the bibliography at the end of the paper.

(3) That is, one of the curves lies between two simple closed polygons, one of which is obtained from the other by dilation from an interior point, the ratio being a number close to 1, and the other curve is a polygon.

of a side of a polygon under certain deformations (displacement of the sides) of the polygon.

The main object of the present paper is the study of the problem in more general cases in which either no restrictions or only weak conditions are placed on the boundary; in particular, the regions considered *need not be bounded by Jordan curves*. The principal tools in this paper are: (i) estimates for the oscillation of the mapping function of the unit circle onto a bounded region, which are valid in the neighborhood of the boundary (see §2); (ii) some auxiliary theorems, in which the mapping function of two regions, one of which is contained in the other, are compared with each other in any circle $|z| \leqq \rho < 1$ (see Lemmas 3 and 6).

As a first result, Marchenko's theorem is extended to a "nearly circular" region which is not necessarily bounded by a Jordan curve (§3). §§4 and 5 deal with the general situation of two arbitrary regions. §6 contains a result on regions bounded by Jordan curves with continuously turning tangents which is easily obtained by the method developed in this paper.

2. **The oscillation of the mapping function at the boundary.** Suppose that R is a simply connected bounded region and that the function $w=f(z)$ maps the circle $|z| < 1$ conformally onto R. Let $|z_0| = 1$. Then we define for $0 < r < 1$

$$\omega(r; z_0) = \sup_{|z_k - z_0| \leqq r} |f(z_1) - f(z_2)| \qquad (|z_1| < 1, \ |z_2| < 1)$$

and

$$\omega(r) = \sup_{|z_0|=1} \omega(r; z_0).$$

We call $\omega(r)$ the *oscillation of* $f(z)$ *at the boundary*, and we are interested in obtaining estimates for $\omega(r)$. Related questions were investigated by J. Wolff [16, pp. 217–218], J. Ferrand [3, pp. 150–154], and M. Lavrientieff [7](4). Their results, however, do not entail direct estimates for $\omega(r)$. In order to obtain such estimates we shall introduce in the following a function $\eta(\delta)$, associated with the boundary of R, in terms of which we shall express our bounds for $\omega(r)$. We shall establish two theorems (Theorems I and II), and in the proof of the first of these we shall make use of the above mentioned result by Wolff and Ferrand (Lemma 1).

(4) J. Ferrand (1942) proved the following: If z_1 and z_2 are points in $\{|z-z_0| < r, |z| < 1\}$, $|z_0| = 1$, $0 < r < 1$, and if $f(0) = w_0$, then $w_1 = f(z_1)$ and $w_2 = f(z_2)$ may be separated from w_0 by a cross-cut in R of length $l \leqq (2\pi A/\log(1/r))^{1/2}$, where A is the area of R. The earlier result of J. Wolff (1934) is similar but not stated quite in this way (see Lemma 1 below). Lavrientieff introduced, for w_1, w_2 in R, a "distance $\rho[w_1, w_2]$ with respect to R and w_0" ($w_0 = f(z_0)$) and showed that for $|z_1| < 1$, $|z_2| < 1$, $\rho[f(z_1); f(z_2)] \leqq \text{const.}/(|\log|z_1 - z_2||)^{1/2}$. The "distance" $\rho[w_1, w_2]$ is defined as min (ρ_1, ρ_2) where ρ_1 is the greatest lower bound of the lengths of all arcs in R, which connect w_1 and w_2, and ρ_2 is the greatest lower bound of the lengths of all cross-cuts of R which separate w_1 and w_2 from w_0.

Several estimates have been given in the literature for the oscillation of the inverse function(5), that is, the function which maps R onto $|z|<1$. We shall make use of one of these results (see Theorem III).

2.1. *Estimate of $\omega(r)$ in the general case.* We begin with the following definition.

DEFINITION. Suppose R is a simply connected bounded region which contains the origin O. Let c be a cross-cut of R which does not pass through O, and let T be the one of the two subregions of R which does not contain O. Denote by λ the diameter of c and by Λ the diameter of T. For any $\delta>0$ consider all possible cross-cuts c of R with $\lambda \leqq \delta$ and define

$$\eta(\delta) = \sup_{|\lambda| \leqq \delta} \Lambda.$$

The function $\eta(\delta)$ is in a certain sense a measure for the "irregularity" of the boundary of R. If the boundary of R is a simple closed curve, then it is easily seen that $\lim_{\delta\to 0} \eta(\delta)=0$. The converse, however, is not necessarily true (consider, for example, the region obtained by removing one radius from the interior of the unit circle). We shall call $\eta(\delta)$ the *structure modulus of the boundary of R.*

THEOREM I. *Suppose that R is a simply connected bounded region which contains the origin and that $w=f(z)$ maps the circle $|z|<1$ conformally onto R such that $f(0)=0$. If A denotes the area of R, then the oscillation of $f(z)$ at the boundary,*

$$\omega(r) \leqq \eta\left(\left(\frac{2\pi A}{\log 1/r}\right)^{1/2}\right), \qquad 0<r<1. \tag{2.11}$$

This theorem is easily proved by means of the following lemma(6).

LEMMA 1. *Suppose that the circle $|z|<1$ is mapped conformally onto a simply connected region R of finite area A. Let z_0 be a point on $|z|=1$ and k_r the part of the circle $|z-z_0|=r$ which is contained in $|z|<1$. Then for every r, $0<r<1$, there exists a ρ_1, $r\leqq\rho_1\leqq r^{1/2}$, such that the image of k_{ρ_1} is a cross-cut of R of length*

$$l_{\rho_1} \leqq \left(\frac{2\pi A}{\log 1/r}\right)^{1/2}. \tag{2.12}$$

Proof. Let $w=f(z)$ map $|z|<1$ conformally onto R. We introduce polar coordinates about z_0 and write, for $0<\rho<1$,

$$l_\rho = \int_{k_\rho} |f'(z)|\,|dz| = \int_{k_\rho} |f'(z_0+\rho e^{i\theta})|\,\rho d\theta.$$

(5) E. Lindelöf [9, 1915, pp. 15–18], M. Lavrientieff [7, 1936, formula (a)], J. Ferrand [3, 1942, pp. 165–171].

(6) This lemma is essentially due to J. Wolff [16, pp. 217–218].

Here $l_\rho \leqq +\infty$. By the inequality of Schwarz:

$$(2.13)\quad l_\rho^2 \leqq \int_{k_\rho} |f'(z_0+\rho e^{i\theta})|^2 \rho d\theta \int_{k_\rho} \rho d\theta \leqq \pi\rho \int_{k_\rho} |f'(z_0+\rho e^{i\theta})|^2 \rho d\theta.$$

Let $0<r<1$. Integrating with respect to ρ from r to $r^{1/2}$ we obtain

$$\int_r^{r^{1/2}} \frac{l_\rho^2}{\rho} d\rho < \int_0^{r^{1/2}} \frac{l_\rho^2}{\rho} d\rho \leqq \pi \int_0^{r^{1/2}} \int_{k_\rho} |f'(z_0+\rho e^{i\theta})|^2 \rho d\theta < \pi A.$$

Hence there exists a ρ_1, $r \leqq \rho_1 \leqq r^{1/2}$, such that

$$l_{\rho_1}^2 \int_r^{r^{1/2}} \frac{d\rho}{\rho} = \frac{1}{2} l_{\rho_1}^2 \log \frac{1}{r} < \pi A.$$

Since the image of k_{ρ_1} has finite length l_{ρ_1} it is easily seen that it forms a cross-cut of R.

Proof of Theorem I. Let T_r denote the image of the region $\{|z-z_0| < r, |z|<1\}$ in the w-plane, and let ρ_1 be determined by Lemma 1 so that (2.12) holds, $r \leqq \rho_1 \leqq r^{1/2}$. Then T_r is contained in T_{ρ_1}. Now k_{ρ_1} (Lemma 1) is mapped onto cross-cut c_{ρ_1} of R whose diameter is not greater than l_{ρ_1}. Since T_{ρ_1} does not contain the origin, it follows from the definition of $\eta(\delta)$ that the diameter of T_{ρ_1} and hence that of T_r does not exceed $\eta(l_{\rho_1}) \leqq \eta((2\pi A/\log(1/r))^{1/2})$.

Now if z_1 and z_2 are points of $|z|<1$ which are in $|z-z_0|<r$, then $f(z_1)$ and $f(z_2)$ are in T_r, and therefore $|f(z_1)-f(z_2)| \leqq \eta(l_{\rho_1})$. This proves (2.11).

2.2. *Estimates of $\omega(r)$ for linear $\eta(\delta)$.* A more accurate estimate for $\omega(r)$ may be obtained if some information regarding the order of magnitude of $\eta(\delta)$ is available. A particularly simple form is

$$(2.21)\qquad \eta(\delta) \leqq \kappa\delta + \eta_0 \qquad \text{for } \delta \leqq \delta_0,$$

for some $\delta_0 > 0$; here κ and η_0 are constants(7), $\kappa>0$, $\eta_0 \geqq 0$. For this case we prove the following theorem.

THEOREM II. *Suppose that R, $w=f(z)$, and $\omega(r)$ are defined as in Theorem I. Suppose furthermore that the structure modulus of the boundary of R satisfies* (2.21) *and that D is the diameter of R. Then for every $\mu \geqq 1$,*

$$(2.22)\qquad \omega(r) \leqq mr^\alpha + \left(1+\frac{2}{\mu^{1/2}}\right)\eta_0 \qquad \left(\alpha = \frac{2}{\pi^2\kappa^2}\right),$$

where m is a constant which depends only on κ, μ, δ_0, and D. In fact, if $h = \min(\delta_0, D/\kappa)$ and $\rho_0 = \exp[-\pi^2 D^2/2\delta_0^2]$, one may take

(7) The case $\eta_0=0$ occurs, for example, when the boundary of R consists of a finite number of Jordan arcs which possess continuously turning tangents and which form a finite number of corners (that is, angles different from 0 and 2π), such as the interior of the unit circle with a finite number of slits: $\arg w=\theta_i$, $1/2 \leqq |w| \leqq 1$, $i=1, 2, \cdots, n$.

$$m = \begin{cases} 2he\kappa\rho_0^{-\alpha}\left(\dfrac{e^\mu}{\mu}\right)^{1/2}, & \textit{when} \quad 0 < r \leqq r_0 = \rho_0 \exp\left[-\dfrac{\pi^2\kappa^2}{4}\mu\right] \\ D\rho_0^{-\alpha}(e^\mu)^{1/2}, & \textit{when} \quad r_0 < r < 1(^8). \end{cases}$$

REMARKS. 1. *The value of m is least when $\mu = 1$.*

2. *One has, from* (2.22), $\lim_{r\to 0}\omega(r) \leqq (1 + 2/\mu^{1/2})\eta_0$ *and since μ is arbitrary one may let $\mu \to \infty$ and one obtains* $\lim_{r\to 0}\omega(r) \leqq \eta_0$.

In the proof of this theorem we shall use the following lemma.

LEMMA 2. *Suppose that the hypotheses of Theorem* II *are satisfied. Let z_0 be a fixed point, $|z_0| = 1$, and let T_ρ denote the image of the region $\{|z - z_0| < \rho, |z| < 1\}$ under the transformation $w = f(z)$. Let Λ_ρ be the diameter of T_ρ and l_ρ the length of the image of the circular arc k_ρ: $\{|z - z_0| = \rho, |z| < 1\}$. Then for $\rho \leqq \rho_0 = \exp[-\pi^2 D^2/2\delta_0^2]$*

$$\Lambda_\rho \leqq \kappa l_\rho + \eta_0(^9). \tag{2.23}$$

Proof. If $l_\rho \leqq \delta_0$, then (2.23) follows from the hypothesis (2.21). If, however, $l_\rho > \delta_0$, then an application of Lemma 1 shows that for every ρ there exists a ρ_1, such that $\rho \leqq \rho_1 \leqq \rho^{1/2}$ and (A is the area of R)

$$l_{\rho_1} \leqq \left(\frac{2\pi A}{\log 1/\rho}\right)^{1/2} \leqq \left(\frac{\pi^2 D^2}{2\log 1/\rho_0}\right)^{1/2} = \delta_0.$$

Hence

$$\Lambda_{\rho_1} \leqq \kappa l_{\rho_1} + \eta_0.$$

But since $T_\rho \subset T_{\rho_1}$, we have $\Lambda_\rho \leqq \Lambda_{\rho_1}$ and on the other hand $l_{\rho_1} \leqq \delta_0 < l_\rho$. Hence

$$\Lambda_\rho \leqq \Lambda_{\rho_1} \leqq \kappa l_{\rho_1} + \eta_0 \leqq \kappa l_\rho + \eta_0.$$

Proof of Theorem II. We use the same notation as in Lemma 1, and repeat the argument leading to inequality (2.13), or

$$\frac{l_\rho^2}{\rho} \leqq \pi \int_{k_\rho} |f'(z)|^2 \rho d\theta \qquad (z = z_0 + \rho e^{i\theta}).$$

Let

$$g(x) = \int_0^x \frac{l_\rho^2}{\rho} d\rho.$$

Then we have

(8) By the method of Theorem II one may obtain bounds for $\omega(r)$ (better than those resulting from Theorem I) for other choices of $\eta(\delta)$, for example, $\eta(\delta) = \delta^\beta$, $0 < \beta < 1$.

(9) If $\delta_0 \geqq D$, then we may let $\delta_0 \to \infty$ (since $\eta(\delta) = \eta(D)$ for $\delta \geqq D$) and take $\rho_0 = 1$. Then Lemma 2 is trivial and the proof of Theorem II is somewhat simplified.

$$g(x) \leqq \pi \int_0^x \int_{k_\rho} |f'(z)|^2 \rho d\theta d\rho.$$

The integral on the right (apart from the factor π) represents the area of T_x, the image of the region $\{|z-z_0|<x, |z|<1\}$ in the w-plane, and it does not exceed therefore the value $\pi(\Lambda_x/2)^2$, where Λ_x is the diameter of T_x. For $x \leqq \rho_0$ we have, by Lemma 2, $\Lambda_x \leqq \kappa l_x + \eta_0$, and therefore

$$g(x) \leqq \frac{\pi^2}{4} (\kappa l_x + \eta_0)^2, \qquad x \leqq \rho_0.$$

The function $g(x)$ is positive, monotone increasing, and continuous for $0 \leqq x \leqq 1$, and, for almost all these x: $l_x^2 = xg'(x)$. Hence

$$g(x) \leqq \frac{\pi^2}{4} (\kappa (xg'(x))^{1/2} + \eta_0)^2.$$

If we set

$$\kappa\eta_1 = \eta_0, \qquad \gamma = \frac{2}{\pi\kappa}, \tag{2.24}$$

then we may write this inequality in the form

$$\gamma(g(x))^{1/2} \leqq (xg'(x))^{1/2} + \eta_1. \tag{2.25}$$

Suppose that there exists a number ξ, $0 \leqq \xi < \rho_0$, such that

(2.26) $\qquad \gamma(g(x))^{1/2} \geqq 2\eta_1 \qquad$ for $x \geqq \xi$ (ξ is the *least* such number).

Then we have, from (2.25), $(\gamma(g(x))^{1/2} - \eta_1)^2 \leqq xg'(x)$, $x \geqq \xi$, or

$$\frac{1}{x} \leqq \frac{g'(x)}{(\gamma(g(x))^{1/2} - \eta_1)^2}.$$

Integration over the interval $\rho \leqq x \leqq \rho_0$, where $\rho \geqq \xi$, yields

$$\log \frac{\rho_0}{\rho} \leqq \int_\rho^{\rho_0} \frac{g'(x)dx}{(\gamma(g(x))^{1/2} - \eta_1)^2}$$

or

$$\frac{\gamma^2}{2} \log \frac{\rho_0}{\rho} \leqq \log \frac{\gamma(g(\rho_0))^{1/2} - \eta_1}{\gamma(g(\rho))^{1/2} - \eta_1} + \eta_1 \left[\frac{1}{\gamma(g(\rho))^{1/2} - \eta_1} - \frac{1}{\gamma(g(\rho_0))^{1/2} - \eta_1} \right].$$

Because of (2.26), the second term on the right does not exceed 1. Hence

$$\left(\frac{\rho_0}{\rho}\right)^{\gamma^2/2} \leqq \frac{\gamma(g(\rho_0))^{1/2} - \eta_1}{\gamma(g(\rho))^{1/2} - \eta_1} e. \tag{2.27}$$

The assumption (2.26) implies that

$$\gamma(g(\rho))^{1/2} - \eta_1 \geqq \frac{\gamma}{2}(g(\rho))^{1/2}, \qquad \rho \geqq \xi. \tag{2.28}$$

Furthermore, by Lemma 2,

$$(g(\rho_0))^{1/2} \leqq \frac{\pi}{2}\Lambda_{\rho_0} \leqq \frac{1}{\gamma}(l_{\rho_0} + \eta_1).$$

If $l_{\rho_0} \leqq \delta_0$, we have

$$\gamma(g(\rho_0))^{1/2} - \eta_1 \leqq \delta_0. \tag{2.29}$$

However, if $l_{\rho_0} > \delta_0$, then there exists, by Lemma 1, a number $\rho_1 \geqq \rho_0$ such that $l_{\rho_1} \leqq \delta_0$. Then we have

$$(g(\rho_0))^{1/2} \leqq (g(\rho_1))^{1/2} \leqq \frac{\pi}{2}\Lambda_{\rho_1} \leqq \frac{1}{\gamma}(l_{\rho_1} + \eta_1) \leqq \frac{1}{\gamma}(\delta_0 + \eta_1),$$

by hypothesis (2.23), and (2.29) is again true.

Thus we find from (2.27) by use of (2.28) and (2.29)

$$\gamma(g(\rho))^{1/2} \leqq 2e\delta_0\left(\frac{\rho}{\rho_0}\right)^{\gamma^2/2}, \qquad \xi \leqq \rho \leqq \rho_0. \tag{2.210}$$

We also note: it follows from (2.27), (2.28), and the inequality $\gamma(g(\rho_0))^{1/2} \leqq \gamma \cdot (\pi/2)D = D/\kappa$ that we may replace, in (2.210), $2e\delta_0$ by $2eD/\kappa$. Thus, we may write, in any case, $2eh$ in place of $2e\delta_0$, where $h = \min(\delta_0, D/\kappa)$.—The inequality (2.210) presupposes $\rho \geqq \xi$. If $\rho < \xi$ then

$$(g(\rho))^{1/2} < \frac{2\eta_1}{\gamma}. \tag{2.211}$$

Hence for every ρ, $0 < \rho \leqq \rho_0$,

$$(g(\rho))^{1/2} \leqq \frac{2eh}{\gamma}\left(\frac{\rho}{\rho_0}\right)^{\gamma^2/2} + \frac{2\eta_1}{\gamma}. \tag{2.212}$$

This inequality has been derived under the assumption that there exists a ξ such that (2.26) holds. However, if there is no such ξ then (2.211) is true for all ρ, $0 < \rho \leqq \rho_0$, and the inequality (2.212) is again valid for all these ρ.

Now let $p > 1$. Then

$$\int_{\rho/p}^{\rho} \frac{l_x^2}{x}\,dx < g(\rho)$$

and hence, for a suitable ρ_1, $\rho/p \leqq \rho_1 \leqq \rho$,

$$l_{\rho_1}^2 \log p \leqq g(\rho).$$

We obtain therefore from (2.212), substituting the values of γ and η_1 from (2.24),

$$l_{\rho_1} \leqq \frac{eh\kappa\pi}{(\log p)^{1/2}}\left(\frac{\rho}{\rho_0}\right)^{\gamma^2/2} + \frac{\pi\eta_0}{(\log p)^{1/2}}.$$

We write now $r=\rho/p$ and take $p=e^{\mu/\gamma^2}$ where μ is not less than 1 and a constant. Then

$$l_{\rho_1} \leqq 2eh\left(\frac{e^{\mu}}{\mu}\right)^{1/2}\left(\frac{r}{\rho_0}\right)^{\gamma^2/2} + \frac{2\eta_0}{\kappa\mu^{1/2}}.$$

Since $\rho_1 \leqq \rho_0$ we have, by Lemma 2, $\Lambda_{\rho_1} \leqq \kappa l_{\rho_1} + \eta_0$. Furthermore, since $r \leqq \rho_1$, it follows that $\omega(r) \leqq \Lambda_r \leqq \Lambda_{\rho_1}$. Thus, for $r \leqq \rho_0 e^{-\mu/\gamma^2}$,

$$\omega(r) \leqq \kappa l_{\rho_1} + \eta_0 \leqq 2eh\kappa\left(\frac{e^{\mu}}{\mu}\right)^{1/2}\left(\frac{r}{\rho_0}\right)^{\gamma^2/2} + \eta_0\left(1+\frac{2}{\mu^{1/2}}\right). \tag{2.213}$$

This proves (2.22) with the value of m as given for $r \leqq r_0$. If $r > r_0$ the value of m is so chosen that $mr^{\alpha} \geqq D$ and (2.22) is obviously true.

2.3. *Oscillation of the inverse mapping function.* In §5 we shall need an estimate for the oscillation of the inverse mapping function. We shall use the following theorem(10).

THEOREM III. *Suppose that R is a simply connected bounded region which contains the origin O and that σ is the distance of O from the boundary B of R. Let $z=\phi(w)$ map R conformally onto the circle $|z|<1$ so that $\phi(0)=0$. If w_1 and w_2 are points in R which are separated from O by a circular cross-cut of R of radius r, $0<r<\sigma/2$, whose center lies on B, then*

$$|\phi(w_1) - \phi(w_2)| \leqq Mr^{1/2}.$$

Here M is a constant which depends only on σ and the diameter of R.

3. **Application to nearly circular regions.** In 1935 A. R. Marchenko [10] established the following theorem:

Suppose C is a closed Jordan curve which lies in the ring $1 \leqq |w| \leqq 1+\epsilon$ for some ϵ, $0<\epsilon<1$. Consider any arc of C which subtends a chord whose length does not exceed ϵ. Let λ be the least upper bound of the diameters of all such arcs, whereby, in each case, the arc with the smaller diameter is chosen. If $w=f(z)$ maps the circle $|z|<1$ conformally onto the interior of C such that $f(0)=0$ and $f'(0)>0$, then there exist two absolute constants K and K_1 such that for $|z| \leqq 1$:

(10) See J. Ferrand [3 (1942) pp. 166–171]. A result which contains this theorem was given earlier (1936) by Lavrientieff [7, formula (a)].

$$\left|f(z)-z\right| \leqq K\epsilon \log \frac{1}{\epsilon} + K_1\lambda.$$

This inequality is the best possible as to the order of magnitude([11]) in ϵ and in λ.

As an application of Theorem II we shall extend this theorem to the case of a "nearly circular" region whose boundary is not necessarily a Jordan curve.

3.1. *Statement of the result.* We shall prove the following

THEOREM IV. *Suppose R is a simply connected region which contains the origin and whose boundary is contained in the ring*

$$1 < |w| < 1 + \epsilon$$

for some ϵ, $0<\epsilon<\log(8/\pi)$. Let λ be a number with the following property: Any two points in R whose distance is less than ϵ may be connected by an arc in R whose diameter does not exceed λ. If $w=f(z)$ maps the circle $|z|<1$ conformally onto R such that $f(0)=0, f'(0)>0$, then

$$\left|f(z)-z\right| \leqq \pi\epsilon \log \frac{1}{\epsilon} + \epsilon k(\epsilon) + 2\lambda\left(1 + a\left(\frac{\epsilon}{\lambda}\right)^{1/3}\right), \tag{3.11}$$

where $a=(e^{\epsilon}\pi)^{1/3}$ and $k(\epsilon)$ is bounded([12]) *for $0<\epsilon<1$.*

Since $\epsilon \leqq \lambda$ and $k(\epsilon)$ is bounded for $0<\epsilon<1$, the right-hand side of (3.11) is clearly of the form $K\epsilon \log(1/\epsilon)+K_1\lambda$([13]).

3.2. *Two lemmas.* The following lemma will be used in the proofs of several theorems in this paper.

LEMMA 3. *Suppose that H is a simply connected region which contains the origin and whose boundary lies in the ring*

$$\frac{1}{1+\epsilon} \leqq |w| \leqq 1+\epsilon \qquad (\epsilon > 0). \tag{3.21}$$

If $h(z)$ maps $|z|<1$ conformally onto H such that $h(0)=0$, $h'(0)>0$, then for $|z|\leqq\rho<1$

$$\left|h(z)-z\right| \leqq \rho\left(1+\frac{2}{\pi}\log\frac{1+\rho}{1-\rho}\right)\epsilon e^{\epsilon}. \tag{3.22}$$

([11]) In [4], 1945, J. Ferrand announced a new proof of Marchenko's Theorem according to which $2/\pi$ is the best possible value for K.

([12]) In fact $k(\epsilon)=1+\epsilon^3+e^{\epsilon}+\pi e^{\epsilon}(1+\log 4+4(1+\epsilon)^2+\epsilon \log 1/\epsilon)$.

([13]) The factor π of $\epsilon \log(1/\epsilon)$ is larger than the value announced by J. Ferrand for Marchenko's theorem. However, the factor of 2λ is "asymptotically" (as $\epsilon\to 0$) the best possible in the sense that it approaches 1 when $\epsilon/\lambda\to 0$; simple examples show that it cannot be less than 1.

Proof. The function $h(z)/z$ is regular for $|z|<1$ if defined as $f'(0)$ for $z=0$. Since the boundary of H lies in (3.21),

$$\frac{1}{1+\epsilon} \leqq \left|\frac{h(z)}{z}\right| \leqq 1+\epsilon \qquad (|z|<1)$$

and therefore

$$\left|\log\left|\frac{h(z)}{z}\right|\right| \leqq \epsilon.$$

The branch of $\log (h(z)/z) = \log (h(z)/z) + i \arg (h(z)/z)$ for which $\arg h'(0)=0$ is single-valued and regular for $|z|<1$. Hence by an inequality of Carathéodory (see, for example, [2, p. 45, (76.3)]):

$$\left|\log \frac{h(z)}{z}\right| \leqq \epsilon\left(1+\frac{2}{\pi}\log\frac{1+\rho}{1-\rho}\right) \qquad \text{for } |z| \leqq \rho < 1.$$

Since for any complex a

$$|e^a - 1| \leqq |a| e^{|\Re a|},$$

we have (taking $a=\log (h(z)/z)$ the inequality (3.22).

For the proof of Theorem IV we shall need one more lemma.

LEMMA 4. *If R is a region which satisfies the hypotheses of Theorem* IV, *then the structure modulus of the boundary of R,*

$$\eta(\delta) \leqq \delta + 2\lambda \qquad \text{for } \delta < 1.$$

Proof. Let c be a cross-cut of R whose diameter $\delta<1$. Then c does not pass through the origin O; c decomposes R into two subregions and we denote by T the one which does not contain O. The arc c may or may not intersect the circle $|w|=1$. If it does, let γ denote the set of all arcs of $|w|=1$ which are in T and whose end points are points of c. Since $\delta<1$, the set γ is contained in an arc of the unit circle which is less than π. Let Γ be the union of c and γ, $\Gamma=c+\gamma$. *Then the diameter of* Γ *is equal to* δ. To see this consider two points w and w' on Γ. If both are on c, then clearly $|w-w'|\leqq\delta$; if both are on γ, then $|w-w'|$ is smaller than the distance of suitable end points of the two arcs on which w and w' lie, and since these end points are on c, we have again $|w-w'|\leqq\delta$. Suppose now that w is on c, w' on γ; let a, b be the end points of the arc of γ on which w' lies. Then the circle about w with radius r equal to the larger of the two numbers $|w-a|$ and $|w-b|$ must contain the (open) arc $[ab]$ ($<\pi$) of the unit circle, as is easily seen from the fact that $r\leqq\delta<1$ and the radius of $[ab]$ is equal to one. Hence again $|w-w'|\leqq\delta$. Thus the diameter of Γ does not exceed δ; but since Γ contains c as subset, it must be equal to δ.

Now the proof of the lemma is easily completed. Let w_0 be a boundary point of T *not* on c. Then there is, in any neighborhood of w_0 a point w_1 in T such that $1<|w_1|<1+\epsilon$. Draw the radius of the unit circle through w_1 and denote by w_2 its intersection with the circle. Since $|w_1-w_2|<\epsilon$ it is possible, by hypothesis, to connect w_1 and w_2 by an arc β in R whose diameter does not exceed λ. This arc may or may not intersect c. If it does intersect c at a point w_3, say, then $|w_1-w_3|\leqq\lambda$. If β does not intersect c, then w_2 is a point of γ, and $|w_1-w_2|\leqq\epsilon\leqq\lambda$. Thus, it is always possible to find a point ω_0 of Γ (and ω_0 is either w_2 or w_3) such that

$$|w_1-\omega_0|\leqq\lambda.$$

If w_0' is another boundary point of T, not on c, then there exists in every neighborhood of w_0' a point w_1' in T and a point ω_0' on Γ such that

$$|w_1'-\omega_0'|\leqq\lambda.$$

Hence $|w_1-w_1'|\leqq|\omega_0-\omega_0'|+2\lambda\leqq\delta+2\lambda$ and therefore also

$$|w_0-w_0'|\leqq\delta+2\lambda.$$

If w_0' is a point on c, then

$$|w_1-w_0'|\leqq|w_1-\omega_0|+|\omega_0-w_0'|\leqq\lambda+\delta,$$

and hence

$$|w_0-w_0'|\leqq\lambda+\delta.$$

Finally, if w_0 and w_0' are points of c then clearly $|w_0-w_0'|\leqq\delta$. Thus, the diameter of the boundary of T and hence that of T itself does not exceed $\delta+2\lambda$. This completes the proof.

3.3. *Proof of Theorem* IV. We apply Theorem II to the function $w=f(z)$ of Theorem IV. By Lemma 4, we may take $\kappa=1$, $\delta_0=1$, $\eta_0=2\lambda$; furthermore $D\leqq2(1+\epsilon)$, $\alpha=2/\pi^2$. Since $2e>D$, we have for $0<r<1$:

$$m\leqq 2e\,e^{4(1+\epsilon)^2}(e^{\mu})^{1/2}=2\exp\left[1+4(1+\epsilon)^2+\frac{\mu}{2}\right]. \tag{3.31}$$

Let $z=\rho e^{i\theta}$, $z_1=\rho_1e^{i\theta}$, $r=1-\rho_1$. We choose r so that $mr^{\alpha}=\epsilon$, or

$$r=1-\rho_1=\left(\frac{\epsilon}{m}\right)^{1/\alpha}<\epsilon^4. \tag{3.32}$$

Then we have by Theorem II, if $\rho_1\leqq\rho<1$:

$$|f(z)-f(z_1)|\leqq\epsilon+\left(1+\frac{2}{\mu^{1/2}}\right)2\lambda. \tag{3.33}$$

Now

$$(3.34)\qquad |f(z)-z| \leqq |f(z)-f(z_1)| + |f(z_1)-z_1| + |z_1-z|.$$

Here $|z-z_1| \leqq 1-\rho_1$. Furthermore, by Lemma 3,

$$|f(z_1)-z_1| \leqq \epsilon e^{\epsilon}\left(1+\frac{2}{\pi}\log\frac{1+\rho_1}{1-\rho_1}\right) \leqq \epsilon e^{\epsilon}\left(1+\frac{2}{\pi}\log\frac{2m^{1/\alpha}}{\epsilon^{1/\alpha}}\right)$$

$$\leqq \epsilon e^{\epsilon} + \pi\epsilon e^{\epsilon}\log\frac{1}{\epsilon} + \pi\epsilon e^{\epsilon}\log(2m).$$

Since $e^{\epsilon} \leqq 1+\epsilon e^{\epsilon}$ we have, using (3.31),

$$(3.35)\qquad \begin{aligned} |f(z_1)-z_1| \leqq{}& \epsilon\pi\log\frac{1}{\epsilon} + \epsilon^2\pi e^{\epsilon}\log\frac{1}{\epsilon} + \epsilon e^{\epsilon} \\ &+ \epsilon e^{\epsilon}\pi\left(\log 4 + 1 + 4(1+\epsilon)^2 + \frac{\mu}{2}\right). \end{aligned}$$

Thus we find from (3.34) by use of (3.33), (3.35), and (3.32)

$$(3.36)\qquad |f(z)-z| \leqq \epsilon\pi\log\frac{1}{\epsilon} + \epsilon k(\epsilon) + \frac{\epsilon e^{\epsilon}\pi\mu}{2} + \left(1+\frac{2}{\mu^{1/2}}\right)2\lambda,$$

where $k(\epsilon) = 1+\epsilon^3+e^{\epsilon}+\pi e^{\epsilon}(1+\log 4+4(1+\epsilon)^2+\epsilon\log(1/\epsilon))$. The sum of the two terms involving μ will be least if $\epsilon e^{\epsilon}\pi\mu/2 = (2/\mu^{1/2})2\lambda$, which gives $\mu = ((4/e^{\epsilon}\pi)\cdot(2\lambda/\epsilon))^{2/3} \geqq 1$. Substituting this value for μ into the right-hand side of (3.36) we obtain (3.11). This proves the theorem for $\rho \leqq |z| < 1$, and by the principle of the maximum modulus it is therefore true for $|z| < 1$.

4. **Arbitrary regions.** We consider now the general case in which the mapping functions of two arbitrary regions are compared with each other.

4.1. *Statement and discussion of results.* Let R_1 and R_2 be two simply connected bounded regions and let B_1 and B_2 denote their boundaries. We define first the "*inner distance*" $D_i(B_1, B_2)$ *of* B_1 *and* B_2: Let P be a point of B_1 *which lies in* R_2 and let $d(P, B_2)$ denote the (shortest) distance of P from B_2. Then we set $d_1 = \max_{P\in B_1\cdot R_2} d(P, B_2)$. Similarly, let Q be any point on B_2, *which lies in* R_1, $d(Q, B_1)$ the (shortest) distance of Q from B_1 and $d_2 = \max_{Q\in B_2\cdot R_1} d(Q, B_1)$. (If B_2 is contained in R_1, then $d_1 = 0$, and similarly if B_1 lies in R_2, then $d_2 = 0$.) We define now[14]

$$D_i(B_1, B_2) = D_i(B_2, B_1) = \max(d_1, d_2).$$

Using this definition we state first the following theorem:

THEOREM V. *Suppose* R_1 *and* R_2 *are simply connected bounded regions which contain the origin. Suppose furthermore that the inner distance of their bound-*

[14] For this definition see [12]. It should be noted that $D_i(B_1, B_2)$ does not necessarily satisfy the "triangle inequality."

aries, $D_i(B_1, B_2)<\epsilon$ for some ϵ, $0<\epsilon<1$, $0<\epsilon<\sigma/64$, where σ is the distance of O from B_1 and B_2. Let $\eta_1(\delta)$ and $\eta_2(\delta)$ denote the structure moduli of B_1 and B_2, respectively.

If $w=f_1(z)$ and $w=f_2(z)$ map the circle $|z|<1$ conformally onto R_1 and R_2, respectively, such that $f_1(0)=f_2(0)=0$, $f_1'(0)>0$, $f_2'(0)>0$, then for $|z|<1$

$$(4.11)\quad |f_1(z)-f_2(z)| \leqq \left(1+\frac{k}{\sigma^{1/2}}\epsilon^{1/4}\log\frac{4}{\epsilon}\right)\left\{\eta_1\left(\left(\frac{8\pi A_1}{\log(1/\epsilon)}\right)^{1/2}\right) + \eta_2\left(\left(\frac{8\pi A_2}{\log(1/\epsilon)}\right)^{1/2}\right)\right\}.$$

Here A_1 and A_2 are the areas of R_1 and R_2, respectively, and k is an absolute constant ($\leqq 16e$).

As a corollary one obtains at once the following result: *Suppose R_n ($n=1, 2, \cdots$) and R_0 are simply connected bounded regions, all of which contain the origin O. Denote by B_n the boundary of R_n ($n=0, 1, 2, \cdots$). Suppose that* (i) *$D_i(B_n, B_0)\to 0$ as $n\to\infty$;* (ii) *the structure moduli of B_n, $\eta_n(\delta)\to 0$ as $\delta\to 0$, uniformly for all $n=0, 1, 2, \cdots$;* (iii) *the areas A_n of R_n are uniformly bounded. If $w=f_n(z)$, normalized by the conditions $f_n(0)=0$, $f_n'(0)>0$, maps $|z|<1$ conformally onto R_n, then*[15]

$$(4.12)\quad f_n(z)\to f_0(z) \qquad \text{as } n\to\infty, \text{ uniformly in } |z|\leqq 1.$$

If the B_n ($n=0, 1, 2, \cdots$) are closed Jordan curves, these three conditions are also necessary for (4.12).

We must prove the necessity. Suppose that $w_n(t)$, $0\leqq t\leqq b$, are parametric representations of B_n ($n=0, 1, 2, \cdots$) and that

$$(4.13)\quad w_n(t)\to w(t) \qquad \text{as } n\to\infty, \text{ uniformly for } 0\leqq t\leqq b$$

(for example, $w_n(t)=f_n(e^{it})$, $0\leqq t\leqq 2\pi$). Then it is clear that conditions (i) and (iii) are satisfied. To prove (ii) we must show that for every $\epsilon>0$ there exists a $\delta_0>0$ such that

$$\eta_n(\delta)<\epsilon \qquad \text{if } 0<\delta\leqq\delta_0 \text{ for all } n=0, 1, 2, \cdots.$$

Suppose this were not true. Then there would exist an $\epsilon_0>0$ and, for every $k=1, 2, \cdots$, a curve B_{n_k} of the sequence with the following property: there is a cross-cut c_{n_k} of R_{n_k} (of the interiors of B_{n_k}), whose diameter is less than $1/k$, such that the subregion T_{n_k} of R_{n_k} formed by c_{n_k} and B_{n_k} which does not contain the origin has the diameter

$$(4.14)\quad \Lambda_{n_k}\geqq\epsilon_0.$$

[15] A somewhat more general sufficient condition for (4.12) is obtained if the hypothesis (ii) is replaced by the following two assumptions: $\lim_{\delta\to 0}\eta_0(\delta)=0$ and $\limsup_{n\to\infty}\eta_n(\delta)\leqq\bar{\eta}(\delta)$ where $\lim_{\delta\to 0}\bar{\eta}(\delta)=0$.

Suppose that the end points of c_{n_k} have the parameter values t_k and t_k', respectively, $0 \leqq t_k < t_k' \leqq b$. We may assume that

$$\lim_{k\to\infty} t_k = \tau, \quad \text{and} \quad \lim_{k\to\infty} t_k' = \tau' \qquad (0 \leqq \tau \leqq \tau' \leqq b)$$

exist. Hence

$$\lim_{k\to\infty} w_{n_k}(t_k) = w_0(\tau), \qquad \lim_{k\to\infty} w_{n_k}(t_k') = w_0(\tau').$$

Since the diameter of c_{n_k} is less than $1/k$ and thus approaches 0 as $k\to\infty$, we have $w_0(\tau) = w_0(\tau')$, hence, $\tau = \tau'$ or $\tau' = \tau + b$, that is, $\tau = 0$, $\tau' = b$. By changing the origin of the t-scale we can avoid the second possibility. Because of (4.13), the arcs γ_k: $w = w_{n_k}(t)$, $t_k \leqq t \leqq t_k'$, will lie in any given neighborhood of $w_0(\tau)$ for sufficiently large k; the same is true of the arcs c_{n_k} and hence also of the subregions of R_{n_k} formed by c_{n_k} and γ_{n_k}. For sufficiently large k, these subregions will *not* contain the origin, and hence will be the T_{n_k}. Thus the diameter of T_{n_k}, $\Lambda_{n_k} \to 0$ as $k\to\infty$, contrary to (4.14).

Some of the known theorems on the convergence of the mapping functions of variable regions bounded by Jordan curves are easily derived from our corollary. We indicate this for T. Radó's theorem [13](16), which states that *a necessary and sufficient condition for* (4.12) *is that the Frechet distance* d_n *between the boundary curves* B_n *and* B_0 *approach* 0 *as* $n\to\infty$. We need to show only the sufficiency. The assumption that $\lim_{n\to\infty} d_n = 0$ implies the existence of parametric representations $w_n(t)$ of B_n such that (4.13) holds, and we have just shown that (4.13) implies the conditions (i), (ii), and (iii).

Next we state a result concerning a more restricted class of regions, for which we obtain a sharper estimate than (4.11).

THEOREM VI. *Suppose* R_1 *and* R_2 *are regions which satisfy the hypotheses of Theorem* V. *Suppose, furthermore, that the structure moduli of their boundaries* B_1, B_2 *satisfy the inequalities*

$$\eta_1(\delta) \leqq \kappa\delta + \eta_1, \qquad \eta_2(\delta) \leqq \kappa\delta + \eta_2, \qquad \delta \leqq \delta_0,$$

for some δ_0, *where* κ, η_1, *and* η_2 *are constants*, $\kappa > 0$, $\eta_1 \geqq 0$, $\eta_2 \geqq 0$. *If* $f_1(z)$ *and* $f_2(z)$ *are defined as in Theorem* V, *then for* $|z| < 1$:

$$(4.15) \qquad |f_1(z) - f_2(z)| \leqq K\left(\epsilon^{1/2} \log \frac{4}{\epsilon}\right)^{\alpha} + K_1(\eta_1 + \eta_2), \qquad \alpha = \frac{2}{\pi^2\kappa^2},$$

where K *and* K_1 *are constants*; K *depends only on* δ_0, κ, *the larger of the diameters of* B_1 *and* B_2, *and the minimal distance* σ *of* O *from* B_1 *and* B_2; K_1 *depends only on* σ.

The proofs of Theorems V and VI will be given in sections 4.3 and 4.4;

(16) Other examples are a theorem of Courant and a theorem of Markoushevitch, both of which—as is shown by Markoushevitch [11, pp. 874–875]—are equivalent to Radó's result.

section 4.2 contains lemmas used in this proof.

4.2. *Two lemmas.* We shall use the following lemma which is an immediate consequence of a lemma due to G. Szegö [14, p. 191, (11)][17].

LEMMA 5. *Suppose that the function $z=\phi(w)$ maps the simply connected region R conformally onto the circle $|z|<1$ so that $\phi(0)=0$. If w is a point in R whose distance from the boundary of R is less than ϵ, then*

$$1-|\phi(w)|\leqq 4(\epsilon|\phi'(0)|)^{1/2},$$

that is, the image $\phi(w)$ lies in the ring

$$1-4(\epsilon|\phi'(0)|)^{1/2}\leqq |z|<1.$$

The following lemma has possibly some interest beyond its immediate use in the proofs of this section.

LEMMA 6. *Suppose R and T are two simply connected bounded regions; T is contained in R and contains the origin O. Let σ be the distance of O from the boundary B of R and let every boundary point of T be within distance ϵ from B, where $\epsilon<\sigma/64$. Suppose that $w=f(z)$ and $w=g(z)$ map $|z|<1$ conformally onto R and T, respectively, so that $f(0)=g(0)=0$ and $f'(0)>0$, $g'(0)>0$. Then for all $|z|\leqq\rho$, $1/2<\rho<1$:*

$$(4.21)\qquad |f(z)-g(z)|\leqq 4e\left(\frac{\epsilon}{\sigma}\right)^{1/2}\left(1+\frac{2}{\pi}\log\frac{1+\rho}{1-\rho}\right)\frac{\omega(2(1-\rho))}{1-\rho}$$

where $\omega(r)$ is the oscillation of $f(z)$ at the boundary.

Proof. Let $\phi(w)$ denote the inverse function of $f(z)$. Since R contains the circle $|w|<\sigma$, we have

$$(4.22)\qquad 0<\phi'(0)\leqq 1/\sigma.$$

The function $\zeta=\phi(w)$ carries T into a subregion H of $|\zeta|<1$ which contains O, and, by Lemma 5 and (4.22), the boundary of H lies in the ring

$$(4.23)\qquad 1-4\left(\frac{\epsilon}{\sigma}\right)^{1/2}\leqq|\zeta|<1.$$

Let $\zeta=h(z)$ map $|z|<1$ conformally onto H such that $h(0)=0$, $h'(0)>0$. We apply Lemma 3 to $h(z)$ (since $4(\epsilon/\sigma)^{1/2}<1/2$ we notice that $1-4(\epsilon/\sigma)^{1/2}>1/(1+8(\epsilon/\sigma)^{1/2})$ and we may replace the ϵ of Lemma 3 by $8(\epsilon/\sigma)^{1/2}$). Thus we find:

$$(4.24)\qquad |h(z)-z|\leqq 8e\left(\frac{\epsilon}{\sigma}\right)^{1/2}\left(1+\frac{2}{\pi}\log\frac{1+\rho}{1-\rho}\right),\qquad |z|\leqq\rho.$$

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(17) The main content of this lemma follows from Theorem III, but it has the advantage that the factor of $\epsilon^{1/2}$ is given numerically.

Now we have $g(z)=f(h(z))$ and therefore, for $|z|=\rho<1$

$$f(z)-g(z)=f(z)-f(h(z))=\int_{h(z)}^{z} f'(\zeta)d\zeta, \tag{4.25}$$

the integration being taken along the straight line segment s from $h(z)$ to z. Since, by the lemma of Schwarz, $|h(z)|\leqq|z|$, s lies in the circle $|z|\leqq\rho$. Hence

$$\max_{\zeta\in s}|f'(\zeta)|\leqq\max_{|\zeta|\leqq\rho}|f'(\zeta)|=|f'(\zeta_1)|$$

for some ζ_1 with $|\zeta_1|=\rho$. Let $r=1-\rho$. Then

$$\pi r^2|f'(\zeta_1)|^2\leqq\int_0^r\int_0^{2\pi}|f'(\zeta_1+Re^{i\theta})|^2Rd\theta dR,$$

and the last integral represents the area of the image of the circle $|z-\zeta_1|<r$ under the transformation $w=f(z)$. If $\zeta_1=\rho e^{i\theta_1}$, then this circle is contained in the region Δ: $\{|z-e^{i\theta_1}|<2r,\ |z|<1\}$, and the double integral is smaller than the area of the image of Δ. Since the diameter of this image does not exceed $\omega(2r)$, we obtain finally

$$\pi r^2|f'(\zeta_1)|^2\leqq\frac{\pi\omega^2(2r)}{4}$$

or

$$|f'(\zeta_1)|\leqq\frac{\omega(2(1-\rho))}{2(1-\rho)}. \tag{4.26}$$

If we note from (4.25) that

$$|f(z)-g(z)|\leqq|h(z)-z||f'(\zeta_1)|,$$

we find (4.21) from (4.24) and (4.26).

Remark. Suppose R and T are two regions as in Lemma 6: T is contained in R and contains the origin O, and every boundary point of T is within a distance of ϵ from the boundary B of R. Suppose that any function $z=\phi(w)$ which maps R onto $|z|<1$ such that $\phi(0)=0$ satisfies the following condition: *If w is a point in R, w_0 a point on B nearest to R, then*

$$1-|\phi(w)|\leqq K|w-w_0|^{\gamma} \tag{4.27}$$

where K and γ are constants, $0<\gamma\leqq 1$. If $f(z)$ and $g(z)$ are defined as in the lemma, and if $2K\epsilon^{\gamma}<1$, then for $|z|\leqq\rho$, $1/2<\rho<1$,

$$|f(z)-g(z)|\leqq K\epsilon^{\gamma}e\left(1+\frac{2}{\pi}\log\frac{1+\rho}{1-\rho}\right)\frac{\omega(2(1-\rho))}{1-\rho}.$$

For the condition (4.27) implies that $\zeta=\phi(w)$ carries T into a region H whose boundary lies in the ring

$$1-K\epsilon^{\gamma}\leqq|\zeta|<1, \tag{4.28}$$

and if this inequality is used in place of (4.23), the proof of this remark is merely a repetition of that of Lemma 6.

4.3. *Proof of* V. Let T be the largest subregion of the intersection of R_1 and R_2 which contains the origin O; T is simply connected and every boundary point of T is within distance ϵ from the boundary of R_1 and of R_2. Let $w=g(z)$, normalized by the condition $g(0)=0$, $g'(0)>0$, map the circle $|z|<1$ conformally onto T. Then by Lemma 6, for $|z_1|\leqq\rho$, $1/2<\rho<1$,

$$|f_k(z_1)-g(z_1)|\leqq 4e\left(\frac{\epsilon}{\sigma}\right)^{1/2}\left(1+\frac{2}{\pi}\log\frac{1+\rho}{1-\rho}\right)\frac{\omega_k(2(1-\rho))}{1-\rho} \qquad (k=1,2) \tag{4.31}$$

where $\omega_k(r)$ denotes the boundary oscillation of $f_k(z)$.

Let z_1 be a fixed point, $|z_1|=\rho$, and let z be such that $\rho<|z|<1$, $\arg z=\arg z_1$. Then, by Theorem I, for $r=1-\rho$,

$$|f_k(z)-f_k(z_1)|\leqq\omega_k(r)\leqq\eta_k\left(\left(\frac{2\pi A_k}{\log(1/r)}\right)^{1/2}\right) \qquad (k=1,2). \tag{4.32}$$

Now

$$\begin{aligned}|f_1(z)-f_2(z)|\leqq{}&|f_1(z)-f_1(z_1)|+|f_1(z_1)-g(z_1)|\\&+|g(z_1)-f_2(z_1)|+|f_2(z_1)-f_2(z)|.\end{aligned} \tag{4.33}$$

By choosing $r=1-\rho=\epsilon^{1/4}/2$ and using the estimates (4.31) and (4.32), we obtain easily the desired inequality (4.11) for all z in $\rho\leqq|z|<1$. By the principle of the maximum modulus it holds then also for all $|z|\leqq\rho$.

4.4. *Proof of* VI. This proof differs from the preceding one only insofar as Theorem II will be used in order to estimate $\omega_k(r)$ and the relation between $r=1-\rho$ and ϵ will be changed. If $g(z)$ has the same meaning as in §4.3 we obtain from (4.31) and Theorem II, applied with $\mu=1$: For $|z_1|\leqq\rho$

$$|f_k(z_1)-g(z_1)|\leqq 8e\left(\frac{\epsilon}{\sigma}\right)^{1/2}\left(1+\frac{2}{\pi}\log\frac{2}{r}\right)\frac{1}{2r}(m(2r)^{\alpha}+3\eta_k), \qquad k=1,2,$$

where m is a constant which depends only on δ_0, κ, and the larger of the diameters of B_1 and B_2. We choose now $2r=\epsilon^{1/2}\log(4/\epsilon)>\epsilon$. Then a short calculation yields the inequality:

$$|f_k(z_1)-g(z_1)|\leqq\frac{14e}{\sigma^{1/2}}\left[m\left(\epsilon^{1/2}\log\frac{4}{\epsilon}\right)^{\alpha}+3\eta_k\right]. \tag{4.41}$$

Let z be a point in $\rho < |z| < 1$ such that $\arg z = \arg z_1$. Then, again by Theorem II,

$$(4.42)\qquad |f_k(z) - f_k(z_1)| \leqq mr^\alpha + 3\eta_k = \frac{m}{2^\alpha}\left(\epsilon^{1/2} \log \frac{4}{\epsilon}\right)^\alpha + 3\eta_k.$$

Using (4.33) in conjunction with (4.41) and (4.42) we obtain the desired inequality (4.15) with $K = 2m(1 + 14e/\sigma^{1/2})$, $K_1 = 3(1 + 14e/\sigma^{1/2})$. It holds for $|z| \geqq \rho$, and hence by the principle of the maximum modulus for all $|z| \leqq \rho$.

5. **Arbitrary regions: inverse mapping function.** We consider now the analogous problem for the inverse function. As is to be expected the result obtained is sharper than the one for the direct mapping function.

THEOREM VII. *Suppose that R and S are two simply connected bounded regions such that $S \subset R$ and $w = 0$ lies in S. Let $\eta(\delta)$ denote the structure modulus of the boundary B_s of S and σ the distance of O from B_s. If B is the boundary of R, suppose that $D_i(B, B_s) < \epsilon$, $0 < \epsilon < 1$, $\epsilon < \sigma/64$.*

Let $z = \phi(w)$ and $z = \psi(w)$, normalized by the condition $\phi(0) = \psi(0) = 0$, $\phi'(0) > 0$, $\psi'(0) > 0$ map R and S conformally onto $|z| < 1$.

(a) *If $\eta(\delta) \leqq \kappa\delta$, then for $w \in S$:*

$$(5.1)\qquad |\phi(w) - \psi(w)| \leqq K\epsilon^{1/2} \log \frac{2}{\epsilon}$$

where K is a constant which depends only on κ, σ, and the diameter of R.

(b) *In the general case, for $w \in S$,*

$$(5.2)\qquad |\phi(w) - \psi(w)| \leqq L(\eta(\epsilon^{1/5}))^{1/2}$$

where L is a constant which depends only on σ and the diameter of R.

Proof. (i) Let C_{ρ_0} denote the level curve $|\psi(w)| = \rho_0 < 1$ of S where $D(C\rho_0, B) < \epsilon$. Let w_0 be a point on C_{ρ_0}, $z_0 = \psi(w_0) = \rho_0 e^{i\theta_0}$. Let $z_1 = \rho_1 e^{i\theta_0}$, $0 < \rho_1 < \rho_0 < 1$. If $z_1 = \psi(w_1)$, then

$$(5.3)\qquad |\psi(w_0) - \psi(w_1)| = \rho_0 - \rho_1 < 1 - \rho_1.$$

Since z_0 and z_1 lie in the region $\{|z - e^{i\theta_0}| \leqq 1 - \rho_1, |z| < 1\}$ it follows that w_0 and w_1 are within a subregion T of S which does not contain O and whose diameter is not greater than η_1, where in case (a), by Theorem II,

$$\eta_1 = m(1 - \rho_1)^\alpha, \qquad \left(\alpha = \frac{2}{\pi^2\kappa^2}\right)$$

and in case (b), by Theorem I,

$$(5.4)\qquad \eta_1 = \eta\left(\left(\frac{2\pi A}{\log (1/(1 - \rho_1))}\right)^{1/2}\right).$$

Here m is a constant which depends only on κ and the diameter D([18]) of R, A is the area of S. Let w_2 be a point on B at a distance not greater than ϵ

from w_0. We assume here at first that $\eta_1+\epsilon<\sigma$. Then there exists a subarc of the circle $|w-w_2|=\eta_1+\epsilon$, which forms a cross-cut c of R and separates w_0 and w_1 from $w=0$. By Theorem III we have, therefore:

$$|\phi(w_0)-\phi(w_1)| \leqq M(\epsilon+\eta_1)^{1/2} \qquad (M = M(\sigma, D)). \tag{5.5}$$

(ii) The function $\zeta=\phi(w)$ carries S into a subregion H of the unit circle which contains the origin. From Lemma 5 and the fact that $\phi'(0)\leqq 1/\sigma$ we infer that the boundary of H is contained in the ring

$$1-4\left(\frac{\epsilon}{\sigma}\right)^{1/2} \leqq |\zeta| < 1.$$

Let $\zeta=h(z)$ map $|z|<1$ conformally onto H so that $h(0)=0$ and $h'(0)>0$. Then by Lemma 3 (applied with the ϵ of the lemma replaced by $8(\epsilon/\sigma)^{1/2}$)

$$|z_1-h(z_1)| \leqq 8e\left(\frac{\epsilon}{\sigma}\right)^{1/2}\left(1+\frac{2}{\pi}\log\frac{1+\rho_1}{1-\rho_1}\right), \qquad |z_1| = \rho_1.$$

It is easily seen that $\phi(w_1)=h(z_1)$. Hence

$$|\psi(w_1)-\phi(w_1)| = |z_1-h(z_1)| \leqq 8e\left(\frac{\epsilon}{\sigma}\right)^{1/2}\left(1+\frac{2}{\pi}\log\frac{1+\rho_1}{1-\rho_1}\right). \tag{5.6}$$

(iii) Now the proof is easily completed. Using the inequality

$$|\phi(w_0)-\psi(w_0)| \leqq |\phi(w_0)-\phi(w_1)| + |\phi(w_1)-\psi(w_1)| + |\psi(w_1)-\psi(w_0)|$$

e obtain from (5.5), (5.3), and (5.6)

$$|\phi(w_0)-\psi(w_0)| \leqq M(\epsilon+\eta_1)^{1/2} + (1-\rho_1) + 8e\left(\frac{\epsilon}{\sigma}\right)^{1/2}\left(1+\frac{2}{\pi}\log\frac{1+\rho_1}{1-\rho_1}\right). \tag{5.7}$$

In the case (a) we choose $m(1-\rho_1)^\alpha=\epsilon$, so that $\eta_1+\epsilon=2\epsilon<\sigma$, and a simple estimate leads to (5.1) for $w\in C\rho_0$. In case (b) we choose ρ_1, so that $2\pi A/|\log(1-\rho_1)|=\epsilon^{2\beta}$ for some $\beta>0$. Then the right-hand side of (5.7) does not exceed

$$M(\epsilon+\eta(\epsilon^\beta))^{1/2} + \frac{\epsilon^{2\beta}}{2\pi A} + \frac{L_1}{\sigma^{1/2}}\epsilon^{1/2-2\beta} \leqq M(2\eta(\epsilon^\beta))^{1/2} + \frac{\epsilon^{2\beta}}{2\pi^2\sigma^2} + \frac{L_1}{\sigma^{1/2}}\epsilon^{1/2-2\beta},$$

where L_1 depends only on the diameter of R. Here we take $\beta=1/5$ and obtain (5.2) for $w\in C_{\rho_0}$, provided $\epsilon+\eta(\epsilon^{1/5})<\sigma$. But if $\epsilon+\eta(\epsilon^{1/5})\geqq\sigma$, then

[18] By Theorem II, m depends on the diameter D_s of S, but since $D_s\leqq D$, the value of m is not decreased if D_s is replaced by D.

$$\left|\phi(w)-\psi(w)\right| \leqq \frac{4}{\sigma}\eta(\epsilon^{1/5}) \leqq \frac{4}{\sigma}D^{1/2}(\eta(\epsilon^{1/5}))^{1/2}.$$

Since C_{ρ_0} may be taken so that ρ_0 is arbitrarily close to 1, the theorem holds for *all* $w \in S$, by the principle of the maximum modulus.

6. **Regions with smooth boundaries.** The estimate for degree of proximity of the mapping functions of two neighboring regions may be sharpened considerably if the boundaries of these regions are Jordan curves possessing continuously turning tangents.

THEOREM VIII. *Suppose that C_1 and C_2 are closed Jordan curves which contain the origin in their interiors and satisfy the hypotheses:*

(a) *C_k $(k=1, 2)$ has continuously turning tangents, and the tangent angle $\alpha_k(s)$, considered as function of the arc length, has the modulus of continuity $\beta(t)$, that is,*

$$\left|\alpha_k(s \pm t)-\alpha_k(s)\right| \leqq \beta(t), \qquad t>0,$$

where $\beta(t)$ is nondecreasing and $\lim_{t\to 0}\beta(t)=0$.

(b) *If w_1 and w_2 are points on C_k and Δs is the (shorter) arc of C_k between them, then there exists a constant a such that $\Delta s/\left|w_1-w_2\right| \leqq a$.*

(c) *The diameter of C_k does not exceed D and the distance of O from C_1 and C_2 is at least σ.*

(d) *$D_i(C_1, C_2)<\epsilon$ for some ϵ, $0<\epsilon<1$.*

If $w=f_k(z)$ maps the circle $|z|<1$ conformally onto the interior R_k of C_k and if $f_k(0)=0$, $f_k'(0)>0$, then there exists for every δ, $0<\delta<1$, a constant M_δ which depends only on δ, a, σ, D, and the function $\beta(t)$—and in no other way upon C_1 and C_2—such that for $|z| \leqq 1$

$$(6.1) \qquad \left|f_1(z)-f_2(z)\right| \leqq M_\delta\epsilon^{1-\delta}.$$

Proof. Under the present assumptions on C_k there exists for every θ, $0<\theta<1$, a B, which depends only on θ, a, σ, D, and the function $\beta(t)$, such that for $|z_0|=1$, $|z| \leqq 1$ (19),

$$(6.2) \qquad \frac{1}{B}\left|z-z_0\right|^{1+\theta} \leqq \left|f_k(z)-f_k(z_0)\right| \leqq B\left|z-z_0\right|^{1-\theta}.$$

Let $z=\phi_k(w)$ be the inverse of $w=f_k(z)$, Then, for any w in R_k, w_0 on C_k

$$(6.3) \quad 1-\left|\phi(w)\right| \leqq \left|\phi(w_0)-\phi(w)\right| \leqq B'\left|w-w_0\right|^{1/(1+\theta)} \quad (B'=B^{1/(1+\theta)}).$$

Now let T be the largest subregion of $R_1 \cdot R_2$ which contains O. T is simply connected and every boundary point of T is within distance ϵ from C_1 and C_2. Let $w=g(z)$, $g(0)=0$, $g'(0)>0$, map $|z|<1$ conformally onto T.

(19) These inequalities are well known, see for example [8, p. 1408]. In [15] it is shown that B depends only on the parameters indicated. Cf. also [6, p. 35 (VIII)].

We assume at first that $2B'\epsilon^{1/(1+\theta)}<1$. Then by the remark following Lemma 6, we have, because of (6.3), for $|z_1|\leqq\rho$, $1/2<\rho<1$,

$$|f_k(z_1)-g(z_1)|\leqq B'\epsilon^{1/(1+\theta)}e\left(1+\frac{2}{\pi}\log\frac{1+\rho_1}{1-\rho_1}\right)\frac{\omega(2(1-\rho_1))}{1-\rho_1}.$$

By the right-hand inequality of (6.2), $\omega(r)\leqq 2Br^{1-\theta}$ and therefore:

$$(6.4)\qquad |f_k(z_1)-g(z_1)|\leqq A\epsilon^{1/(1+\theta)}\left(1+\frac{2}{\pi}\log\frac{1+\rho_1}{1-\rho_1}\right)(1-\rho_1)^{-\theta}$$

where $A=4BB'e$. We choose now $1-\rho_1=\epsilon$. Then we note first that

$$1+\frac{2}{\pi}\log\frac{1+\rho_1}{1-\rho_1}<2\log\frac{e}{\epsilon}=2\epsilon^{-\theta}\left(\epsilon^{\theta}\log\frac{e}{\epsilon}\right)\leqq\frac{2\epsilon^{-\theta}}{\theta}.$$

Thus we obtain from (6.4)

$$(6.5)\qquad |f_k(z_1)-g(z_1)|\leqq\frac{2A}{\theta}\epsilon^{(1-2\theta-2\theta^2)/(1+\theta)}.$$

Let $|z_0|=1$, $\arg z_0=\arg z_1$. Then by (6.2)

$$(6.6)\qquad |f_k(z_0)-f_k(z_1)|\leqq B(1-\rho_1)^{1-\theta}=B\epsilon^{1-\theta}.$$

Given δ, choose θ so small that $(1-2\theta-2\theta^2)/(1+\theta)=\delta$. Then using (4.33) and applying (6.5) and (6.6), we obtain the desired result (6.1) for the case that $2B'\epsilon^{1/(1+\theta)}<1$. If $\epsilon\geqq(2B')^{-(1+\theta)}$, then we have trivially for $|z|\leqq 1$

$$|f_1(z)-f_2(z)|\leqq[2D(2B')^{1+\theta}]\epsilon.$$

Bibliography

1. L. Bieberbach, *Über die Kreisabbildung von nahezu kreisförmigen Berreichen*, Preuss. Akad. Wiss. Sitzungsber. (1924) pp. 181–188.

2. C. Carathéodory, *Conformal representation*, Cambridge University Tracts, No. 28, 1932.

3. J. Ferrand, *Étude de la correspondance entre les frontières dans la représentation conforme*, Bull. Soc. Math. France vol. 70 (1942) pp. 143–174.

4. ———, *Sur la déformation analytique d'un domain*, C. R. Acad. Sci. Paris vol. 221 (1945) pp. 132–134.

5. C. Gattegno and A. Ostrowski, *Représentation conforme à la frontière: domaines généraux*, Memorial des Sciences Mathématiques, vol. 109, 1949.

6. ———, *Représentation conforme à la frontière: domaines particuliers*, Memorial des Sciences Mathématiques, vol. 110, 1949.

7. M. Lavrientieff, *Sur la continuité des fonctions univalentes*, C. R. (Doklady) Acad. Sci. USSR. vol. IV (1936) pp. 215–217.

8. ———, *Sur la représentation conforme*, C. R. Acad. Sci. Paris vol. 184 (1927) pp. 1407–1409.

9. E. Lindelöf, *Sur un principe général de l'analyse et ses applications à la représentation conforme*, Acta Societatis Scientiarum Fennicae vol. 46, No. 4 (1915) pp. 1–35.

10. A. R. Marchenko, *Sur la représentation conforme*, C. R. (Doklady) Acad. Sci. USSR. vol. 1 (1935) pp. 289–290.

11. A. Markoushevitch, *Sur la représentation conforme des domaines à frontières variables*, Matematicheskij Sbornik vol. 43 (1936) pp. 864–886.

12. F. J. Polansky, *Some approximation theorems in conformal mapping*, Thesis, University of Minnesota.

13. T. Radó, *Sur la représentation conforme des domaines variables*, Acta Litterarum ac Scientiarum, Universitatis Francisco-Josephinae vol. 1 (1923) pp. 180–186.

14. G. Szegö, *Zur Theorie der schlichten Abbildungen*, Math. Ann. vol. 100 (1928) pp. 188–211.

15. S. E. Warschawski, *Conformal mapping of regions bounded by smooth curves*, to appear in Proceedings of the American Mathematical Society.

16. J. Wolff, *Sur la représentation conforme des bandes*, Compositio Math. vol. 1 (1934) pp. 217–222.

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