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DRUKARNIA UNIWERSYTETU JAGIELLOŃSKIEGO W KRAKOWIE

On the density character of compact topological groups

by

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1. Introduction. In 1943, S. Kakutani [8] proved that, for a compact Abelian group G, if $w(G) = \mathfrak{w}$ is the least cardinal of a basis of open sets for G then the density character (see Hewitt [2]) of G is $\mathfrak{n}$, where $\mathfrak{n}$ is the least cardinal number satisfying $2^{\mathfrak{n}} \geqslant \mathfrak{w}$. In that paper Kakutani also showed that the cardinal of G is $|G| = 2^{\mathfrak{w}}$. Thus one could state this theorem in the following form: Let $\mathfrak{n}$ be the least cardinal number such that card $G \leqslant 2^{2^{\mathfrak{n}}}$: then there is a dense subgroup in G of cardinal $\mathfrak{n}$.

In 1958 S. Hartman and A. Hulanicki [1] gave a simpler proof of this theorem, based on a theorem of Hewitt [2] on the density character of a certain product of topological spaces. In a further paper Hulanicki [4] extended Hewitt's theorem to prove that a certain product of measurable spaces $P_{t\in T} X_t$, $w(X_t) \leqslant \aleph_0$, card $T \leqslant 2^{\mathfrak{n}}$, had a dense set of full outer measure of cardinal $\mathfrak{n}^{\aleph_0}$. With this result he was able to prove that if $\mathfrak{n}$ is the least cardinal number such that card $G \leqslant 2^{2^{\mathfrak{n}}}$, G being a compact Abelian group, then G contains a dense subgroup of full outer measure and cardinal $\mathfrak{n}^{\aleph_0}$.

G. L. Itzkowitz [6], using again a theorem similar to Hewitt's, extended this result to prove that such a G contains a dense pseudo-compact subgroup of cardinal $\mathfrak{n}^{\aleph_0}$ (hence of full outer measure in G). H. Wilcox [12], using a result of N. Th. Varopoulos, proved that this result may be extended to include all compact groups. Left unanswered was Kakutani's question in [8]: Is $\mathfrak{n}$ the density character of G when G is a compact group and $\mathfrak{n}$ is the least cardinal such that card $G \leqslant 2^{2^{\mathfrak{n}}}$? Wilcox's theorem shows that the density character of G is at most $\mathfrak{n}^{\aleph_0}$.

It is the purpose of this note to answer Kakutani's question in the affirmative. Our method of proof is essentially the same as the method followed by H. Wilcox. For completeness we will give a new proof, also based on Hewitt's theorem, of Kakutani's theorem for compact Abelian groups.

2. The density character in groups. Let $w(G)$ denote the least cardinal of a basis of open sets of G ($w(G)$ is also called the density character of G). If $\mathfrak{B}$ is a collection of sets, we shall say that the collection of sets $\mathfrak{C}$ is a *weak base for* $\mathfrak{B}$ if, given $B \in \mathfrak{B}$, there is a $C \in \mathfrak{C}$ such that $C \subset B$.

We recall the theorem of Hewitt, [2].

2.1. THEOREM (Hewitt). *Let $X = \boldsymbol{P}_{t\in T} X_t$, where $\{(X_t, \tau_t)\colon t \in T\}$ is a family of topological spaces such that $w(X_t) \leqslant \mathfrak{n}$, for each $t \in T$, and card $T \leqslant 2^{\mathfrak{n}}$. Then the product space (X, τ) has a weak base $\mathfrak{B}$ for the open sets for which card $\mathfrak{B} \leqslant \mathfrak{n}$. (Note that $w(X) = 2^{\mathfrak{n}}$.)*

2.2. COROLLARY. *Let $G = \boldsymbol{P}_{t\in T} H_t$, where each H_t is a compact separable topological group, and card $T = 2^{\mathfrak{n}}$. Then $w(G) = 2^{\mathfrak{n}}$, and there is a weak base for the open sets of G having cardinal $\mathfrak{n}$.*

2.3. THEOREM (Vilenkin [12]). *Let G be a compact Abelian group. For some cardinal number $\mathfrak{m}$, there is a continuous mapping of $\{-1, 1\}^{\mathfrak{m}}$ onto G; $\mathfrak{m}$ can be taken to be $\max[\aleph_0, r]$, where r is the rank of the character group of G.*

Note. If card $G > \aleph_0$ then $r =$ card G.

2.4. THEOREM (Kakutani). *Let G be an infinite compact Abelian group satisfying $w(G) = \mathfrak{m}$, and let $\mathfrak{n}$ be the least cardinal such that $\mathfrak{m} \leqslant 2^{\mathfrak{n}}$. Then G has a dense subgroup J of cardinal $\mathfrak{n}$.*

Proof. Pontryagin [10] has shown that if G is a locally compact Abelian group then $w(G) = w(\hat{G})$, where $\hat{G}$ is the character group of G. Thus 2.3 implies that G is the continuous image of $G' = \{-1, 1\}^{\mathfrak{m}}$. Evidently G' has a weak base for its open sets of cardinal $\mathfrak{n}$, and so G has a weak base $\mathfrak{B}$ for its open sets of cardinal $\mathfrak{n}$. Let J be the subgroup of G generated by the set obtained by selecting one point from each element of the weak base $\mathfrak{B}$.

Hulanicki [5], Ivanovskii [7], and Kuzminov [9] have shown that a 0-dimensional infinite topological group G that is compact is homeomorphic with the space $\{-1, 1\}^{\mathfrak{m}}$, where $\{-1, 1\}$ is discrete and $\mathfrak{m} = w(G)$. Thus an immediate corollary of Hewitt's Theorem 2.1 that we have cited is:

2.5. THEOREM. *Let G be an infinite compact 0-dimensional topological group satisfying $w(G) = \mathfrak{m}$, and let $\mathfrak{n}$ be the least cardinal such that $\mathfrak{m} \leqslant 2^{\mathfrak{n}}$. Then G has a dense subgroup J of cardinal $\mathfrak{n}$.*

Varopoulos has shown in [11] that if G is a connected compact group and Z is its centre, then G/Z has the form $\boldsymbol{P}_{\alpha\in A} M_\alpha$ with all the M_α compact metrizable groups. Thus we may state the following theorem:

2.6. THEOREM. *Let G be an infinite compact connected topological group satisfying $w(G) = \mathfrak{m}$, and let $\mathfrak{n}$ be the least cardinal such that $\mathfrak{m} \leqslant 2^{\mathfrak{n}}$. Then G has a dense subgroup J of cardinal $\mathfrak{n}$.*

Proof. Let Z be the centre of G: then it is evident that $w(Z) \leqslant \mathfrak{m}$ and that 2.4 applies to Z. It is also clear that $G/Z = \boldsymbol{P}_{\alpha\in A} M_\alpha$ is of the proper form for an application of 2.1. Thus we observe that both Z and G/Z have dense subsets of cardinal at most $\mathfrak{n}$. It is now an elementary exercise (see Hewitt and Ross, 5.38(f)) to show that G has a dense subgroup of cardinal $\mathfrak{n}$.

2.7. THEOREM. *Let G be an infinite compact topological group satisfying $w(G) = \mathfrak{m}$, and let $\mathfrak{n}$ be the least cardinal such that $\mathfrak{m} \leqslant 2^{\mathfrak{n}}$. Then G has a dense subgroup J of cardinal $\mathfrak{n}$.*

Proof. Let C be the component of the identity in G. Then G/C is 0-dimensional. Obviously $w(G/C) \leqslant \mathfrak{m}$, and $w(C) < \mathfrak{m}$, so that Theorems 2.5 and 2.6 apply to G/C and C respectively. Thus again we may conclude that G contains a dense subgroup of cardinal $\mathfrak{n}$.

Remark. The theorem in Hewitt and Ross, 5.38(f), is concerned with a dense set of the desired size; however, it is clear that the group generated by this set also has cardinal $\mathfrak{n}$.

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