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On the density of ranges of generalized divisor functions

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Abstract: The range of the divisor function σ_{-1} is dense in the interval $[1, \infty)$. However, although the range of the function σ_{-2} is a subset of the interval $\left[1, \frac{\pi^2}{6}\right)$, we will see that the range of σ_{-2} is not dense in $\left[1, \frac{\pi^2}{6}\right)$. We begin by generalizing the divisor functions to a class of functions σ_t for all real t. We then define a constant $\eta \approx 1.8877909$ and show that if $r \in (1, \infty)$, then the range of the function σ_{-r} is dense in the interval $[1, \zeta(r))$ if and only if $r \leq \eta$. We end with an open problem.
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1 Introduction

Throughout this paper, we will let $\mathbb{N}$ denote the set of positive integers, and we will let p_i denote the i^{th} prime number.

For any integer t, the divisor function σ_t is a multiplicative arithmetic function defined by $\sigma_t(n) = \sum_{\substack{d|n \\ d>0}} d^t$ for all positive integers n. The value of $\sigma_1(n)$ is the sum of the positive divisors of n, while the value of $\sigma_0(n)$ is simply the number of positive divisors of n.

Another interesting divisor function is σ_{-1}, which is often known as the abundancy index. One may show [2] that the range of σ_{-1} is a subset of the interval $[1,\infty)$ that is dense in $[1,\infty)$. If $t<-1$, then the range of σ_t is a subset of the interval $[1,\zeta(-t))$, where ζ denotes the Riemann zeta function. This is because, for any positive integer n, $\sigma_t(n)=\sum_{\substack{d\mid n\\ d>0}} d^t<\sum_{i=1}^{\infty} i^t=\zeta(-t)$. For example, the range of the function σ_{-2} is a subset of the interval $\left[1,\frac{\pi^2}{6}\right)$. However, it is interesting to note that the range of the function σ_{-2} is not dense in the interval $\left[1,\frac{\pi^2}{6}\right)$. To see this, let n be a positive integer. If $2|n$, then $\sigma_{-2}(n)\geq\frac{1}{1^2}+\frac{1}{2^2}=\frac{5}{4}$. On the other hand, if $2\nmid n$, then $\sigma_{-2}(n)<\sum_{d\in\mathbb{N}\setminus(2\mathbb{N})}\frac{1}{d^2}=\frac{\zeta(2)}{\left(\frac{1}{1-2^{-2}}\right)}=\frac{\pi^2}{8}$. As $\frac{\pi^2}{8}<\frac{5}{4}$, we see that there is a "gap" in the range of σ_{-2}. In other words, there are no positive integers n such that $\sigma_{-2}(n)\in\left(\frac{\pi^2}{8},\frac{5}{4}\right)$.

Our first goal is to generalize the divisor functions to allow for nonintegral subscripts. For example, we might consider the function $\sigma_{-\sqrt{2}}$, defined by $\sigma_{-\sqrt{2}}(n)=\sum_{\substack{d\mid n\\ d>0}} d^{-\sqrt{2}}$. We formalize this idea in the following definition.

Definition 1.1. For a real number t, define the function $\sigma_t\colon\mathbb{N}\to\mathbb{R}$ by $\sigma_t(n)=\sum_{\substack{d\mid n\\ d>0}} d^t$ for all $n\in\mathbb{N}$. Also, we will let $\log\sigma_t=\log\circ\sigma_t$.

In analyzing the ranges of these generalized divisor functions, we will find a constant which serves as a "boundary" between divisor functions with dense ranges and divisor functions with ranges that have gaps. Note that, for any real number t, we may write $\sigma_t=I_0*I_t$, where I_0 and I_t are arithmetic functions defined by $I_0(n)=1$ and $I_t(n)=n^t$. As I_0 and I_t are multiplicative, we find that σ_t is multiplicative.

2 The ranges of functions σ_{-r}

Theorem 2.1. *Let r be a real number greater than 1. The range of σ_{-r} is dense in the interval $[1,\zeta(r))$ if and only if $1+\frac{1}{p_m^r}\leq\prod_{i=m+1}^{\infty}\left(\sum_{j=0}^{\infty}\frac{1}{p_i^{jr}}\right)$ for all positive integers m.*

Proof. First, suppose that $1+\frac{1}{p_m^r}\leq\prod_{i=m+1}^{\infty}\left(\sum_{j=0}^{\infty}\frac{1}{p_i^{jr}}\right)$ for all positive integers m. We will show that the range of $\log\sigma_{-r}$ is dense in the interval $[0,\log(\zeta(r)))$, which will imply that the range of σ_{-r} is dense in $[1,\zeta(r))$. Choose some arbitrary $x\in(0,\log(\zeta(r)))$, and define $X_0=0$. For each positive integer n, we define α_n and X_n in the following manner. If $X_{n-1} + \log\left(\sum_{j=0}^{\infty} \frac{1}{p_n^{jr}}\right) \leq x$, define $\alpha_n = -1$. If $X_{n-1} + \log\left(\sum_{j=0}^{\infty} \frac{1}{p_n^{jr}}\right) > x$, define α_n to be the largest nonnegative integer that satisfies $X_{n-1} + \log\left(\sum_{j=0}^{\alpha_n} \frac{1}{p_n^{jr}}\right) \leq x$. Define X_n by

$$X_n = \begin{cases} X_{n-1} + \log\left(\sum_{j=0}^{\alpha_n} \frac{1}{p_n^{jr}}\right), & \text{if } \alpha_n \geq 0; \\ X_{n-1} + \log\left(\sum_{j=0}^{\infty} \frac{1}{p_n^{jr}}\right), & \text{if } \alpha_n = -1. \end{cases}$$

Also, for each $n \in \mathbb{N}$, define D_n by

$$D_n = \begin{cases} \log\left(\sum_{j=0}^{\infty} \frac{1}{p_n^{jr}}\right) - \log\left(\sum_{j=0}^{\alpha_n} \frac{1}{p_n^{jr}}\right), & \text{if } \alpha_n \geq 0; \\ 0, & \text{if } \alpha_n = -1, \end{cases}$$

and let $E_n = \sum_{i=1}^{n} D_i$. Note that

$$\lim_{n\to\infty}(X_n + E_n) = \lim_{n\to\infty}\left(X_n + \sum_{i=1}^{n} D_i\right)$$
$$= \lim_{n\to\infty}\sum_{i=1}^{n}\log\left(\sum_{j=0}^{\infty}\frac{1}{p_i^{jr}}\right) = \log(\zeta(r)).$$

Now, because the sequence $(X_n)_{n=1}^{\infty}$ is bounded and monotonic, we know that there exists some real number γ such that $\lim_{n\to\infty} X_n = \gamma$. We wish to show that $\gamma = x$.

Notice that we defined the sequence $(X_n)_{n=1}^{\infty}$ so that $X_n \leq x$ for all $n \in \mathbb{N}$. Hence, we know that $\gamma \leq x$. Now, suppose $\gamma < x$. Then $\lim_{n\to\infty} E_n = \log(\zeta(r)) - \gamma > \log(\zeta(r)) - x$. This implies that there exists some positive integer N such that $E_n > \log(\zeta(r)) - x$ for all integers $n \geq N$. Let m be the smallest positive integer that satisfies $E_m > \log(\zeta(r)) - x$. If $\alpha_m = -1$ and $m > 1$, then $D_m = 0$, so $E_{m-1} = E_m > \log(\zeta(r)) - x$. However, this contradicts the minimality of m. If $\alpha_m = -1$ and $m = 1$, then $0 = D_m = E_m > \log(\zeta(r)) - x$, which is also a contradiction. Thus, we conclude that $\alpha_m \geq 0$. This means that $X_m + D_m = X_{m-1} + \log\left(\sum_{j=0}^{\infty}\frac{1}{p_m^{jr}}\right) > x$, so $D_m > x - X_m$. Furthermore,

$$\log\left(\prod_{i=m+1}^{\infty}\left(\sum_{j=0}^{\infty}\frac{1}{p_i^{jr}}\right)\right) = \sum_{i=m+1}^{\infty}\log\left(\sum_{j=0}^{\infty}\frac{1}{p_i^{jr}}\right)$$
$$= \log(\zeta(r)) - \sum_{i=1}^{m}\log\left(\sum_{j=0}^{\infty}\frac{1}{p_i^{jr}}\right)$$

$$= \log(\zeta(r)) - E_m - X_m < x - X_m < D_m, \tag{1}$$

and we originally assumed that $1 + \frac{1}{p_m^r} \leq \prod_{i=m+1}^{\infty} \left(\sum_{j=0}^{\infty} \frac{1}{p_i^{jr}} \right)$. This means that $\log\left(1 + \frac{1}{p_m^r}\right) < D_m = \log\left(\sum_{j=0}^{\infty} \frac{1}{p_m^{jr}}\right) - \log\left(\sum_{j=0}^{\alpha_m} \frac{1}{p_m^{jr}}\right)$, or, equivalently, $\log\left(1 + \frac{1}{p_m^r}\right) + \log\left(\sum_{j=0}^{\alpha_m} \frac{1}{p_m^{jr}}\right) < \log\left(\frac{p_m^r}{p_m^r - 1}\right)$. If $\alpha_m > 0$, we have

$$\log\left(\left(1 + \frac{1}{p_m^r}\right)^2\right) \leq \log\left(1 + \frac{1}{p_m^r}\right) + \log\left(\sum_{j=0}^{\alpha_m} \frac{1}{p_m^{jr}}\right) < \log\left(\frac{p_m^r}{p_m^r - 1}\right),$$

so $\left(1 + \frac{1}{p_m^r}\right)^2 < \frac{p_m^r}{p_m^r - 1}$. We may write this as $1 + \frac{2}{p_m^r} + \frac{1}{p_m^{2r}} < 1 + \frac{1}{p_m^r - 1}$, so $2 < \frac{p_m^r}{p_m^r - 1} = 1 + \frac{1}{p_m^r - 1}$. As $p_m^r > 2$, this is a contradiction. Hence, $\alpha_m = 0$. By the definitions of α_m and X_m, this implies that $X_{m-1} + \log\left(1 + \frac{1}{p_m^r}\right) > x$ and that $X_m = X_{m-1}$. Therefore, $\log\left(1 + \frac{1}{p_m^r}\right) > x - X_{m-1} = x - X_m$. However, recalling from (1) that

$$\sum_{i=m+1}^{\infty} \log\left(\sum_{j=0}^{\infty} \frac{1}{p_i^{jr}}\right) < x - X_m,$$

we find that

$$\sum_{i=m+1}^{\infty} \log\left(\sum_{j=0}^{\infty} \frac{1}{p_i^{jr}}\right) < \log\left(1 + \frac{1}{p_m^r}\right),$$

which is a contradiction because we originally assumed that $1 + \frac{1}{p_m^r} \leq \prod_{i=m+1}^{\infty} \left(\sum_{j=0}^{\infty} \frac{1}{p_i^{jr}} \right)$. Therefore, $\gamma = x$.

We now know that $\lim_{n\to\infty} X_n = x$. To show that the range of $\log \sigma_{-r}$ is dense in $[0, \log(\zeta(r)))$, we need to construct a sequence $(C_n)_{n=1}^{\infty}$ of elements of the range of $\log \sigma_{-r}$ that satisfies $\lim_{n\to\infty} C_n = x$. We do so in the following fashion. For each positive integer n, write

$$Y_n = \begin{cases} 1, & \text{if } \alpha_n \geq 0; \\ 0, & \text{if } \alpha_n = -1, \end{cases}$$

$$Z_n = \begin{cases} 0, & \text{if } \alpha_n \geq 0; \\ 1, & \text{if } \alpha_n = -1, \end{cases}$$

and

$$\beta_n = \begin{cases} \alpha_n, & \text{if } \alpha_n \geq 0; \\ 0, & \text{if } \alpha_n = -1. \end{cases}$$

Now, for each positive integer n, define C_n by

$$C_n = \sum_{k=1}^{n} \left(Y_k \log \left(\sum_{j=0}^{\beta_k} \frac{1}{p_k^{jr}} \right) + Z_k \log \left(\sum_{j=0}^{n} \frac{1}{p_k^{jr}} \right) \right).$$

Notice that, by the way we defined X_n, we have

$$X_n = \sum_{k=1}^{n} \left(Y_k \log \left(\sum_{j=0}^{\beta_k} \frac{1}{p_k^{jr}} \right) + Z_k \log \left(\sum_{j=0}^{\infty} \frac{1}{p_k^{jr}} \right) \right).$$

Therefore, $\lim_{n\to\infty} C_n = \lim_{n\to\infty} X_n = x$. All we need to do now is show that each C_n is in the range of $\log \sigma_{-r}$. We have

$$\begin{aligned}
C_n &= \sum_{k=1}^{n} \left(Y_k \log \left(\sum_{j=0}^{\beta_k} \frac{1}{p_k^{jr}} \right) + Z_k \log \left(\sum_{j=0}^{n} \frac{1}{p_k^{jr}} \right) \right) \\
&= \sum_{\substack{k\in\mathbb{N} \\ k\le n \\ \alpha_k \ge 0}} \log \left(\sum_{j=0}^{\alpha_k} \frac{1}{p_k^{jr}} \right) + \sum_{\substack{k\in\mathbb{N} \\ k\le n \\ \alpha_k = -1}} \log \left(\sum_{j=0}^{n} \frac{1}{p_k^{jr}} \right) \\
&= \log \left[\left(\prod_{\substack{k\in\mathbb{N} \\ k\le n \\ \alpha_k \ge 0}} \sigma_{-r}(p_k^{\alpha_k}) \right) \left(\prod_{\substack{k\in\mathbb{N} \\ k\le n \\ \alpha_k = -1}} \sigma_{-r}(p_k^{n}) \right) \right] \\
&= \log \sigma_{-r} \left(\left(\prod_{\substack{k\in\mathbb{N} \\ k\le n \\ \alpha_k \ge 0}} p_k^{\alpha_k} \right) \left(\prod_{\substack{k\in\mathbb{N} \\ k\le n \\ \alpha_k \ge 0}} p_k^{n} \right) \right).
\end{aligned}$$

We finally conclude that if $1 + \frac{1}{p_m^r} \le \prod_{i=m+1}^{\infty} \left(\sum_{j=0}^{\infty} \frac{1}{p_i^{jr}} \right)$ for all positive integers m, then the range of σ_{-r} is dense in the interval $[1, \zeta(r))$.

Conversely, suppose that there exists some positive integer m such that $1 + \frac{1}{p_m^r} > \prod_{i=m+1}^{\infty} \left(\sum_{j=0}^{\infty} \frac{1}{p_i^{jr}} \right)$. Fix some $N \in \mathbb{N}$, and let $N = \prod_{i=1}^{v} q_i^{\gamma_i}$ be the canonical prime factorization of N. If $p_s | N$ for some $s \in \{1, 2, \ldots, m\}$, then

$$\sigma_{-r}(N) \ge 1 + \frac{1}{p_s^r} \ge 1 + \frac{1}{p_m^r}.$$

On the other hand, if $p_s \nmid N$ for all $s \in \{1, 2, \ldots, m\}$, then

$$\sigma_{-r}(N) = \prod_{i=1}^{v} \sigma_{-r}(q_i^{\gamma_i}) = \prod_{i=1}^{v} \left(\sum_{j=0}^{\gamma_i} \frac{1}{q_i^{jr}} \right)$$

$$< \prod_{i=1}^{v}\left(\sum_{j=0}^{\infty}\frac{1}{q_i^{jr}}\right) < \prod_{i=m+1}^{\infty}\left(\sum_{j=0}^{\infty}\frac{1}{p_i^{jr}}\right).$$

Because N was arbitrary, this shows that there is no element of the range of σ_{-r} in the interval $\left[\prod_{i=m+1}^{\infty}\left(\sum_{j=0}^{\infty}\frac{1}{p_i^{jr}}\right), 1+\frac{1}{p_m^r}\right)$, which means that the range of σ_{-r} is not dense in $[1,\zeta(r))$. □

Theorem 2.1 provides us with a method to determine values of $r > 1$ with the property that the range of σ_{-r} is dense in $[1,\zeta(r))$. However, doing so is still a somewhat difficult task. Luckily, for $r \in (1,2]$, we may greatly simplify the problem with the help of the following theorem. First, we need a short lemma.

Lemma 2.1. *If* $j \in \mathbb{N}\backslash\{1,2,4\}$*, then* $\frac{p_{j+1}}{p_j} < \sqrt{2}$.

Proof. Pierre Dusart [1] has shown that, for $x \geq 396\,738$, there must be at least one prime in the interval $\left[x, x+\frac{x}{25\log^2 x}\right]$. Therefore, whenever $p_j > 396\,738$, we may set $x = p_j + 1$ to get $p_{j+1} \leq (p_j+1) + \frac{p_j+1}{25\log^2(p_j+1)} < \sqrt{2}p_j$. Using Mathematica 9.0 [3], we may quickly search through all the primes less than $396\,738$ to conclude the desired result. □

Theorem 2.2. *Let* r *be a real number in the interval* $(1,2]$*. The range of* σ_{-r} *is dense in the interval* $[1,\zeta(r))$ *if and only if* $1+\frac{1}{p_m^r} \leq \prod_{i=m+1}^{\infty}\left(\sum_{j=0}^{\infty}\frac{1}{p_i^{jr}}\right)$ *for all* $m \in \{1,2,4\}$.

Proof. Let $F(m,r) = \left(1+\frac{1}{p_m^r}\right)\prod_{i=1}^{m}\left(\sum_{j=0}^{\infty}\frac{1}{p_i^{jr}}\right)$ so that the inequality $1+\frac{1}{p_m^r} \leq \prod_{i=m+1}^{\infty}\left(\sum_{j=0}^{\infty}\frac{1}{p_i^{jr}}\right)$ is equivalent to $F(m,r) \leq \zeta(r)$. In light of Theorem 2.1, it suffices to show that if $F(m,r) \leq \zeta(r)$ for all $m \in \{1,2,4\}$, then $F(m,r) \leq \zeta(r)$ for all $m \in \mathbb{N}$. Thus, let us assume that r is such that $F(m,r) \leq \zeta(r)$ for all $m \in \{1,2,4\}$. If $m \in \mathbb{N}\backslash\{1,2,4\}$, then Lemma 2.1 tells us that $\frac{p_{m+1}}{p_m} < \sqrt{2} \leq \sqrt[r]{2}$, which implies that $\frac{2}{p_{m+1}^r} > \frac{1}{p_m^r}$. We then have

$$F(m+1,r) = \left(1+\frac{1}{p_{m+1}^r}\right)\prod_{i=1}^{m+1}\left(\sum_{j=0}^{\infty}\frac{1}{p_i^{jr}}\right) > \left(1+\frac{1}{p_{m+1}^r}\right)^2\prod_{i=1}^{m}\left(\sum_{j=0}^{\infty}\frac{1}{p_i^{jr}}\right)$$

$$> \left(1+\frac{2}{p_{m+1}^r}\right)\prod_{i=1}^{m}\left(\sum_{j=0}^{\infty}\frac{1}{p_i^{jr}}\right) > \left(1+\frac{1}{p_m^r}\right)\prod_{i=1}^{m}\left(\sum_{j=0}^{\infty}\frac{1}{p_i^{jr}}\right) = F(m,r)$$

for all $m \in \mathbb{N}\backslash\{1,2,4\}$. This means that $F(3,r) < F(4,r) \leq \zeta(r)$. Furthermore, $F(m,r) < \zeta(r)$ for all integers $m \geq 5$ because $(F(m,r))_{m=5}^{\infty}$ is a strictly increasing sequence and $\lim_{m\to\infty} F(m,r) = \zeta(r)$. □

We have seen that, for $r \in (1, 2]$, the range of σ_{-r} is dense in $[1, \zeta(r))$ if and only if $F(m, r) \leq \zeta(r)$ for all $m \in \{1, 2, 4\}$. Using Mathematica 9.0, one may plot a function $g_m(r) = F(m, r) - \zeta(r)$ for each $m \in \{1, 2, 4\}$. It is then easy to verify that g_2 has precisely one root, say η, in the interval $(1, 2]$ (for anyone seeking a more rigorous proof of this fact, we mention that it is fairly simple to show that $g_2'(r) > 0$ for all $r \in (1, 2]$). Furthermore, one may confirm that $g_1(r), g_2(r), g_4(r) \leq 0$ for all $r \in (1, \eta]$ and that $g_2(r) > 0$ for all $r \in (\eta, 3]$. Hence, we have proven (or at least left the reader to verify) the first part of the following theorem.

Theorem 2.3. *Let η be the unique number in the interval $(1, 2]$ that satisfies the equation*

$$\left(\frac{2^\eta}{2^\eta - 1}\right)\left(\frac{3^\eta + 1}{3^\eta - 1}\right) = \zeta(\eta).$$

If $r \in (1, \infty)$, then the range of the function σ_{-r} is dense in the interval $[1, \zeta(r))$ if and only if $r \leq \eta$.

Proof. In virtue of the preceding paragraph, we know from the fact that

$$g_2(\eta) = F(2, \eta) - \zeta(\eta) = \left(\frac{2^\eta}{2^\eta - 1}\right)\left(\frac{3^\eta + 1}{3^\eta - 1}\right) - \zeta(\eta) = 0$$

that if $r \in (1, 3]$, then the range of σ_{-r} is dense in $[1, \zeta(r))$ if and only if $r \leq \eta$. We now show that the range of σ_{-r} is not dense in $[1, \zeta(r))$ if $r > 3$. To do so, we merely need to show that $F(1, r) > \zeta(r)$ for all $r > 3$. For $r > 3$, we have

$$F(1, r) = \left(1 + \frac{1}{2^r}\right)\sum_{j=0}^{\infty}\frac{1}{2^{jr}} > \left(1 + \frac{1}{2^r}\right)^2 = 1 + \frac{1}{2^r} + \frac{3}{4}\left(\frac{1}{2^{r-1}}\right)$$

$$> 1 + \frac{1}{2^r} + \frac{1}{(r-1)2^{r-1}} = 1 + \frac{1}{2^r} + \int_2^\infty \frac{1}{x^r}dx > \zeta(r).$$

□

3 An open problem

We end by acknowledging that it might be of interest to consider the number of "gaps" in the range of σ_{-r} for various r. For example, for which values of $r \in (1, \infty)$ is there precisely one gap in the range of σ_{-r}? More generally, if we are given a positive integer L, then, for what values of $r > 1$ is the closure of the range of σ_{-r} a union of exactly L disjoint subintervals of $[1, \zeta(r)]$?

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