Fall 2: Für $\mathfrak{P}^e = 7S$ mit $7 | e$ ist wegen $\overline{w}_i \in D(K)$ und (*) in 2.

$$\mathfrak{d}_1(S)\frac{\overline{w}_i}{7} \subset SdS \quad \text{für} \quad i = 1, 2, 3.$$

Für $7 \nmid e$ wird ähnlich wie beim Nachweis von $\overline{w}_i \in D(K)$ argumentiert: Da $\frac{u}{\sigma^3}$ oder $\frac{\sigma^3}{u}$ in S liegt, gilt wegen (i) und (ii)

$$\mathfrak{d}_1(S)\frac{\overline{w}_1}{7} = \mathfrak{d}_1(S)\frac{w_1}{\sigma^3} = \mathfrak{d}_1(S)\frac{y_1}{7}\cdot\frac{\sigma^3}{u}\cdot\frac{dy_1}{y_1} = \mathfrak{d}_1(S)\frac{u}{\sigma^3}\cdot\frac{du}{u} \subset SdS.$$

Wie bei der Berechnung von $D(K)$ sind jetzt nur noch die S mit $z, v_1 \in \mathfrak{P}, (x_1 - 3y_1) \sim y_1^2$ interessant. Und es genügt auch hier zu zeigen, daß für diese S

$$\mathfrak{d}_1(S)\frac{\overline{w}_3}{7} \subset SdS$$

gilt. Das aber ist wegen

$$\mathfrak{d}_1(S)\frac{\overline{w}_3}{7} = \frac{\mathfrak{d}_1(S)}{7}v_1 z dz$$

nach (ii) der Fall.

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On the density of the zeros of the Dedekind Zeta-function

by

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1. Introduction. The Dedekind Zeta-function is defined by

$$\zeta_K(s) = \sum 1/(N\mathfrak{a})^s$$

where K is an algebraic number field, the summation is over all integral ideals $\mathfrak{a}$ of K, $N\mathfrak{a}$ denotes the norm of $\mathfrak{a}$ and s is an arbitrary complex variable. The series converges absolutely for $\sigma > 1$, where $s = \sigma + it$ in the usual notation; moreover the function has an analytic continuation to the whole complex plane, from which one sees in particular, that it is regular except for a simple pole at $s = 1$. The Dedekind Zeta-function is a generalization of the Riemann Zeta-function, $\zeta(s)$, which is $\zeta_K(s)$ with $K = \boldsymbol{Q}$, the rational field; as is well known the Riemann Zeta-function gives information about the distribution of the rational primes, and the Dedekind Zeta-function can be used similarly to furnish results on the distribution of the prime ideals in K. For the basic properties of $\zeta_K(s)$ we refer to Landau's tract [11].

Our main object is to establish an estimate for the density of zeros of the Dedekind Zeta-function in the range $\frac{1}{2} \leqslant \sigma \leqslant 1$, which is better than any given hitherto. Let $N_K(\sigma, T)$ be the number of zeros $\varrho = \beta + i\gamma$ of $\zeta_K(s)$ with $|\gamma| \leqslant T$, $\beta \geqslant \sigma$, the zeros being counted according to multiplicity. It is well known, see [11], that

$$N_K(0, T) \sim \frac{k}{\pi} T \log T \quad \text{as} \quad T \to \infty, \tag{1}$$

where k is the degree of K, and so in particular

$$N_K(\tfrac{1}{2}, T) \ll T \log T, \tag{2}$$

where, as later, constants implied by $\ll$ (or $\gg$) depend only on K. We shall prove

THEOREM 1. *For any $\varepsilon > 0$ there exists $C = C(\varepsilon, K)$ such that*

$$N_K(\sigma, T) \ll T^{(k+\varepsilon)(1-\sigma)}(\log T)^C$$

uniformly for $\frac{1}{2} \leqslant \sigma \leqslant 1$, where $k \geqslant 3$.

The theorem improves upon a result of Sokolovsky [13], who was apparently the first to apply Ingham's techniques [7] to the Dedekind Zeta-function. He proved in fact that Theorem 1 holds with the weaker exponent

$$k+2-C/\big(k^2\log(k+2)\big)$$

in place of $k+\varepsilon$, C here being a positive absolute constant. Sokolovsky's result applied also to the cases $k = 1$ and $k = 2$, and here again we now have stronger estimates.

THEOREM 2. *For any quadratic field K, and for any $\varepsilon > 0$, there exists $C = C(\varepsilon, K)$ such that*

$$N_K(\sigma, T) \ll T^{2(1-\sigma)/\sigma}(\log T)^C$$

uniformly for $\frac{3}{4} \leqslant \sigma \leqslant 1-\varepsilon$; and

$$N_K(\sigma, T) \ll T^{4(1-\sigma)/(3-2\sigma)}(\log T)^C$$

uniformly for $\frac{1}{2}+\varepsilon \leqslant \sigma \leqslant \frac{3}{4}$.

Note that (2) gives

$$N_K(\sigma, T) \ll T^{(2+6\varepsilon)(1-\sigma)}$$

uniformly for $\frac{1}{2} \leqslant \sigma \leqslant \frac{1}{2}+\varepsilon$, if $0 < \varepsilon < 1/6$. Further, we shall prove below as a special case of Theorem 3 that, for quadratic fields K

$$N_K(\sigma, T) \ll T^{(2+\varepsilon)(1-\sigma)}(\log T)^C$$

uniformly for $1-\varepsilon \leqslant \sigma \leqslant 1$, if $0 < \varepsilon < 1/10$. Thus Theorem 2 implies, in particular, that Theorem 1 holds, in the case $k = 2$, with the exponent $k+\varepsilon$ replaced by $8/3$. In the case $k = 1$, that is for the Riemann Zeta-function, results of Ingham [7], Montgomery [12], and Huxley [6] show that Theorem 1 holds with $k+\varepsilon$ replaced by $12/5$ and this is the best result obtained to date.

It is clear from (1), together with the symmetry of the zeros about the line $\sigma = \frac{1}{2}$, that one cannot replace the exponent $k+\varepsilon$ in Theorem 1 by any number less than 2. This leads one to conjecture that, for all k, and for any $\varepsilon > 0$

$$N_K(\sigma, T) \ll T^{(2+\varepsilon)(1-\sigma)}(\log T)^C \tag{3}$$

uniformly for $\frac{1}{2} \leqslant \sigma \leqslant 1$, where C depends only on K and ε. This can be regarded as a density hypothesis for the Dedekind Zeta-function in analogy to the familiar conjecture made in the rational case. At present

it is known [9] that the density hypothesis holds in the case $k = 1$ for $\sigma \geqslant 11/14$, and we shall prove here that it holds in the case $k = 2$ for $\sigma \geqslant 111/124 = 0.895\ldots$

THEOREM 3. *For quadratic fields K, (3) holds uniformly for $\sigma \geqslant \frac{3}{4}(1-\mu)^{-1}$ where μ, ν are absolute constants such that*

$$\zeta(\tfrac{1}{2}+it) \ll t^{\mu}(\log t)^{\nu} \quad (t \geqslant 2) \tag{4}$$

and C depends only on μ, ν, ε and K.

Haneke [4] has shown that (4) holds with $\mu = 6/37$, $\nu = 1$, and this gives the lower bound for σ, in the case $k = 2$, quoted above. We are unable, however, to prove a result of the same precision as (3) for fields of degree $k \geqslant 3$ even over a restricted range of σ.

We can use Theorems 1 and 2 to give a result about prime ideals in K. In analogy with the prime number theorem one has (see [11])

$$\pi_K(x) \sim \frac{x}{\log x} \quad \text{as} \quad x \to \infty,$$

where $\pi_K(x)$ denotes the number of prime ideals with norm $\leqslant x$. We give here an asymptotic formula for the number of prime ideals whose norms lie in an interval; the theorem improves upon an earlier result of Sokolovsky [13].

THEOREM 4. *Suppose that $1 \geqslant a > 1-1/k$ if $k \geqslant 3$ and $1 \geqslant a > 5/8$ if $k = 2$. Then*

$$\pi_K(x+y) - \pi_K(x) \sim \frac{y}{\log x} \quad \text{as} \quad x \to \infty$$

uniformly for $x^a \leqslant y \leqslant x$. In particular when $x \geqslant x(a)$, there exists a prime ideal $\mathfrak{p}$ of K with $x < N\mathfrak{p} \leqslant x+x^a$.

The proofs of Theorems 1, 2 and 3 depend on the techniques of Montgomery [12], together with the later developments of Jutila [8]. In addition we shall use the mean-value theorem for $|\zeta_K(s)|^2$ at $\sigma = 1-1/k$ as obtained by Chandrasekharan and Narasimhan [3], and also, to deal with Theorem 3, a modified form of the lemma of Halász [6]. Theorem 4 is an immediate consequence of Theorems 1 and 2 above, together with Theorem 3 of Sokolovsky [13].

In the final section we shall indicate how, on the assumption of Artin's conjecture for non-abelian L-functions, the estimates established here can be improved somewhat; in fact our considerations lead to some unconditional improvements for small values of k.

2. Preliminaries. We record here some properties of the Dedekind Zeta-function that will be required frequently later.

First (see [11], Satz 155), we have

$$\zeta_K(\sigma+it) \ll |t|^A \tag{5}$$

in the region $\sigma \geqslant \frac{1}{4}$, $|t| \geqslant 2$, where A depends only on K; also (ibid., Satz 171)

$$N_K(\tfrac{1}{2}, T+1) - N_K(\tfrac{1}{2}, T) \ll \log T \qquad (T \geqslant 2). \tag{6}$$

Further we note that, for $\sigma > 1$ we can write

$$\zeta_K(s) = \sum_{n=1}^{\infty} a_n/n^s, \qquad 1/\zeta_K(s) = \sum_{n=1}^{\infty} b_n/n^s \tag{7}$$

where $a_1 = b_1 = 1$ and $|b_n| \leqslant a_n \leqslant \big(d(n)\big)^k$, $d(n)$ being the number of divisors of n. Indeed by the Euler product (see [11], Satz 141) $b_n = \sum(-1)^\nu$ where the sum is over all representations of n as $N(\mathfrak{p}_1\mathfrak{p}_2\ldots\mathfrak{p}_\nu)$ with distinct prime ideals $\mathfrak{p}_i$ of K, and a_n is the number of representations of n in the above form, where now the $\mathfrak{p}_i$ are not necessarily distinct; this gives $|b_n| \leqslant a_n$. Further since both a_n and $\big(d(n)\big)^k$ are multiplicative it suffices to prove $a_n \leqslant \big(d(n)\big)^k$ for $n = p^e$, where p is a prime; now if $\mathfrak{p}_1 \ldots \mathfrak{p}_m$ are the distinct prime ideals dividing p, and $N\mathfrak{p}_i = p^{e_i}$, then $m \leqslant k$ and plainly a_n is the number of solutions of

$$e = e_1x_1 + \ldots + e_mx_m$$

in non-negative integers $x_1, \ldots, x_m$, whence, since $x_i \leqslant e$ we have

$$a_n \leqslant (e+1)^m = \big(d(p^e)\big)^m \leqslant \big(d(n)\big)^k$$

as required.

We shall need also two results which show that a Dirichlet polynomial

$$S(s) = \sum_{n=N+1}^{2N} a_n n^{-s} \qquad (N \geqslant 2),$$

cannot be large at too many well-spaced points. Suppose that $\mathscr{S}$ is a finite set of complex numbers $s = \sigma+it$ satisfying $\sigma \geqslant \sigma_0$, $T_0 \leqslant t \leqslant T_0+T$ for some σ_0, T_0 and $T \geqslant 2$, and satisfying also the spacing condition $|t-t'| \geqslant 1$ for all distinct s, s' in $\mathscr{S}$. Suppose further that $V > 0$ and that $|S(s)| \geqslant V$ for all s in $\mathscr{S}$. Then by Theorem 7.5 of [12] with $Q = 1$, $\chi = 1$, $\delta = 1$, we have

$$|\mathscr{S}| \ll (T+N)\, UV^{-2}\log N, \tag{8}$$

where $|\mathscr{S}|$ denotes the number of elements in $\mathscr{S}$ and, for brevity, we have written

$$U = \sum_{n=N+1}^{2N} |a_n|^2 n^{-2\sigma_0}.$$

The second result is a modified form of the lemma of Halász referred to in § 1.

LEMMA. *Suppose μ, ν are positive constants such that*

$$\zeta(\sigma+it) \ll |t|^\mu (\log|t|)^\nu$$

uniformly for $|t| \geqslant 2$, $\sigma \geqslant \theta$, where $0 \leqslant \theta \leqslant 1$. Then

$$|\mathscr{S}| \ll NUV^{-2} + TN^{1+\theta/\mu}U^{1+1/\mu}V^{-2-2/\mu}(\log T)^{\nu/\mu}.$$

Proof. We shall apply Theorem 8.4 of [12] to a subinterval of $[T_0, T_0+T]$ of length T'; denoting by $\mathscr{S}'$ the subset of $\mathscr{S}$ consisting of points with t in the subinterval, this gives

$$|\mathscr{S}'| \leqslant C\big(N + |\mathscr{S}'|N^\theta T'^\mu(\log T)^\nu\big)\, UV^{-2}$$

for some constant C. Thus $|\mathscr{S}'| \leqslant 2CNUV^{-2}$, provided that

$$T'^\mu \leqslant (2C)^{-1}N^{-\theta}U^{-1}V^2(\log T)^{-\nu}. \tag{9}$$

If (9) holds for $T' = T$ then we can take $\mathscr{S}' = \mathscr{S}$ and the lemma is proved. Otherwise we divide the interval $[T_0, T_0+T]$ into m subintervals of length T/m, where m $(\geqslant 2)$ is the least integer such that (9) holds with $T' = T/m$. This gives

$$m-1 \leqslant T\{2CN^\theta UV^{-2}(\log T)^\nu\}^{1/\mu}.$$

Now $|\mathscr{S}'| \ll NUV^{-2}$ for each subinterval, and so $|\mathscr{S}| \ll mNUV^{-2}$; this proves the lemma.

3. Basic density estimates. Our object in this section is to reduce the problem of estimating $N_K(\sigma, T)$ to that of estimating the number of points at which a Dirichlet polynomial of the kind considered in § 2 can be large.

We introduce the function

$$M_X(s) = \sum_{n \leqslant X} b_n n^{-s}$$

where the b_n are the coefficients appearing in (7). Then $\zeta_K(s)M_X(s) \sim 1$ as $X \to \infty$ for $\sigma > 1$ and indeed

$$\zeta_K(s)M_X(s) = \sum_{n=1}^{\infty} C_n n^{-s}$$

where $C_1 = 1$ and $C_n = C_n(X) = 0$ for $2 \leqslant n \leqslant X$. Further, by a well known Mellin transform we have, for any $Y > 0$ and any $\sigma > -1$,

$$\sum_{n=1}^{\infty} C_n n^{-s} e^{-n/Y} = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} f(s, w)\, dw, \tag{10}$$

where $f(s, w) = \zeta_K(s+w)M_X(s+w)\,Y^w\Gamma(w)$. The sum on the left can be written

$$e^{-1/Y}+\sum_{n>X} C_n n^{-s}e^{-n/Y},$$

and here $e^{-1/Y}\sim 1$ as $Y\to\infty$. To estimate the integral in (10) we move the line of integration to $\operatorname{re} w = a$, where $a = a(s) = 1-\sigma-1/k$. Assuming that $1-1/k < \sigma < 1$ we pass simple poles at $w = 0$ and $w = 1-s$, and thus, on denoting by λ the residue of the pole of $\zeta_K(s)$ at $s = 1$, the expression on the right of (10) becomes

$$\zeta_K(s)M_X(s)+\lambda M_X(1)\,Y^{1-s}\Gamma(1-s)+\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} f(s, w)\,dw. \tag{11}$$

We now take $s = \varrho$, a zero of $\zeta_K(s)$; then the first term of (11) vanishes, and, since $|e^{-1/Y}-1|\leqslant\frac14$ for $Y\geqslant 4$, as we now assume, this implies that at least one of the following expressions has absolute value at least $\frac14$:

$$\lambda M_X(1)\,Y^{1-\varrho}\Gamma(1-\varrho);\qquad \frac{1}{2\pi i}\int_{a(\varrho)-i\infty}^{a(\varrho)+i\infty} f(\varrho, w)\,dw;\qquad \sum_{n>X} C_n n^{-\varrho}e^{-n/Y}. \tag{12}$$

Let N_1, N_2, N_3 be the number of distinct $\varrho = \beta+i\gamma$ with $\beta\geqslant\sigma$, $|\gamma|\leqslant T$, for which the expressions in (12) are, in modulus, at least $\frac14$ respectively. Since each zero has multiplicity $\ll \log T$ we have

$$N_K(\sigma, T) \ll (N_1+N_2+N_3)\log T.$$

We proceed to estimate N_1, N_2 and N_3; for this purpose we select $X = T^\eta \leqslant Y\leqslant T^{3k}$, where $0<\eta<1$.

N_1 can be estimated trivially. Since

$$|\Gamma(x+iy)| \ll e^{-|y|}\quad\text{for}\quad |y|\geqslant 1,\ 0\leqslant x\leqslant 1, \tag{13}$$

where the implied constant is absolute, we have

$$|\Gamma(1-\varrho)| \ll e^{-|\gamma|}\quad\text{for}\quad |\varrho|\geqslant 2.$$

Further $|Y^{1-\varrho}|\leqslant T^{3k}$ and, using the estimate $|b_n|\leqslant (d(n))^k\leqslant n^k$ obtained in § 2, we get

$$|M_X(1)|\leqslant \sum_{n\leqslant X}|b_n|n^{-1}\leqslant X^k,$$

whence

$$|\lambda M_X(1)\,Y^{1-\varrho}\Gamma(1-\varrho)| \ll T^{(\eta+3)k}e^{-|\gamma|},$$

where the implied constant depends now on K. Hence, if the expression on the left is at least $\frac14$, it follows that $|\gamma| \ll \log T$ and so, by (6), $N_1\ll(\log T)^2$ which is plainly less than any of the bounds required for $N_K(\sigma, T)$.

To estimate N_2 we note first that there exists $C = C(K, \eta) > 0$ such that

$$\left|\frac{1}{2\pi i}\int_{a(\varrho)+iC\log T}^{a(\varrho)+i\infty} f(\varrho, w)\,dw\right|\leqslant\frac{1}{16},$$

and similarly for the integral with $v < -C\log T$, where $v = \operatorname{im} w$. For we have

$$|M_X(\varrho+w)| \ll \sum_{n\leqslant X}(d(n))^k n^{1/k-1} \ll X,$$

where, as above, the implied constant depends only on K, and, by (5) and (13)

$$|\zeta_K(\varrho+w)\,Y^w\Gamma(w)| \ll |\gamma+v|^A e^{-|v|},$$

hence

$$|f(\varrho, w)|\leqslant e^{-\frac12|v|}$$

if $|v|>C\log T$ with C sufficiently large, and the assertion follows. Now we have $|\Gamma(w)| \ll \log T$ provided that $|w|>1/\log T$ and so

$$|M_X(\varrho+w)\,Y^w\Gamma(w)| \ll XY^{a(\varrho)}\log T.$$

Thus if the second expression in (12) is, in modulus, at least $\frac14$, then

$$\frac{1}{2\pi}\int_{\gamma-C\log T}^{\gamma+C\log T}\left|\zeta_K\left(1-\frac1k+iu\right)\right|du \gg (XY^{a(\varrho)}\log T)^{-1},$$

assuming that $a(\sigma) < -1/\log T$; if the latter condition does not hold, that is, if $\sigma<1-1/k+1/\log T$, then by (2), we see that Theorems 1 and 2 hold trivially. Now by the Cauchy–Schwarz inequality we obtain

$$\int_{\gamma-C\log T}^{\gamma+C\log T}\left|\zeta_K\left(1-\frac1k+iu\right)\right|^2du \gg X^{-2}\,Y^{-2a(\varrho)}(\log T)^{-3},$$

and on summing over all ϱ contributing to N_2 this gives

$$\int_{T-C\log T}^{T+C\log T}\left|\zeta_K\left(1-\frac1k+iu\right)\right|^2 n(u)\,du \gg N_2X^{-2}\,Y^{-2a(\sigma)}(\log T)^{-3}, \tag{14}$$

where $n(u)$ is the number of ϱ with $|\gamma-u|\leqslant C\log T$. By (6) we have $n(u) \ll (\log T)^2$. Further by the mean-value theorem of Chandrasekharan and Narasimhan [3] we have

$$\int_{-T}^{+T}\left|\zeta_K\left(1-\frac1k+iu\right)\right|^2du \ll T(\log T)^k$$

and so the left hand side of (14) is $\ll T(\log T)^{k+2}$. On comparing estimates this gives

$$N_2 \ll X^2 Y^{2a(\sigma)} T(\log T)^{k+5} \ll T^{1+3\eta} Y^{2(1-1/k-\sigma)}. \tag{15}$$

Finally we estimate N_3. First we note that there exists $C = C(K) > 0$ such that

$$\sum_{n\geq CY\log T} |C_n| e^{-n/Y} \leq 1/8;$$

for we have

$$C_n = \sum_{\substack{d|n\\ d\leq X}} b_d a_{n/d}$$

and since $|b_d| \leq (d(n))^k$, $|a_{n/d}| \leq (d(n))^k$ it follows that

$$|C_n| \leq (d(n))^{2k+1} \leq n^{2k+1} < e^{n/(2Y)}$$

provided that $n \gg Y\log T$. Hence we have

$$\Big|\sum_{X<n<CY\log T} C_n n^{-\varrho} e^{-n/Y}\Big| \geq 1/8 \tag{16}$$

for at least N_3 zeros $\varrho = \beta + i\gamma$ with $\beta \geq \sigma$, $|\gamma| \leq T$. We now use an argument due to Jutila. We write (16) in the form

$$|\Sigma_1 + \Sigma_2 + \ldots + \Sigma_m| \geq 1/8;$$

where m is the largest integer with $Y^{1/m} > X$, Σ_l $(2\leq l<m)$ denotes the sum over n with $Y^{1/(l+1)} < n \leq Y^{1/l}$, Σ_1 refers to the range $Y^{1/2}<n<CY\log T$, and Σ_m refers to the remaining range namely $X<n\leq Y^{1/m}$. By Dirichlet's box principle there exists an l such that $|\Sigma_l| \geq 1/(8m)$ for at least N_3/m zeros ϱ. Further since $X = T^\eta \leq Y \leq T^{3k}$ we have $m < 3k/\eta$ and so

$$|\Sigma_l^l| \geq (8m)^{-l} \gg 1 \tag{17}$$

where, as later, the constant implied by $\gg$, or $\ll$ depends only on K and η. Now, on noting that $X^m \geq Y^{m/(m+1)} \geq Y^{1/2}$ we obtain

$$\Sigma_l^l = \sum_{Y^{1/2}<n<CY\log T} d_n n^{-\varrho}$$

for some coefficients d_n. Since, as above, $|C_n| \leq (d(n))^{2k+1}$, and furthermore $e^{-n/Y}<1$, we see that

$$|d_n| \leq (d(n))^{(2k+1)m} d_m(n),$$

where $d_m(n)$ is the number of ways of expressing n as the product of m factors; and on recalling that $m \ll 1$ and $d_m(n) \leq (d(n))^m$ we get $|d_n|$

$\leq (d(n))^\varkappa$ for some $\varkappa$ depending only on η and K. We now write (17) in the form

$$|\Sigma^{(1)}+\Sigma^{(2)}+\ldots+\Sigma^{(j)}| \gg 1,$$

where $\Sigma^{(i)}$ $(1\leq i\leq j)$ is the sum of terms $d_n n^{-\varrho}$ with

$$Y^{1/2}2^{i-1} < n \leq Y^{1/2}2^i$$

and j is the smallest integer such that $Y^{1/2}2^j \geq CY\log T$; here d_n is defined as zero for $n\geq CY\log T$. By Dirichlet's box principle there exists an i such that $|\Sigma^{(i)}| \gg 1/j$ for at least $N_3/(mj)$ zeros ϱ. We write $N = Y^{1/2}2^{i-1}$, for brevity, and put

$$S(\varrho) = \Sigma^{(i)} = \sum_{n=N+1}^{2N} d_n n^{-\varrho}.$$

Then $|S(\varrho)| \geq V$, where $V \gg 1/j$, and since $2^{j-1} \leq CY\log T$ we have $V \gg 1/\log T$. Further we define

$$U = \sum_{n=N+1}^{2N} |d_n|^2 n^{-2\sigma};$$

then from the well known estimate (Hua [5], p. 17)

$$\sum_{n\leq x} (d(n))^r \ll x(\log x)^{2^r-1},$$

together with the bound $|d_n| \leq (d(n))^\varkappa$ established above, we have $U \ll N^{1-2\sigma}(\log T)^C$ where $C = 4^\varkappa - 1$. Finally we select a subset $\mathscr{S}$ of the $N_3/(mj)$ zeros ϱ, such that $|\gamma-\gamma'|\geq 1$ for distinct ϱ, ϱ' in $\mathscr{S}$, and $|\mathscr{S}| \gg N_3/(mj\log T)$; this is possible by (6).

We are now in a position to apply the results of § 2. First, since $m\ll 1$ and $j \ll \log T$, we have from (8)

$$N_3 \ll |\mathscr{S}|(\log T)^2 \ll (T+N)N^{1-2\sigma}(\log T)^{C+5}. \tag{18}$$

Secondly on taking $\theta = 0$, $\mu = \frac{1}{2}$, $\nu = 1$, in the lemma of § 2, as is possible (see Titchmarsh [14], p. 20), we obtain

$$N_3 \ll (N^{2-2\sigma} + TN^{4-6\sigma})(\log T)^{3C+10}. \tag{19}$$

Thirdly on taking $\theta = \frac{1}{2}$ in the lemma we get

$$N_3 \ll (N^{2-2\sigma}+TN^{2-2\sigma-(4\sigma-3)/(2\mu)})(\log T)^{C'} \tag{20}$$

for some C' depending only on K and η. The estimate (19) will be used to prove Theorem 1, (18) and (19) for Theorem 2, and (20) for Theorem 3.

4. Proof of the theorems. It remain to choose Y so that the estimates for N_2+N_3 are as sharp as possible; we recall that there is no longer any need to consider N_1. We recall also that we can suppose, for the proof of Theorem 1 that $\sigma \geqslant 2/3$, since it holds trivially if $\sigma < 1-1/k$.

Since $Y^{1/2} \leqslant N \ll Y \log T$ we have

$$N^{2-2\sigma} \ll Y^{2-2\sigma} \log T, \quad N^{4-6\sigma} \ll Y^{2-3\sigma}.$$

Thus (15) and (19) give

$$N_2+N_3 \ll T^{1+3\eta} Y^{2(1-1/k-\sigma)} + (Y^{2-2\sigma}+TY^{2-3\sigma})(\log T)^{3C+11}. \tag{21}$$

We now take $Y = T^{(1+3\eta)k/2}$; then

$$Y^{2-2\sigma} = T^{1+3\eta} Y^{2(1-1/k-\sigma)} = T^{(1+3\eta)(1-\sigma)k}.$$

Further, for $\sigma \geqslant 2/3$, $k \geqslant 3$, we have $TY^{-\sigma} \leqslant 1$. Hence we obtain

$$N_2+N_3 \ll T^{(1+3\eta)(1-\sigma)k} (\log T)^{3C+11},$$

and Theorem 1 follows on taking $\eta = \varepsilon/(3k)$.

For the proof of Theorem 2 we assume first that $\sigma \geqslant 2/3$ and take $Y = T^{1/\sigma}$ in (21), then

$$Y^{2-2\sigma} = TY^{2-3\sigma} = T^{2(1-\sigma)/\sigma}.$$

Further, in the case $k = 2$, we have $T^{1+3\eta} Y^{-2/k} \leqslant 1$, provided that $1+3\eta < 1/\sigma$, and the condition certainly holds if $\sigma \leqslant 1-\varepsilon$ and $\eta = \varepsilon/3$; this establishes the first part of Theorem 2. For the second part of Theorem 2, that is for $\sigma \leqslant 3/4$, we use (15) with $k = 2$ and (18); these give

$$N_2+N_3 \ll T^{1+3\eta} Y^{1-2\sigma} + (Y^{2-2\sigma}+TY^{1/2-\sigma})(\log T)^{C+6}.$$

On taking $Y = T^{2/(3-2\sigma)}$ we obtain

$$Y^{2-2\sigma} = TY^{1/2-\sigma} = T^{4(1-\sigma)/(3-2\sigma)};$$

further $T^{1+3\eta} Y^{-1} \leqslant 1$ provided that $1+3\eta < 2/(3-2\sigma)$, and the condition holds if $\sigma \geqslant \frac{1}{2}+\varepsilon$ and $\eta = \varepsilon/3$. This completes the proof of Theorem 2.

Finally, for Theorem 3, we use (15) with $k = 2$ and (20). Since by hypothesis

$$2-2\sigma-(4\sigma-3)/(2\mu) \leqslant 2-4\sigma$$

and furthermore, $\sigma \geqslant 3/4$ so that $2-4\sigma < 0$, we obtain

$$N_2+N_3 \ll T^{1+3\eta} Y^{1-2\sigma} + (Y^{2-2\sigma}+TY^{1-2\sigma})(\log T)^{C'+1}.$$

On taking $Y = T^{1+3\eta}$ we get

$$Y^{2-2\sigma} = T^{1+3\eta} Y^{1-2\sigma} = T^{(1+3\eta)(2-2\sigma)},$$

and this gives (3) on putting $\eta = \varepsilon/6$.

5. *L*-functions. Let F be an algebraic number field which is a Galois extension of Q and let G be its Galois group. Let χ be a character of a matrix representation ϱ of G and let k be the degree of χ. The L-function introduced by Artin is given by

$$L(s, \chi) = L(s, \chi, F/Q) = \prod \left(\det\{I - p^{-s} \varrho((F/p))\}\right)^{-1},$$

where the product is taken over all primes p which do not ramify in F, I is the unit $k\times k$ matrix and (F/p) denotes the Frobenius automorphism $\left(\frac{F/Q}{\mathfrak{p}}\right)$. We omit the factors corresponding to ramified primes; this will not affect the number of zeros in the strip $\frac{1}{2} \leqslant \sigma \leqslant 1$. If K is a subfield of F, then $\zeta_K(s) = L(s, \chi, F/Q)$ for some χ, so that the Artin L-function includes the Dedekind Zeta-function as a special case.

Let now $N_\chi(\sigma, T)$ be the number of zeros $\varrho = \beta+i\gamma$ of $L(s, \chi)$ with $|\gamma| \leqslant T$, $\beta \geqslant \sigma$, the zeros being counted according to multiplicity. Then, providing that (A) to (E) below hold, the proofs of Theorems 1, 2 and 3 go through verbatim for $L(s, \chi)$, with the definition of k above; the only proviso is that $L(s, \chi)$ may be regular at $s = 1$ in which case the second term of (11) does not occur, so that $N_1 = 0$. Thus for example Theorem 3 becomes:

For characters χ of degree 2

$$N_\chi(\sigma, T) \ll T^{(2+\varepsilon)(1-\sigma)} (\log T)^C$$

uniformly for $\sigma \geqslant \frac{3}{4}(1-\mu)^{-1}$, where μ, ν are absolute constants such that (4) *holds and where C depends only on μ, ν, ε and F.*

The properties required of $L(s, \chi)$ are the following:

(A) $L(s, \chi)$ is analytic in the region $\sigma \geqslant \frac{1}{4}$

except possibly for a simple pole at $s = 1$.

(B) $L(\sigma+it, \chi) \ll |t|^A \quad (\sigma \geqslant \frac{1}{4},\ |t| \geqslant 2)$,

where the constant A depends only on F.

(C) $N_\chi(\frac{1}{2}, T+1) - N_\chi(\frac{1}{2}, T) \ll \log T \quad (T \geqslant 2)$.

(D) $L(s, \chi) = \sum_1^\infty a_n n^{-s}, \quad L^{-1}(s, \chi) = \sum_1^\infty b_n n^{-s}$,

where $a_1 = b_1 = 1$ and $|a_n|, |b_n| \leqslant (d(n))^B$ for some constant B, depending only on F.

(E) For fixed $\varepsilon > 0$ and $k \geqslant 2$

$$\int_{-T}^{+T} \left|L\left(1-\frac{1}{k}+it, \chi\right)\right|^2 dt \ll T^{1+\varepsilon};$$

also if $k = 1$

$$\int_{-T}^{+T} |L(\tfrac{1}{2}+it, \chi)|^4 dt \ll T^{1+\varepsilon}.$$

Since we may suppose that χ is irreducible, $L(s, \chi)$ is a Dirichlet L-function when χ is abelian. In this case (A) to (E) are well known [6], [12]. But for non-abelian characters it is not even known that (A) holds, this being the unproved conjecture of Artin. However it is known [2] that $L(s, \chi)$ is meromorphic, and that when $G \leqslant S_4$, $L(s, \chi)$ is regular for non-unit, irreducible χ [1]. Further, (B) and (E) may be proved on the assumption that (A) holds, while (C) and (D) hold unconditionally. In fact (C) follows from the corresponding result for general abelian L-functions $L(s, \chi, E/K)$ ([10], Satz 71), since, by [2], $L(s, \chi)$ may be written as the product of such functions and their inverses. Furthermore (D) may be proved by an argument similar to that given for the coefficients in (7). Hence (A) to (E) may be replaced by the single hypothesis (A). We shall give complete proofs of these latter assertions elsewhere.

$\zeta_K(s)$ may be factorised as $\zeta(s)L(s, \chi, E/Q)$ where E is the normal closure of K/Q and χ has degree $[K: Q]-1$. Then

$$N_K(\sigma, T) \leqslant N_Q(\sigma, T) + N_\chi(\sigma, T).$$

If we now assume the truth of Artin's conjecture, and estimate $N_\chi(\sigma, T)$ as indicated above, we get a sharper bound for $N_K(\sigma, T)$ than previously, since k has, in effect been reduced from $[K: Q]$ to $[K: Q]-1$. More precisely if Artin's conjecture is true then

$$N_K(\sigma, T) \ll T^{(k-1+\varepsilon)(1-\sigma)}(\log T)^C$$

for fields of degree $k \geqslant 4$, and the estimates of Theorems 2 and 3 now hold for cubic fields K. These results could be further sharpened in special cases, for example when K/Q is normal, by decomposing $\zeta_K(s)$ into a product of $\zeta(s)$ and more than one L-function. For quadratic fields K we have $\zeta_K(s) = \zeta(s)L(s, \chi, K/Q)$ where χ is a Dirichlet character. Hence the above considerations show that the estimates [6], [9] previously established for the rational field in fact hold also for the quadratic fields. Thus for instance the condition $\sigma \geqslant 0.895\ldots$ obtained in Theorem 3 can be weakened to $\sigma \geqslant 0.785\ldots = 11/14$.

In the case when K has degree $\leqslant 4$ the Galois group of E/Q will be a subgroup of S_4. In this case Artin's conjecture is known to hold [1]; thus, in fact, the results stated above for cubic and quartic fields hold unconditionally.

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