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ON THE DEPARTURE FROM NORMALITY OF A CERTAIN CLASS OF MARTINGALES

C. C. HEYDE AND B. M. BROWN

Australian National University and La Trobe University

Let $\{X_n, \mathscr{F}_n, n = 0, 1, 2, \cdots\}$ be a martingale with $X_0 = 0$ a.s., $X_n = \sum_{i=1}^n Y_i$, $n \geqq 1$, and $\mathscr{F}_n$ the σ-field generated by $X_0, X_1, \cdots, X_n$. Write

$$\sigma_n^2 = E(Y_n^2 \mid \mathscr{F}_{n-1}), \qquad s_n^2 = \sum_{i=1}^n E\sigma_i^2,$$

and suppose that there is a constant δ, with $0 < \delta \leqq 1$, such that $E|Y_n|^{2+2\delta} < \infty$, $n = 1, 2, \cdots$. It is the object of this paper to establish the following theorem on departure from normality.

THEOREM. *There exist finite constants K_1, K_2 depending only on δ, such that*

$$\begin{aligned} &\sup_x |P(X_n \leqq s_n x) - \Phi(x)| \\ (1) \qquad &\leqq K_1\{s_n^{-2-2\delta}(\textstyle\sum_{i=1}^n E|Y_i|^{2+2\delta} + E|(\sum_{i=1}^n \sigma_i^2) - s_n^2|^{1+\delta})\}^{1/(3+2\delta)} \\ &\leqq K_2\{s_n^{-2-2\delta}(\textstyle\sum_{i=1}^n E|Y_i|^{2+2\delta} + E|(\sum_{i=1}^n Y_i^2) - s_n^2|^{1+\delta})\}^{1/(3+2\delta)}, \end{aligned}$$

where

$$\Phi(x) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^x e^{-\frac{1}{2}u^2}\, du.$$

Thus, if

$$(2) \qquad \lim_{n\to\infty} s_n^{-2-2\delta} \sum_{i=1}^n E|Y_i|^{2+2\delta} = 0, \qquad \textit{and}$$

$$(3) \qquad \lim_{n\to\infty} E|(s_n^{-2} \sum_{i=1}^n Y_i^2) - 1|^{1+\delta} = 0$$

or more generally, (2) *and*

$$(4) \qquad \lim_{n\to\infty} E|(s_n^{-2} \sum_{i=1}^n \sigma_i^2) - 1|^{1+\delta} = 0,$$

then $\lim_{n\to\infty} P(X_n \leqq s_n x) = \Phi(x)$, *and a bound on the rate of convergence is given by* (1).

The interesting feature of this result is the bound on departure from normality. More general central limit results for martingales are known, under conditions related to (2), (3) and (4) (see for example Brown [1]), but rates of convergence are not available.

We note that if $Y_1, Y_2, \cdots$ are independent, or more generally if $\sigma_1^2, \sigma_2^2, \cdots$ are constants a.s., then $s_n^{-2}\sum_{i=1}^n \sigma_i^2 = 1$ a.s., (4) is trivially true, and the first bound in (1) assumes a simplified form, since its second term vanishes. The utility of the second bound in (1) is that it does not depend on the conditioning by the sequence of σ-fields $\{\mathscr{F}_n\}$.

The proof of the theorem is based on a martingale form of the Skorokhod representation theorem which we state as a lemma in the interests of clarity.

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LEMMA. (Strassen [3], Theorem 4.3) *Let* $Z_1, Z_2, \cdots$ *be random variables such that for all* n, $E(Z_n^2 \mid Z_{n-1}, \cdots, Z_1)$ *is defined and* $E(Z_n \mid Z_{n-1}, \cdots, Z_1) = 0$ a.s. *Then, there is a Brownian motion* W *together with a sequence of nonnegative random variables* $T_1, T_2, \cdots$ *such that*

$$\sum_{i=1}^n Z_i = W(\sum_{i=1}^n T_i) \quad \text{a.s.}$$

for all n. *Moreover, if* $\mathscr{G}_n$ *is the* σ*-field generated by* $Z_1, Z_2, \cdots, Z_n$ *and* $W(t)$ *for* $0 \leqq t \leqq \sum_{i=1}^n T_i$, *then* T_n *is* $\mathscr{G}_n$*-measurable,* $E(T_n \mid \mathscr{G}_{n-1})$ *is defined and*

$$E(T_n \mid \mathscr{G}_{n-1}) = E(Z_n^2 \mid \mathscr{G}_{n-1}) = E(Z_n^2 \mid Z_{n-1}, \cdots, Z_1) \quad \text{a.s.}$$

If k *is a real number* > 1 *and* $E(|Z_n|^{2k} \mid Z_{n-1}, \cdots, Z_1)$ *is defined, then* $E(T_n^k \mid \mathscr{G}_{n-1})$ *is also defined and*

$$\begin{aligned} E(T_n^k \mid \mathscr{G}_{n-1}) &\leqq L_k E(|Z_n|^{2k} \mid \mathscr{G}_{n-1}) \\ &= L_k E(|Z_n|^{2k} \mid Z_{n-1}, \cdots, Z_1) \quad \text{a.s.,} \end{aligned}$$

where L_k *are constants which depend only on* k.

Applying the lemma to the sequence of random variables $s_n^{-1} Y_1, \cdots, s_n^{-1} Y_n$, we note that there exists a Brownian motion W and nonnegative random variables $T_{n1}, \cdots, T_{nn}$ such that

$$s_n^{-1} X_n = W(\sum_{i=1}^n T_{ni}) \quad \text{a.s.} \tag{5}$$

for all n. Moreover, if $\mathscr{G}_{ni}$ is the σ-field generated by $Y_1, \cdots, Y_i$ and $W(t)$ for $0 \leqq t \leqq \sum_{j=1}^i T_{nj}$, $1 \leqq i \leqq n$, then T_{ni} is $\mathscr{G}_{ni}$-measurable and

$$\begin{aligned} E(T_{ni} \mid \mathscr{G}_{n,i-1}) &= s_n^{-2} E(Y_i^2 \mid \mathscr{G}_{n,i-1}) \\ &= s_n^{-2} \sigma_i^2 \quad \text{a.s.} \end{aligned} \tag{6}$$

Now define

$$Z_{nn} = \sum_{i=1}^n T_{ni} - 1, \tag{7}$$

noting that $EZ_{nn} = 0$, from (6). Then, from (5) and (7),

$$P(X_n \leqq s_n x) = P(W(1+Z_{nn}) \leqq x),$$

so that

$$|P(X_n \leqq s_n x) - \Phi(x)| = |P(W(1+Z_{nn}) \leqq x) - P(W(1) \leqq x)|. \tag{8}$$

Next, let $\{\varepsilon_n\}$ be a sequence with $0 < \varepsilon_n < 1$ and $\lim_{n\to\infty} \varepsilon_n = 0$. We have

$$\begin{aligned} &P(W(1+Z_{nn}) \leqq x) \\ &\quad = P(W(1+Z_{nn}) \leqq x; |Z_{nn}| \leqq \varepsilon_n) + P(W(1+Z_{nn}) \leqq x; |Z_{nn}| > \varepsilon_n) \\ &\quad \leqq P(\inf_{|t| \leqq \varepsilon_n} W(1+t) \leqq x; |Z_{nn}| \leqq \varepsilon_n) + P(|Z_{nn}| > \varepsilon_n) \\ &\quad \leqq P(\inf_{|t| \leqq \varepsilon_n} W(1+t) \leqq x) + P(|Z_{nn}| > \varepsilon_n), \end{aligned} \tag{9}$$

and similarly,

$$P(W(1+Z_{nn}) > x) \leqq P(\sup_{|t| \leqq \varepsilon_n} W(1+t) > x) + P(|Z_{nn}| > \varepsilon_n),$$

or equivalently

$$P(W(1+Z_{nn}) \leqq x) \geqq P(\sup_{|t| \leqq \varepsilon_n} W(1+t) \leqq x) - P(|Z_{nn}| > \varepsilon_n). \tag{10}$$

Now,

$$\begin{aligned} P(\inf_{|t| \leqq \varepsilon_n} W(1+t) \leqq x) &= P(\inf_{|t| \leqq \varepsilon_n} [W(1+t) - W(1-\varepsilon_n)] + W(1-\varepsilon_n) \leqq x) \\ &= P(W(1-\varepsilon_n) + \inf_{0 \leqq t \leqq 2\varepsilon_n} [W(1+t) - W(1)] \leqq x) \\ &= \int_{-\infty}^{0} P(W(1-\varepsilon_n) \leqq x - y)\, dP(\inf_{0 \leqq t \leqq 2\varepsilon_n} W(t) \leqq y) \\ &= (\pi\varepsilon_n)^{-\frac{1}{2}} \int_0^\infty \Phi((x+y)(1-\varepsilon_n)^{-\frac{1}{2}}) \exp(-y^2/4\varepsilon_n)\, dy \\ &= \pi^{-\frac{1}{2}} \int_0^\infty \Phi((x+\varepsilon_n^{\frac{1}{2}} z)(1-\varepsilon_n)^{-\frac{1}{2}})\, e^{-z^2/4}\, dz, \end{aligned} \tag{11}$$

and also, using a similar procedure,

$$P(\sup_{|t| \leqq \varepsilon_n} W(1+t) \leqq x) = \pi^{-\frac{1}{2}} \int_0^\infty \Phi((x-\varepsilon_n^{\frac{1}{2}} z)(1-\varepsilon_n)^{-\frac{1}{2}})\, e^{-z^2/4}\, dz. \tag{12}$$

But,

$$|\Phi((x+\varepsilon_n^{\frac{1}{2}} z)(1-\varepsilon_n)^{-\frac{1}{2}}) - \Phi(x(1-\varepsilon_n)^{-\frac{1}{2}})| \leqq (2\pi)^{-\frac{1}{2}} |z|\, \varepsilon_n^{\frac{1}{2}}(1-\varepsilon_n)^{-\frac{1}{2}}, \tag{13}$$

so that, using (13) in (11) and (12), we obtain

$$\begin{aligned} \pi^{-\frac{1}{2}} \int_0^\infty [\Phi(x(1-\varepsilon_n)^{-\frac{1}{2}}) - (2\pi)^{-\frac{1}{2}} z\varepsilon_n^{\frac{1}{2}}(1-\varepsilon_n)^{-\frac{1}{2}}]\, e^{-z^2/4}\, dz \\ \leqq P(\sup_{|t| \leqq \varepsilon_n} W(1+t) \leqq x) \leqq P(\inf_{|t| \leqq \varepsilon_n} W(1+t) \leqq x) \\ \leqq \pi^{-\frac{1}{2}} \int_0^\infty [\Phi(x(1-\varepsilon_n)^{-\frac{1}{2}}) + (2\pi)^{-\frac{1}{2}} z\varepsilon_n^{\frac{1}{2}}(1-\varepsilon_n)^{-\frac{1}{2}}]\, e^{-z^2/4}\, dz, \end{aligned}$$

which yields, for suitably large n,

$$\begin{aligned} P(W(1-\varepsilon_n) \leqq x) - \varepsilon_n^{\frac{1}{2}} \\ \leqq P(\sup_{|t| \leqq \varepsilon_n} W(1+t) \leqq x) \leqq P(\inf_{|t| \leqq \varepsilon_n} W(1+t) \leqq x) \\ \leqq P(W(1-\varepsilon_n) \leqq x) + \varepsilon_n^{\frac{1}{2}}. \end{aligned} \tag{14}$$

Combining (9) and (10) with (14) then gives

$$|P(W(1+Z_{nn}) \leqq x) - P(W(1-\varepsilon_n) \leqq x)| \leqq \varepsilon_n^{\frac{1}{2}} + P(|Z_{nn}| > \varepsilon_n). \tag{15}$$

Also, from a Taylor series expansion for $\varepsilon_n < \frac{1}{2}$, and trivially for $\varepsilon_n \geqq \frac{1}{2}$, we have

$$|P(W(1-\varepsilon_n) \leqq x) - P(W(1) \leqq x) \leqq 2\varepsilon_n. \tag{16}$$

Consequently, (15) and (16) imply that, for n sufficiently large,

$$\begin{aligned} |P(W(1+Z_{nn}) \leqq x) - P(W(1) \leqq x)| &\leqq 2\varepsilon_n^{\frac{1}{2}} + P(|Z_{nn}| > \varepsilon_n) \\ &\leqq 2\varepsilon_n^{\frac{1}{2}} + \varepsilon_n^{-1-\delta} E|Z_{nn}|^{1+\delta} \end{aligned} \tag{17}$$

by Markov's inequality. The sequence $\{\varepsilon_n\}$ is then chosen to provide the best possible order in (17). Take $\varepsilon_n = (E|Z_{nn}|^{1+\delta})^{2/(3+2\delta)}$, so that

$$|P(W(1+Z_{nn}) \leqq x) - P(W(1) \leqq x)| \leqq 3(E|Z_{nn}|^{1+\delta})^{1/(3+2\delta)}, \tag{18}$$

and the problem reduces to that of producing a bound on $E|Z_{nn}|^{1+\delta}$.

Write $Z_{nn} = A_{nn} + B_n + C_n$, where

$$A_{ni} = \sum_{j=1}^{i}(T_{nj} - s_n^{-2}\sigma_j^2), \qquad 1 \leqq i \leqq n,$$
$$B_n = s_n^{-2}\sum_{i=1}^{n}(\sigma_i^2 - Y_i^2), \qquad C_n = s_n^{-2}\sum_{i=1}^{n} Y_i^2 - 1,$$

and note that $\{A_{ni}, \mathscr{G}_{ni}, 1 \leqq i \leqq n\}$ and $\{s_n^2 B_n, \mathscr{F}_n, n \geqq 1\}$ are both martingale sequences. Therefore, by Burkholder's martingale extension of the Marcinkiewicz–Zygmund inequality (Theorem 9 of [2]), there exists a finite universal constant $M_{1+\delta}$ such that

$$\begin{aligned} E|A_{nn}|^{1+\delta} &\leqq M_{1+\delta} E(\textstyle\sum_{i=1}^{n}(T_{ni} - s_n^{-2}\sigma_i^2)^2)^{\frac{1}{2}(1+\delta)} \\ &\leqq M_{1+\delta} E\textstyle\sum_{i=1}^{n}|T_{ni} - s_n^{-2}\sigma_i^2|^{1+\delta}, \qquad \text{since } \tfrac{1}{2}(1+\delta) \leqq 1, \\ &\leqq M_{1+\delta} 2^{\delta}\textstyle\sum_{i=1}^{n}(ET_{ni}^{1+\delta} + s_n^{-2-2\delta}E\sigma_i^{2+2\delta}) \\ &\leqq M_{1+\delta} 2^{\delta} s_n^{-2-2\delta}\textstyle\sum_{i=1}^{n}(L_{1+\delta}E|Y_i|^{2+2\delta} + E\sigma_i^{2+2\delta}) \end{aligned} \tag{19}$$

using the lemma in the last step. Similarly,

$$E|B_n|^{1+\delta} \leqq M_{1+\delta} 2^{\delta} s_n^{-2-2\delta}\sum_{i=1}^{n}(E\sigma_i^{2+2\delta} + E|Y_i|^{2+2\delta}), \tag{20}$$

so that an application of the inequality

$$E\sigma_i^{2+2\delta} = E[E(Y_i^2 \mid \mathscr{F}_{i-1})]^{1+\delta} \leqq E[E(|Y_i|^{2+2\delta} \mid \mathscr{F}_{i-1})] = E|Y_i|^{2+2\delta}$$

to (19) and (20) gives

$$E|A_{nn}|^{1+\delta} \leqq G_\delta s_n^{-2-2\delta}\sum_{i=1}^{n} E|Y_i|^{2+2\delta} \qquad \text{and} \tag{21}$$

$$E|B_n|^{1+\delta} \leqq G_\delta s_n^{-2-2\delta}\sum_{i=1}^{n} E|Y_i|^{2+2\delta} \tag{22}$$

for all n, where G_δ is a finite constant depending only on δ. Consequently, applying the inequality $(\sum_1^2|a_i|)^{1+\delta} \leqq 2^{\delta}\sum_1^2|a_i|^{1+\delta}$, we have

$$\begin{aligned} E|Z_{nn}|^{1+\delta} &\leqq E|A_{nn} + (B_n + C_n)|^{1+\delta} \\ &\leqq 2^{\delta}(E|A_{nn}|^{1+\delta} + E|B_n + C_n|^{1+\delta}) \\ &\leqq 2^{\delta}[G_\delta s_n^{-2-2\delta}\textstyle\sum_{i=1}^{n} E|Y_i|^{2+2\delta} \\ &\quad + E|(s_n^{-2}\textstyle\sum_{i=1}^{n}\sigma_i^2) - 1|^{1+\delta}], \end{aligned} \tag{23}$$

from (21), and the first part of (1) follows from (8), (18) and (23). To obtain the second part of (1), note that

$$\begin{aligned} E|(s_n^{-2}\textstyle\sum_{i=1}^{n}\sigma_i^2) - 1|^{1+\delta} &= E|B_n + C_n|^{1+\delta} \\ &\leqq 2^{\delta}(E|B_n|^{1+\delta} + E|C_n|^{1+\delta}) \\ &\leqq 2^{\delta}[G_\delta s_n^{-2-2\delta}\textstyle\sum_{i=1}^{n} E|Y_i|^{2+2\delta} \\ &\quad + E|(s_n^{-2}\textstyle\sum_{i=1}^{n} Y_i^2) - 1|^{1+\delta}], \end{aligned} \tag{24}$$

from (22), so that the required result follows from (24) and the first part of (1).

REMARK. It is evident from the proof of the theorem that a sufficient condition for $\lim_{n\to\infty} P(X_n \leqq s_n x) = \Phi(x)$ for all x, is the condition $Z_{nn} \to_p 0$ as $n \to \infty$, i.e.

$$\sum_{i=1}^{n} T_{ni} \to_p 1 \quad \text{as} \quad n \to \infty,$$

which in turn is equivalent to the mean convergence

$$\lim_{n\to\infty} E\left|(\sum_{i=1}^{n} T_{ni}) - 1\right| = 0,$$

since $Z_{nn} \geqq -1$ a.s. and $EZ_{nn} = 0$. These conditions, though not explicitly in terms of the martingale differences $\{Y_n\}$, are in a form much simpler than other known sufficient conditions for the central limit theorem for martingales (cf. [1]).

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