

On the Derivative of a Polynomial

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ABSTRACT

Certain refinements and generalizations of some well known inequalities concerning the polynomials and their derivatives are obtained.

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1. Introduction to the Statement of Results

Let $P_n(z)$ denote the space of all complex polynomials

$P(z) = \sum_{j=1}^n a_j z^j$ of degree n . If $P \in P_n$, then

$$\max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)| \quad (1)$$

and

$$\max_{|z|=R>1} |P(z)| \leq R^n \max_{|z|=1} |P(z)|. \quad (2)$$

Inequality (1) is an immediate consequence of S. Bernstein's theorem (see [1]) on the derivative of a trigonometric polynomial. Inequality (2) is a simple deduction from the maximum modulus principle (see [2, p. 346] or [3, p. 137]).

Both the inequalities (1) and (2) are sharp and the equality in (1) and (2) holds if and only if $P(z)$ has all its zeros at the origin. It was shown by Frappier, Rahman and Ruscheweyh [4, Theorem 8] that if $P \in P_n$, then

$$\max_{|z|=1} |P'(z)| \leq n \max_{1 \leq k \leq 2n} |P(e^{ik\pi/n})|. \quad (3)$$

Clearly (3) represents a refinement of (1), since the maximum of $|P(z)|$ on $|z|=1$ may be larger than the maximum of $|P(z)|$ taken over $(2n)^{th}$ roots of unity, as is shown by the simple example $P(z) = z^n + ia$, $a > 0$.

A. Aziz [5] showed that the bound in (3) can be considerably improved. In fact proved that if $P \in P_n$, then for every given real α ,

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} (M_\alpha + M_{\alpha+\pi}) \quad (4)$$

where

$$M_\alpha = \max_{1 \leq k \leq n} |P(e^{i(\alpha+2k\pi)/n})| \quad (5)$$

and $M_{\alpha+\pi}$ is obtained by replacing α by $\alpha+\pi$. The result is best possible and equality in (4) holds for $P(z) = z^n + re^{i\alpha}$, $-1 \leq r \leq 1$.

Clearly inequality (4) is an interesting refinement of inequality (3) and hence of Bernstein inequality (1) as well.

If we restrict ourselves to the class of polynomials $P \in P_n$ having no zero in $|z| < 1$, then the inequality (1) can be sharpened. In fact, P. Erdős conjectured and later P. D. Lax [6] (see also [7]) verified that if $P(z) \neq 0$ for $|z| < 1$, then (1) can be replaced by

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \max_{|z|=1} |P(z)|. \quad (6)$$

In this connection A. Aziz [5], improved the inequality (4) by showing that if $P \in P_n$ and $P(z)$ does not vanish in $|z| < 1$, then for every real α ,

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} (M_\alpha^2 + M_{\alpha+\pi}^2)^{1/2} \quad (7)$$

where M_α is defined by (5). The result is best possible and equality in (7) holds for $P(z) = z^n + e^{i\alpha}$.

A. Aziz [5] also proved that if $P \in P_n$ and $P(z) \neq 0$ in $|z| < 1$, then for every real α and $R > 1$,

$$\max_{|z|=1} |P(Rz) - P(z)| \leq \frac{R^n - 1}{2} (M_\alpha^2 + M_{\alpha+\pi}^2)^{1/2} \quad (8)$$

In this paper, we first present the following result which is a refinement of inequality (7).

Theorem 1. *If $P \in P_n$, $P(z)$ does not vanish in $|z| < 1$ and $m = \min_{|z|=1} |P(z)|$, then for every real α ,*

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} (M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2)^{1/2}. \quad (9)$$

where M_α is defined by (5). The result is best possible and equality in (9) holds for $P(z) = z^n + e^{i\alpha}$.

As an application of Theorem 1, we mention the corresponding improvement of (8).

Theorem 2. *If $P \in P_n$, and $P(z) \neq 0$ for $|z| < 1$ and $m = \min_{|z|=1} |P(z)|$ then for every real α and $R > 1$,*

$$|P(Rz) - P(z)| \leq \frac{R^n - 1}{2} (M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2)^{1/2} \quad (10)$$

where M_α is defined by (5). The result is best possible and equality in (10) holds for $P(z) = z^n + e^{i\alpha}$.

Here we also consider the class of polynomials $P \in P_n$ having no zero in $|z| < k$, $k > 0$ and present some generalizations of the inequalities (9) and (10). First we consider the case $k \geq 1$ and prove the following result which is a generalization of inequality (9).

Theorem 3. *If $P \in P_n$ does not vanish in $|z| < k$, $k \geq 1$ and $m = \min_{|z|=k} |P(z)|$, then for every real α ,*

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{\sqrt{2(1+k^2)}} (M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2)^{1/2} \quad (11)$$

where M_α is defined by (5).

Next result is a corresponding generalization of the inequality (10).

Theorem 4. *If $P \in P_n$ does not vanish in $|z| < k$, $k \geq 1$ and $m = \min_{|z|=k} |P(z)|$, then for every real α and $R > 1$,*

$$|P(Rz) - P(z)| \leq \frac{R^n - 1}{\sqrt{2(1+k^2)}} (M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2)^{1/2} \quad (12)$$

where M_α is defined by (5).

Remark 1. For $k = 1$, Theorem 3 and Theorem 4 reduces to the Theorem 1 and Theorem 2 respectively.

For the case $k \leq 1$, we have been able to prove:

Theorem 5. *If $P \in P_n$, $P(z)$ has no zero in $|z| < k$, $k \leq 1$ and $m = \min_{|z|=k} |P(z)|$, then for every real α ,*

$$\begin{aligned} & \max_{|z|=1} |P'(z)| \\ & \leq \frac{n}{\sqrt{2(1+k^{2n})}} (M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2)^{1/2}, \end{aligned} \quad (13)$$

provided $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z|=1$ where $Q(z) = z^n \overline{P(1/\bar{z})}$. The result is best possible and equality in (13) holds for $P(z) = z^n + k^n$.

Theorem 6. *If $P \in P_n$, $P(z)$ has no zero in $|z| < k$, $k \leq 1$ and $m = \min_{|z|=k} |P(z)|$, then for every real α and $R > 1$,*

$$|P(Rz) - P(z)| \leq \frac{R^n - 1}{\sqrt{2(1+k^{2n})}} (M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2)^{1/2}, \quad (14)$$

provided $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z|=1$ where $Q(z) = z^n \overline{P(1/\bar{z})}$. The result is best possible and equality in (14) holds for $P(z) = z^n + k^n$.

2. Lemmas

For the proofs of these theorems, we need the following lemmas. The first Lemma is due to A. Aziz [5].

Lemma 1. *If $P \in P_n$, then for $|z|=1$ and for every real α ,*

$$|P'(z)|^2 + |nP(z) - zP'(z)|^2 \leq \frac{n^2}{2} (M_\alpha^2 + M_{\alpha+\pi}^2) \quad (15)$$

where M_α is defined by (5).

Lemma 2. *If $P \in P_n$ and $P(z) \neq 0$ for $|z| < k$, $k \geq 1$, then for $|z|=1$,*

$$k|P'(z)| \leq |nP(z) - zP'(z)| - nm$$

where $m = \min_{|z|=k} |P(z)|$.

Lemma 2 is a special cases of a result due to A. Aziz and N. A. Rather [8, Lemma 5].

Lemma 3. *If $P \in P_n$ does not vanish in $|z| < k$, $k \leq 1$, then*

$$k^n |P'(z)| \leq \max_{|z|=1} |Q'(z)| \text{ for } |z|=1$$

where $Q(z) = z^n \overline{P(1/\bar{z})}$.

This Lemma is due to N. K. Govil [9].

Lemma 4. *If $P(z)$ is a polynomial of degree n which does not vanish in $|z| < k$, $k \leq 1$, then for $|z|=1$*

$$k^n |P'(z)| + n \min_{|z|=k} |P(z)| \leq \max_{|z|=1} |Q'(z)|$$

where $Q(z) = z^n \overline{P(1/\bar{z})}$.

Proof of Lemma 4. Let $m = \min_{|z|=k} |P(z)|$. If $P(z)$ has a zero on $|z|=k$, then $m = 0$ and the result follows from Lemma 3. Henceforth we assume that $P(z)$ has no zero on $|z|=k$, therefore $m > 0$ and

$$m \leq |P(z)| \text{ for } |z|=k.$$

If α is any real or complex number with $|\alpha| < 1$, then for $|z|=k$,

$$|\alpha m z^n / k^n| < |P(z)|.$$

By Rouché's Theorem, it follows that the polynomial $F(z) = P(z) - \alpha m z^n / k^n$ does not vanish in $|z| < k$, for every real or complex number α with $|\alpha| < 1$. Apply-

ing Lemma 3 to the polynomial $F(z)$, we get

$$k^n |F'(z)| \leq \max_{|z|=1} |G'(z)| \text{ for } |z|=1. \tag{16}$$

where

$$\begin{aligned} G(z) &= z^n \overline{P(1/\bar{z})} = z^n \overline{P(1/\bar{z})} - \bar{\alpha}m/k^n \\ &= Q(z) - \bar{\alpha}m/k^n. \end{aligned}$$

Replacing $F(z)$ by $P(z) - \alpha m z^n / k^n$ and $G(z)$ by $Q(z) - \bar{\alpha}m/k^n$, we obtain from (16) for $|z|=1$,

$$k^n |P'(z) - n\alpha m z^{n-1} / k^n| \leq \max_{|z|=1} |Q'(z)|. \tag{17}$$

Now choosing the argument of α in the left hand side of (17) such that

$$|P'(z) - n\alpha m z^{n-1} / k^n| = |P'(z)| + |nm\alpha / k^n|$$

we obtain for $|z|=1$,

$$k^n |P'(z)| + |\alpha| nm \leq \max_{|z|=1} |Q'(z)|.$$

Letting $|\alpha| \rightarrow 1$, we get the desired result. This proves Lemma 4.

3. Proof of the Theorems

Proof of Theorem 1. By hypothesis $P(z)$ does not vanish in $|z| < 1$ and $m = \min_{|z|=k} |P(z)|$, therefore, by Lemma 2 with $k = 1$, we have

$$(|P'(z)| + nm)^2 \leq |nP(z) - zP'(z)|^2 \text{ for } |z|=1.$$

This gives with the help of Lemma 1

$$\begin{aligned} |P'(z)|^2 + (|P'(z)| + nm)^2 &\leq |P'(z)|^2 + |nP(z) - zP'(z)|^2 \\ &\leq \frac{n^2}{2} (M_\alpha^2 + M_{\alpha+\pi}^2). \end{aligned}$$

Since

$$\begin{aligned} (|P'(z)| + nm)^2 &= |P'(z)|^2 + n^2 m^2 + 2nm |P'(z)| \\ &\geq |P'(z)|^2 + n^2 m^2, \end{aligned}$$

it follows that

$$2|P'(z)|^2 + n^2 m^2 \leq \frac{n^2}{2} (M_\alpha^2 + M_{\alpha+\pi}^2),$$

which implies for $|z|=1$

$$|P'(z)| \leq \frac{n}{2} (M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2)^{1/2}$$

and hence

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} (M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2)^{1/2}.$$

This completes the proof of Theorem 1.

Proof of Theorem 2. Applying (2) to the polynomial $P'(z)$ which is of degree $n-1$ and using Theorem 1, we obtain for $t \geq 1$ and $0 \leq \theta < 2\pi$,

$$\begin{aligned} |P'(te^{i\theta})| &\leq t^{n-1} \max_{|z|=1} |P'(z)| \\ &\leq \frac{n}{2} t^{n-1} (M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2)^{1/2}. \end{aligned}$$

Hence for each $\theta, 0 \leq \theta < 2\pi$ and $R > 1$, we have

$$\begin{aligned} |P(Re^{i\theta}) - P(e^{i\theta})| &= \left| \int_1^R e^{i\theta} P'(te^{i\theta}) dt \right| \leq \int_1^R |P'(te^{i\theta})| dt \\ &\leq \frac{1}{2} (M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2)^{1/2} \int_1^R n t^{n-1} dt \\ &= \frac{1}{2} (M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2)^{1/2} (R^n - 1). \end{aligned}$$

This implies for $|z|=1$ and $R > 1$,

$$|P(Rz) - P(z)| \leq \frac{R^n - 1}{2} (M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2)^{1/2},$$

which proves Theorem 2.

The proof of the Theorem 3 and 4 follows on the same lines as that of Theorems 1 and 2, so we omit the details.

Proof of Theorem 5. Since all the zeros of $P(z)$ lie in $|z| \geq k$, where $k \leq 1$, $m = \min_{|z|=k} |P(z)|$, by Lemma 4, we have

$$k^n \max_{|z|=1} |P'(z)| + nm \leq \max_{|z|=1} |Q'(z)|, \tag{18}$$

where $Q(z) = z^n \overline{P(1/\bar{z})}$. Also by hypothesis $|P'(z)|$ and $|Q'(z)|$ become maximum at the same point on $|z|=1$, if

$$\max_{|z|=1} |P'(z)| = |P'(e^{i\alpha})|, 0 \leq \alpha < 2\pi, \tag{19}$$

then

$$\max_{|z|=1} |Q'(z)| = |Q'(e^{i\alpha})|, 0 \leq \alpha < 2\pi \tag{20}$$

and it can be easily verified that

$$|Q'(z)| = |nP(z) - zP'(z)| \text{ for } |z|=1.$$

Therefore, by Lemma 1

$$\begin{aligned} |P'(e^{i\alpha})|^2 + |Q'(e^{i\alpha})|^2 &= |P'(e^{i\alpha})|^2 + |nP(e^{i\alpha}) - e^{i\alpha} P'(e^{i\alpha})|^2 \\ &\leq \frac{n^2}{2} (M_\alpha^2 + M_{\alpha+\pi}^2). \end{aligned}$$

This gives with the help of (18), (19) and (20) that

$$\begin{aligned} |P'(e^{i\alpha})|^2 + (k^n |P'(e^{i\alpha})| + nm)^2 &\leq |P'(e^{i\alpha})|^2 + |Q'(e^{i\alpha})|^2 \leq \frac{n^2}{2} (M_\alpha^2 + M_{\alpha+\pi}^2), \end{aligned}$$

which implies,

$$\left|P'(e^{i\alpha})\right|^2 + k^{2n} \left|P'(e^{i\alpha})\right|^2 + n^2 m^2 \leq \frac{n^2}{2} (M_\alpha^2 + M_{\alpha+\pi}^2).$$

Equivalently,

$$\left|P'(e^{i\alpha})\right|^2 \leq \frac{n^2}{2(1+k^{2n})} (M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2)$$

and hence

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{\sqrt{2(1+k^{2n})}} (M_\alpha^2 + M_{\alpha+\pi}^2 - 2m^2)^{1/2}.$$

This completes the proof of Theorem 5.

Theorem 6 follows on the same lines as that of Theorem 2, so we omit the details.

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