ON THE DESIGN OF ONE-DIMENSIONAL SPARSE ARRAYS WITH APODIZED END ELEMENTS

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ABSTRACT

In ultrasound scanners, a two-way radiation pattern is generated by a transmit array and a receive array. The design of such arrays is simplified by treating the problem as the design of an "effective aperture function" $w_{eff}(n)$ which is given by the convolution of the transmit aperture function $w_T(n)$ and the receive aperture function $w_R(n)$. In this paper we consider the design of one-dimensional sparse transmit and receive arrays with an apodized effective aperture function. The design approach is based on the factorization of a polynomial representation of the effective aperture function.

1. INTRODUCTION

The far-field radiation pattern $P(u)$ at an angle θ away from the broadside of a linear array of N isotropic, equispaced elements is given by

$$P(u) = \sum_{n=0}^{\infty} w(n) e^{j[2\pi(u/\lambda)d]n}, \quad (1)$$

where d is the inter-element spacing, $w(n)$ is the complex excitation or weight of the n-th element, λ is the wavelength and $u = \sin\theta$. The sequence $\{w(n)\}$ is the array element weighting as a function of the element position n and is known as the aperture function. A more convenient representation of the aperture function is obtained from Eq. (1) by substituting $x = e^{j[2\pi(u/\lambda)d]n}$ resulting in the function

$$P(x) = \sum_{n=0}^{N-1} w(n) x^n, \quad (2)$$

which is a polynomial in x.

Sparse arrays with fewer elements are obtained by removing some of the antenna elements which increases the inter-element spacing between some consecutive pairs of elements to more than λ/2. The removal of elements usually results in an increase of the side lobe levels and may result in the appearance of grating lobes in the radiation pattern [1]. In some applications, a two-way radiation pattern is generated by a transmit array and a receive array [2]. In such cases, it is possible to use sparse transmit and receive arrays whose effective aperture function is equivalent to that of a single array with no missing elements and hence, is not associated with the above problems.

If $w_T(n)$ and $w_R(n)$ denote, respectively, the transmit and receive aperture functions, the effective aperture function $w_{eff}(n)$ is then given by the convolution sum:

$$w_{eff}(n) = w_T(n) * w_R(n). \quad (3)$$

If the number of elements (including missing elements) in the transmit and receive arrays are, respectively, L and M, then the number of elements N in a single array with an aperture function $w_{eff}(n)$ is $L+M-1$. The design problem thus reduces to the problem of determining $w_T(n)$ and $w_R(n)$ for a desired $w_{eff}(n)$.

In an earlier paper, we outlined a method of designing sparse transmit and receive arrays with a uniform effective aperture function [3]. The uniform aperture function has a radiation pattern with the narrowest main lobe width, but the peak of its first side lobe is about 13 dB below the peak of the main lobe. An increase in the side lobe rejection can be obtained with an array pair having a tapered effective aperture function. In this paper we outline two different approaches to the design of an array with a tapered effective aperture function.

The paper is organized as follows. Section 2 considers the design of arrays with linearly tapered effective aperture functions and Section 3 treats the design of arrays with staircase effective aperture functions. Two specific design issues concerning sparse arrays are discussed in Section 4.

2. ARRAYS WITH A LINEARLY TAPERED EFFECTIVE APERTURE FUNCTION

We first consider the design of a sparse array pair with a linearly tapered effective aperture function $P_{eff}(x)$. To this end we choose

$$P_{eff}(x) = P_1(x) P_2(x), \quad (4)$$

where

$$P_1(x) = \frac{1}{R}\sum_{i=0}^{R-1} x^i, \quad P_2(x) = \sum_{i=0}^{S-1} x^i. \quad (5)$$

The number of elements in the effective aperture function is then $N = R + S - 1$. The number of apodized elements is equal to $2(R-1)$, with $(R-1)$ apodized elements at the beginning of $P_{eff}(x)$ and $(R-1)$ apodized elements at the end. The values of the apodized elements are $\frac{1}{R}, \frac{2}{R}, \ldots, \frac{R-1}{R}$. The parameter S must satisfy the condition $S > R - 1$. Furthermore, if R and S are positive integers that are power-of-2, we can design sparse transmit and receive antenna arrays using the polynomial factorization approach proposed elsewhere [3]. We illustrate the design of sparse arrays with apodized end elements in the following two examples.

Example 1: $R = 2$ and $S = 16$. Here $P_1(x) = \frac{1}{2}(1+x)$ and $P_2(x) = \sum_{i=0}^{15} x^i$. Using the polynomial factorization approach [3] $P_2(x)$ can be expressed as

$$P_2(x) = (1+x)(1+x^2)(1+x^4)(1+x^8),$$

and hence, $P_{eff}(x)$ is of the form

$$P_{eff}(x) = \tfrac{1}{2}(1+x)(1+x)(1+x^2)(1+x^4)(1+x^8)$$
$$= 0.5 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10} + x^{11} + x^{15} + 0.5x^{16}.$$

A plot of the effective aperture function is shown in Figure 1.

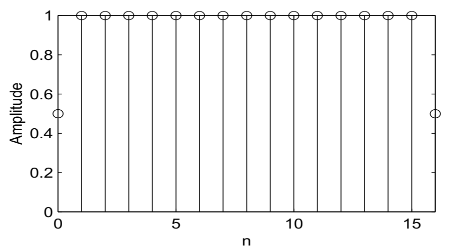

Figure 1: Effective aperture function of Example 1.

One possible design for the transmit and receive arrays obtained using appropriate factors of $P_{eff}(x)$ is given by:

$$P_T(x) = (1+x)(1+x^4)(1+x^8)$$
$$= 1 + x + x^4 + x^5 + x^8 + x^9 + x^{12} + x^{13},$$
$$P_R(x) = \tfrac{1}{2}(1+x)(1+x^2) = \tfrac{1}{2}(1+x+x^2+x^3).$$

Example 2: $R = 3$, $S = 16$. Here $P_{eff}(x)$ is given by

$$P_{eff}(x) = \tfrac{1}{3}(1+x+x^2)(1+x)(1+x^2)(1+x^4)(1+x^8)$$
$$= \tfrac{1}{3} + \tfrac{2}{3}x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10} + x^{11} + x^{12} + x^{13} + x^{14} + x^{15} + \tfrac{2}{3}x^{16} + \tfrac{1}{3}x^{17}.$$

A plot of the effective aperture function is shown in Figure 2. A possible design for the transmit and receive arrays is

$$P_T(x) = (1+x)(1+x^2)(1+x^4)$$
$$= 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7,$$
$$P_R(x) = \tfrac{1}{3}(1+x+x^2)(1+x^8) = \tfrac{1}{3}(1+x+x^2+x^8+x^9+x^{10}).$$

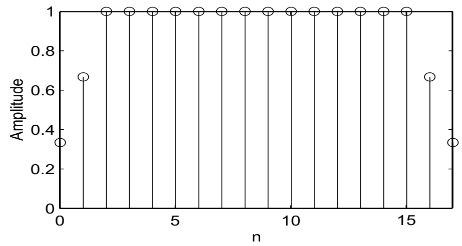

Figure 2: Effective aperture function of Example 2.

3. ARRAYS WITH STAIRCASE APERTURE FUNCTION

For the design of array pair with a staircase effective aperture function, there are two possible forms of the factor of Eq. (1) as described next.

3.1. **Case A** – even number of factors: In this case, the factor $P_1(x)$ is of the form

$$P_1(x) = \tfrac{1}{2\ell+1}[1 + x^{k_1}(1 + x^{k_2}(1 + \ldots + x^{k_\ell}(1 + \ldots + x^{k_2}(1 + x^{k_1})\ldots))))]. \quad (8)$$

The number R of elements (including zero-valued ones) in $P_1(x)$ is given by $R = 2\sum_{i=1}^{\ell} k_i + 1$.

For a staircase effective aperture function $P_{eff}(x)$, the number S of elements in $P_2(x)$ of Eq. (2) must satisfy the condition $S > 2\sum_{i=1}^{\ell} k_i$. The number of elements in the effective aperture function in this case is then $N = 2\sum_{i=1}^{\ell} k_i + S$. The number of apodized elements is equal to $4\sum_{i=1}^{\ell} k_i$. The values of the apodized elements are $\frac{1}{2\ell+1}, \ldots, \frac{\ell}{2\ell+1}, \frac{\ell+1}{2\ell+1}, \ldots, \frac{2\ell}{2\ell+1}$. For a given set of $\{k_i\}$, $i = 1, 2, \ldots, \ell$, and S, $w_{eff}(n)$ is of staircase form. If in addition, S is a power-of-2 positive integer, we can design sparse transmit and receive antenna arrays using the polynomial factorization approach [3]. We illustrate the design of sparse arrays with a staircase effective aperture function in the following example.

Example 3: $k_1 = 2$, $k_2 = 3$, $\ell = 2$, and $S = 16$. Here $P_1(x) = \frac{1}{5}[1 + x^2(1 + x^3(1 + x^3(1 + x^2)))] = \frac{1}{5}(1 + x^2 + x^5 + x^8 + x^{10})$ and $P_2(x) = \sum_{i=0}^{15} x^i$. Therefore $P_{eff}(x)$ is given by

$$P_{eff}(x) = 0.2(1+x) + 0.4x^2(1+x+x^2) + 0.6x^5(1+x+x^2) + 0.8x^8(1+x) + x^{10}(1+x+x^2+x^3+x^4+x^5) + 0.8x^{16}(1+x) + 0.6x^{18}(1+x+x^2) + 0.4x^{21}(1+x+x^2) + 0.2x^{24}(1+x).$$

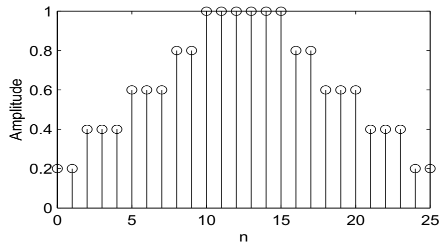

Figure 3: Effective aperture function of Example 3.

A plot of the above effective aperture function is shown in Figure 3. One possible efficient factorization of $P_{eff}(x)$ is $P_T(x) = (1+x^2+x^5+x^8+x^{10})(1+x^4) = 1+x^2+x^4+x^5+x^6+x^8+x^9+x^{10}+x^{12}+x^{14}$ and $P_R(x) = \frac{1}{5}(1+x+x^2+x^3+x^8+x^9+x^{10}+x^{11})$.

3.2. Case B – odd number of factors: In this case, the factor $P_1(x)$ is of the form

$$P_1(x) = \frac{1}{2\ell}[1+x^{k_1}(1+x^{k_2}(1+\ldots+x^{k_\ell}(1+x^{k_{\ell-1}}(1+\ldots+x^{k_2}(1+x^{k_1})\ldots))))]. \quad (12)$$

The number R of elements (including zero-valued ones) in $P_1(x)$ is given by $R = 2\sum_{i=1}^{\ell-1} k_i + k_\ell + 1$. For a staircase effective aperture function $P_{eff}(x)$, the number S of elements in $P_2(x)$ of Eq. (2) must satisfy the condition $S > 2\sum_{i=1}^{\ell-1} k_i + k_\ell$. The number of elements in the effective aperture function in this case is then $N = 2\sum_{i=1}^{\ell-1} k_i + k_\ell + S$. The number of apodized elements is equal to $2(2\sum_{i=1}^{\ell} k_i + k_\ell)$. The values of the apodized elements are $\frac{1}{2\ell}, \frac{2}{2\ell}, \ldots, \frac{2\ell-1}{2\ell}$. For a given set of $\{k_i\}$, $i = 1, 2, \ldots, \ell$, and S, $w_{eff}(n)$ is of the staircase form. If in addition, S is a power-of-2 positive integer, we can design sparse transmit and receive antenna arrays using the polynomial factorization approach proposed [3]. We illustrate the design of sparse arrays with staircase effective aperture function in the following example.

Example 4: $k_1 = 2, k_2 = 3, k_3 = 4, \ell = 3, S = 16$. Here $P_1(x) = \frac{1}{6}[1+x^2(1+x^3(1+x^4(1+x^3(1+x^2))))] = \frac{1}{6}(1+x^2+x^5+x^9+x^{12}+x^{14})$ and $P_2(x) = \sum_{i=0}^{15} x^i$. Therefore $P_{eff}(x)$ is given by

$$P_{eff}(x) = \tfrac{1}{6}(1+x) + \tfrac{1}{3}x^2(1+x+x^2) + \tfrac{1}{2}x^5(1+x+x^2+x^3) + \tfrac{2}{3}x^9(1+x+x^2) + \tfrac{5}{6}x^{12}(1+x) + x^{14}(1+x) + \tfrac{5}{6}x^{16}(1+x) + \tfrac{2}{3}x^{18}(1+x+x^2) + \tfrac{1}{2}x^{21}(1+x+x^2+x^3) + \tfrac{1}{3}x^{25}(1+x+x^2) + \tfrac{1}{6}x^{28}(1+x).$$

A plot of the above effective aperture function is shown in Figure 4. One possible efficient factorization of $P_{eff}(x)$ is $P_T(x) = (1+x^2+x^5+x^9+x^{12}+x^{14}) \cdot (1+x^8) = 1+x^2+x^5+x^8+x^9+x^{10}+x^{12}+x^{13}+x^{14}+x^{17}+x^{20}+x^{22}$ and $P_R(x) = (1+x)(1+x^2)(1+x^4) = \sum_{i=0}^{7} x^{2i}$.

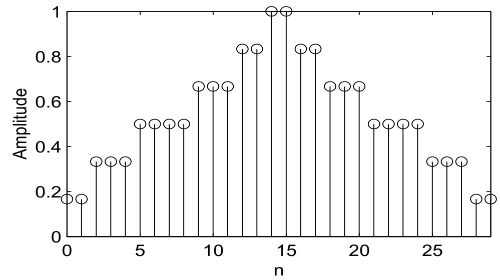

Figure 4: Effective aperture function of Example 4.

Example 5: Consider first $k_r = 6, r = 1, \ldots, 5$. Here $P_1(x) = \frac{1}{6}(1+x^6+x^{12}+x^{18}+x^{24}+x^{30})$ and $P_2(x) = \sum_{i=0}^{50} x^i$, which corresponds to $R = 51$. Possible efficient factorization of $P_{eff}(x)$ is given by $P_T(x) = \frac{1}{6}\sum_{i=0}^{16} x^{3i}$ and $P_R(x) = 1+x+x^2+x^6+x^7+x^8+x^{12}+x^{13}+x^{14}+x^{18}+x^{19}+x^{20}+x^{24}+x^{25}+x^{26}+x^{30}+x^{31}+x^{32}$.

Example 6: Consider next $k_r = 12, r = 1, \ldots, 4$. Here $P_1(x) = \frac{1}{5}\sum_{i=0}^{4} x^{12i}$ and $P_2(x) = \sum_{i=0}^{84} x^i$ ($R = 85$). Possible efficient factorization of $P_{eff}(x)$ is given by $P_T(x) = \frac{1}{5}\sum_{i=0}^{16} x^{5i}$ and $P_R(x) = 1+x+x^2+x^3+x^4+x^{12}+x^{13}+x^{14}+x^{15}+x^{16}+x^{24}+x^{25}+x^{26}+x^{27}+x^{28}+x^{36}+x^{37}+x^{38}+x^{39}+x^{40}+x^{48}+x^{49}+x^{50}+x^{51}+x^{52}$.

4. DESIGN ISSUES

There are two issues that need to be considered in the design of sparse arrays with a linearly tapered and staircase effective aperture function: Element reduction factor and the side lobe rejection. The *element reduction factor* provides a measure of the efficiency of the factorization of the effective aperture function and is given by $(N_1 + N_2)/(N_T + N_R)$, where N_1, N_2, N_T, and N_R are the number of non-zero terms in $P_1(x)$, $P_2(x)$, $P_T(x)$ and $P_R(x)$, respectively. The *sidelobe rejection* is the difference in the gain level in dB between the height of the mainlobe and that of the sidelobe with the largest height. In this section we examine these two issues with respect to the design examples given in the previous section.

4.1. Arrays with a Linearly Tapered Effective Aperture Function

The design example 1 with R = 2 and S = 16 has a sidelobe rejection of -13.49 dB and an element reduction factor of $(2+16)/(4+8)=1.5$. Likewise, the design example 2 with R = 3 and S = 16 has a sidelobe rejection of -14.08 dB and an element reduction factor of $(3+16)/(8+6)=1.357$. In general, in the case of a linearly tapered effective aperture function, it is not possible get more than about -13 dB of sidelobe rejection if a more efficient factorization of the effective aperture function is also desired.

4.2. Arrays with Staircase Aperture Function

In this case there exists two different possible shapes of the effective aperture function and it is possible to get a higher sidelobe rejection along with a larger element reduction factor.

In the case of Example 3, the sidelobe rejection is -24.1 dB. The element reduction factor achieved here is only (5+16)/(10+8) = 1.167. A plot of its radiation pattern is shown in Figure 5.

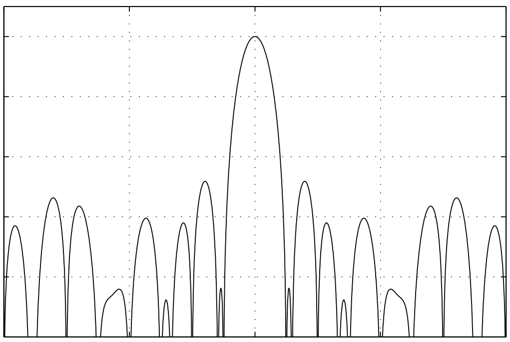

Figure 5: Radiation pattern of the array of Example 3.

In Example 4, the sidelobe rejection is -20.93 dB and the largest element reduction factor possible is only (6+16)/(12+8) = 1.1. A plot of the corresponding radiation pattern is shown in Figure 6.

In Example 5, the sidelobe rejection is -28.12 dB and the largest element reduction factor that can be achieved is (6+51)/(17+18) = 1.629.

In Example 6, the sidelobe rejection is -28.3 dB and the largest element reduction factor possible is (5 + 85)/(17 + 25) = 2.143.

5. CONCLUDING REMARKS

As the examples show the best compromise with respect to getting higher sidelobe rejection of more than -13 dB and also an element reduction factor of greater than 1 is obtained only with an array having a staircase effective aperture function. However, with any type of tapered effective aperture function, the price to pay is an increase in the width of the main lobe. Work is continuing to develop explicit design guidelines.

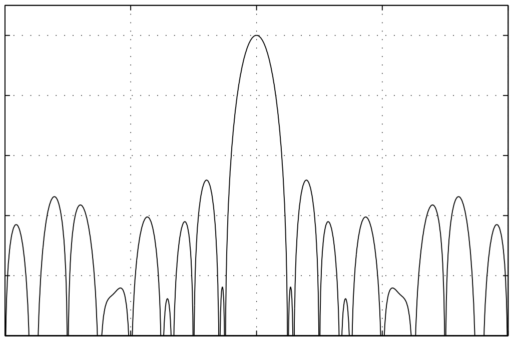

Figure 6: Radiation pattern of the array of Example 4.

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