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ON THE DETERMINANT OF ELLIPTIC BOUNDARY VALUE PROBLEMS ON A LINE SEGMENT

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ABSTRACT. In this paper we present a formula for the determinant of a matrix-valued elliptic differential operator of even order on a line segment $[0, T]$ with boundary conditions.

1. INTRODUCTION AND SUMMARY OF THE RESULTS

In this paper we present a formula for the determinant of a matrix-valued elliptic differential operator of even order on a line segment $[0, T]$ with boundary conditions. In order to state our results we introduce the following notation:

(1) Denote by $\mathscr{A} = \sum_{k=0}^{2n} a_k(x)D^k$ a differential operator, $D = D_x = -i\frac{d}{dx}$, where the coefficients are complex-valued $r\times r$ matrices depending smoothly on x, $0 \leq x \leq T$. The leading coefficient $a_{2n}(x)$ is assumed to be nonsingular and to have θ as a principal angle, i.e. $R_\theta \cap \operatorname{Spec} a_{2n}(x) = \phi$ for $0 \leq x \leq T$, where $R_\theta := \{\rho e^{i\theta} \in \mathbb{C} \mid 0 \leq \rho < \infty\}$.

(2) We impose boundary conditions of the form

$$\ell_j u(T) = 0d, \quad m_j u(0) = 0 \quad (1 \leq j \leq n) \tag{1.1}$$

where $u \in C^\infty([0, T]; \mathbb{C}^r)$ and ℓ_j, m_j are differential operators of the form

$$\ell_j := \sum_{k=0}^{\alpha_j} b_{jk} d_x^k, \quad m_j := \sum_{k=0}^{\beta_j} c_{jk} d_x^k \quad \left(d_x = \frac{d}{dx}\right)$$

such that b_{jk}, c_{jk} are constant $r \times r$ matrices with $b_{j\alpha_j} = c_{j\beta_j} = \mathrm{Id}$ and such that the integers α_j, β_j satisfy

$$0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_n \leq 2n-1, \quad 0 \leq \beta_1 < \beta_2 < \cdots < \beta_n \leq 2n-1. \tag{1.2}$$

Example 1. Dirichlet boundary conditions: $\alpha_D = \beta_D = (0, 1, \dots, n-1)$

$$b_{D,jk} = c_{D,jk} := \begin{cases} \mathrm{Id} & \text{if } 1 \leq j \leq n,\ k = j-1, \\ 0 & \text{otherwise.} \end{cases}$$

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Example 2. Neumann boundary conditions: $\alpha_N = \beta_N = (n, n+1, \dots, 2n-1)$

$$b_{N,jk} = c_{N,jk} := \begin{cases} \mathrm{Id} & \text{if } 1 \le j \le n,\ k = n+j-1, \\ 0 & \text{otherwise.} \end{cases}$$

For convenience we write $\alpha = (\alpha_1, \dots, \alpha_n)$, $|\alpha| = \sum_{j=1}^n \alpha_j$ and similarly β and $|\beta|$. Boundary conditions of the above form are usually called separated. Let $B = (B_{jk})$ and $C = (C_{jk})$, $1 \le j \le 2n$, $0 \le k \le 2n-1$, be $2n \times 2n$ matrices whose entries are the following $r \times r$ matrices

$$B_{jk} := \begin{cases} b_{jk} & \text{if } 1 \le j \le n \text{ and } 0 \le k \le \alpha_j, \\ 0 & \text{otherwise;} \end{cases}$$

$$C_{jk} := \begin{cases} c_{j-n,k} & \text{if } n+1 \le j \le 2n \text{ and } 0 \le k \le \beta_{j-n}, \\ 0 & \text{otherwise.} \end{cases}$$

We denote by $A = A_{B,C}$ the operator $\mathscr{A}$ restricted to the space of smooth functions $u : [0, T] \to \mathbb{C}^r$ satisfying the boundary conditions (1.1).

(3) ζ-regularized determinant $\operatorname{Det}_\theta A$. In the case where A is not 1–1, define $\operatorname{Det}_\theta A = 0$. In the case A is 1–1, one proceeds as follows. As the coefficient $a_{2n}(x)$ has θ as a principal angle, there exists $\varepsilon > 0$ so that $L_{(\theta-\varepsilon, \theta+\varepsilon)} \cap \operatorname{Spec} a_{2n}(x) = \emptyset$, $0 \le x \le T$, where $L_{(\alpha,\beta)} := \{z \in \mathbb{C} \mid \alpha \le \arg z \le \beta\}$. Then the spectrum of A, $\operatorname{Spec} A$, is discrete, $\operatorname{Spec} A = \{\lambda_j, j \in \mathbb{N}\}$, $|\lambda_j| \to \infty$, and $\operatorname{Spec} A \cap L_{(\theta-\varepsilon', \theta+\varepsilon')}$ for any $0 < \varepsilon < \varepsilon'$ is finite.

If $R_\theta \cap \operatorname{Spec} A = \phi$, we define $\zeta_{A,\theta}(s) = \sum_{j\ge1} \lambda_j^{-s} = Tr\, A^{-s}$ where $s \in \mathbb{C}$, $\operatorname{Re} s > 1/2n$ and where the complex powers are defined with respect to the angle θ. It is a well-known fact that $\zeta_{A,\theta}(s)$ admits a meromorphic extension to $\mathbb{C}$ with $s = 0$ being a regular point. According to Ray and Singer [RS] one defines $\log \operatorname{Det}_\theta A := -\frac{d}{ds}|_{s=0} \zeta_{A,\theta}(s)$. If $R_\theta \cap \operatorname{Spec} A \ne \emptyset$, then choose $\theta' \in (\theta - \varepsilon, \theta + \varepsilon)$ so that $R_{\theta'} \cap \operatorname{Spec} A = \emptyset$, and define $\operatorname{Det}_\theta A := \operatorname{Det}_{\theta'}(A)$. It can be easily checked (cf. [BFK1]) that the definition is independent of the choise of θ' in $=$ $(\theta - \varepsilon, \theta + \varepsilon)$.

(4) The fundamental matrix $Y(x) = Y(x, \mathscr{A})$. Denote by $Y(x) = (y_{k\ell}(x))$ $(x \in \mathbb{R})$ the fundamental matrix for $\mathscr{A}$. Note that $Y(x)$ is a $2n \times 2n$ matrix whose entries $y_{k\ell}(x)$ $(0 \le k, \ell \le 2n-1)$ are $r \times r$ matrices defined by

$$y_{k\ell}(x) := d_x^k y_\ell(x)$$

where $y_\ell(x)$ denotes the solution of the Cauchy problem $\mathscr{A} y_\ell(x) = 0$, $y_{k\ell}(0) = \delta_{k\ell} \mathrm{Id}$. Of particular interest is the $2n \times 2n$ matrix $Y(T)$, the evaluation of the fundamental matrix at $x = T$.

(5) Introduce the quantities

$$g_\alpha := \frac{1}{2}\left(\frac{|\alpha|}{n} - n + \frac{1}{2}\right), \quad h_\alpha = \det \begin{pmatrix} w_1^{\alpha_1} & \cdots & w_n^{\alpha_1} \\ \vdots & & \vdots \\ w_1^{\alpha_n} & \cdots & w_n^{\alpha_n} \end{pmatrix}$$

where $w_1, \dots, w_n$ denote the $2n$th roots of $(-1)^{n+1}$ with $\operatorname{Re} w > 0$ given by $w_k = \exp\left\{\frac{2k-n-1}{2n}\pi i\right\}$. For a $r \times r$ matrix a with principal angle θ and eigenvalues $\lambda_1, \dots, \lambda_r$, denote $(\det a)_\theta^{g_\alpha} = \prod_{j=1}^r |\lambda_j|^{g_\alpha} \exp\{i g_\alpha \arg \lambda_j\}$ where $\theta - 2\pi < \arg \lambda_j < \theta$.

Example 1. Dirichlet boundary conditions:

$$g_{\alpha_D} = -n/4, \;\; h_{\alpha_D} = h_n := \prod_{i>j}(w_i - w_j).$$

Example 2. Neumann boundary conditions:

$$g_{\alpha_N} = n/4, \;\; h_{\alpha_N} = (-1)^n h_n.$$

The main result of this paper is

Theorem.

$$\mathrm{Det}_\theta A = K_\theta \exp\left\{ \frac{i}{2}\int_0^T \mathrm{tr}(a_{2n}^{-1}(x) a_{2n-1}(x))\, dx \right\} \det(BY(T) - C)$$

where $K_\theta \equiv K_\theta(\alpha, \beta)$ *is given by*

$$K_\theta = ((-1)^{|\beta|}(2n)^n h_\alpha^{-1} h_\beta^{-1})^r (\det a_{2n}(0))_\theta^{g_\beta} (\det a_{2n}(T))_\theta^{g_\alpha}.$$

Example 1. Dirichlet boundary conditions: $|\alpha_D| = \frac{n(n-1)}{2}$,

$$K_\theta = ((-1)^{|\alpha_D|}(2n)^n h_n^{-2})^r (\det a_{2n}(0))_\theta^{-\frac{n}{4}} (\det a_{2n}(T))_\theta^{-\frac{n}{4}}.$$

Example 2. Neumann boundary conditions: $|\alpha_N| = \frac{n(n-1)}{2}$,

$$K_\theta = ((-1)^{|\alpha_D|}(2n)^n h_{\alpha_N}^{-2})^r (\det a_{2n}(0))_\theta^{\frac{n}{4}} (\det a_{2n}(T))_\theta^{\frac{n}{4}}.$$

Corollary. $\mathrm{Det}_\theta A$ *is a complex number independent of* θ *up to multiplication with a* $2n$ *th root of unity.*

Remark 1. In the formula above all terms except the matrix $Y(T)$ are easily computable from the coefficients of $\mathscr{A}$, ℓ_i and m_j. The matrix $Y(T)$ requires the knowledge of the fundamental solutions. The matrix $Y(T)$ and therefore $\det(BY(T)-C)$ can be calculated numerically within arbitrary accuracy by solving a finite difference equation approximating $\mathscr{A}$. So the determinant $\mathrm{Det}_\theta A$ can be calculated numerically within arbitrary accuracy.

Remark 2. Theorem is a companion of the corresponding result on the circle instead of the interval $[0, T]$ which was treated in an earlier paper [BFK1]. Again, the proof of Theorem relies on a deformation argument and explicit computations for certain special operators and special boundary conditions.

Remark 3. Introduce a spectral parameter λ, and denote the fundamental matrix of $\mathscr{A}+\lambda$ by $Y(x, \lambda) = Y(x, \mathscr{A}+\lambda)$. One then verifies $\det(BY(T;\lambda)-C) = 0$ iff $\mathrm{Det}_\theta(A+\lambda) = 0$, i.e. iff $-\lambda$ is an eigenvalue of $A = A_{B,C}$.

Remark 4. First results of the type described in Theorem are due to Dreyfus and Dym [DD] and to Forman [Fo1] (cf. also [Fo2]). Forman proved by different methods that the quotient $\mathrm{Det}_\theta A/\mathrm{Det}(BY(T)-C)$ only depends on the principal and subprincipal symbols of $\mathscr{A}$, and the principal symbol of the boundary operators ℓ_j, m_j $(1 \le j \le n)$. Our Theorem provides a formula for this quotient.

Remark 5. Analogous to results obtained in [BFK2], Theorem can be extended to the case where $\mathscr{A}$ is a pseudodifferential operator. The determinant $\operatorname{Det}_\theta A$ can be written as a product of local invariants with a Fredholm determinant of a pseudodifferential operator of determinant class, canonically associated to A. The Fredholm determinant corresponds to $\det(BY(T) - C)$ in the case when $\mathscr{A}$ is a differential operator.

2. Auxiliary results

In this section we collect some auxiliary results needed for the proof of Theorem. First we introduce some additional notation. Denote by $EDO_{2n} \equiv EDO_{2n,r}$ the set of all elliptic differential operators $\mathscr{A}$ of order $2n$ on $[0, T]$ as introduced in Section 1. We identify EDO_{2n} with the open set $\{(a_{2n}, \dots, a_0) \in C^\infty([0, T], \operatorname{End}\mathbb{C}^r)^{2n+1} : \det(a_{2n}(x)) \neq 0,\ 0 \leq x \leq T\}$ of the Frechet space $C^\infty([0, T], \operatorname{End}\mathbb{C}^r)^{2n+1}$. Further define $EDO_{2n,\theta} := \{\mathscr{A} \in EDO_{2n} : \theta$ is principal angle for $a_{2n}\}$. Clearly $EDO_{2n;\theta}$ is an open connected subset in EDO_{2n}. Given $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$ with $0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_n \leq 2n-1$, we introduce the space BDO_α of operators used to define the boundary conditions:

$$BDO_\alpha := \{B = (b_{jk})_{0 \leq j,k \leq 2n-1} : \quad b_{jk} \in \operatorname{End}\mathbb{C}^r,\ b_{j\alpha_j} = \operatorname{Id}, \quad b_{jk} = 0 \text{ if } k \geq \alpha_j + 1\}.$$

Given α, β, we introduce the space

$$EDO_{2n;\alpha;\beta} := \{A_{B,C} : \mathscr{A} \in EDO_{2n},\ B \in BDO_\alpha,\ C \in BDO_\beta\}$$

where $A_{B,C}$ is the restriction of $\mathscr{A}$ to the subspace of functions $u \in C^\infty([0, T]; \mathbb{C}^r)$ satisfying the boundary conditions defined by B and C. Similarly introduce $EDO_{2n,\theta;\alpha,\beta} = \{A_{B,C} \in EDO_{2n;\alpha,\beta} : \mathscr{A} \in EDO_{2n;\theta}\}$. Observe that $\{A_{B,C} \in EDO_{2n;\theta;\alpha;\beta} : A_{B,C} \text{ is } 1\text{–}1\}$ is open.

Further, denote by $\widehat{EDO}_{2n;\alpha,\beta}$ the open subset of $EDO_{2n;\alpha,\beta} \times S^1$ consisting of pairs $(A_{B,C},\ \theta)$ with $A_{B,C} \in EDO_{2n;\theta;\alpha;\beta}$. As in [BFK1] we have the following

Proposition 2.1. (1) $\operatorname{Det}_\theta(A_{B,C})$ *is a smooth function on* $\widehat{EDO}_{2n;\alpha;\beta}$ *and is locally constant in* θ.

(2) $\operatorname{Det}_\theta(A_{B,C})$ *is holomorphic when considered as a function on the open subset of injective operators in* $EDO_{2n;\theta;\alpha;\beta}$.

(3) $\det(BY(T, \mathscr{A}) - C)$ *is holomorphic on* $EDO_{2n} \times BDO_\alpha \times BDO_\beta$.

Observe that a necessary and sufficient condition for $A_{B,C}$ to have zero as an eigenvalue is that $\det(BY(T) - C) = 0$, which in view of Proposition 2.1 (3) implies that the subsets of $EDO_{2n;\theta;\alpha;\beta}$ and $EDO_{2n,\alpha,\beta}$ consisting of injective operators are open (as we already noticed) and connected, and therefore, $\widehat{EDO}_{2n;\alpha;\beta}$ is open and connected as well.

Let $s : [0, T] \to GL(\mathbb{C}^r)$ be a smooth map. Given $\mathscr{A} \in EDO_{2n}$ and boundary operators ℓ_j, m_j $(1 \leq j \leq n)$ introduce $\mathscr{A}_1 := s(x)^{-1}\mathscr{A}s(x)$, $\ell_{1j} := s(T)^{-1}\ell_j s(x) \restriction_{x=T}$, and $m_{1j} := s(0)^{-1} m_j s(x) \restriction_{x=0}$. Denote by (B_{1jk}) and (C_{1jk}) the matrices introduced in Section 1 corresponding to the boundary operators $(\ell_{1j}, m_{1j})_{1 \leq j \leq n}$ and write $Y_1(x) = Y(x, \mathscr{A}_1)$ for short.

Proposition 2.2. $\det(B_1 Y_1(T) - C_1) = (\det s(0)s(T)^{-1})^n \det(BY(T) - C)$.

Proof. Let $L = L(x)$ be a $2n \times 2n$ matrix with entries $L_{k\ell}$ which are the following $r \times r$ matrices $(0 \le k, \ell \le 2n-1)$

$$L_{k\ell} := \binom{k}{\ell} d_x^{k-\ell} s(x) \quad \text{if } k \ge \ell; \quad L_{k\ell} = 0 \quad \text{if } k < \ell.$$

Thus we obtain

$$B_1 = \operatorname{diag}(s(T)^{-1}, \ldots, s(T)^{-1}) B L(T)$$

where $\operatorname{diag}(s(T)^{-1}, \ldots, s(T)^{-1})$ is a $2n \times 2n$ diagonal matrix whose entries on the diagonal are all equal to the $r \times r$ matrix $s(T)^{-1}$. Similarly, one obtains

$$C_1 = \operatorname{diag}(s(0)^{-1}, \ldots, s(0)^{-1}) C L(0).$$

Further, by a straightforward computation, Y_1 is given by

$$Y_1(x) = L(x)^{-1} Y(x) L(0).$$

Thus

$$\begin{aligned} B_1 Y_1(T) - C_1 = {} & \operatorname{diag}(s(T)^{-1}, \ldots, s(T)^{-1}, s(0)^{-1}, \ldots, s(0)^{-1}) \\ & \cdot [BY(T) - C] L(0). \end{aligned}$$

Now observe that $\det L(0) = (\det s(0))^{2n}$ as $L(0)$ is lower triangular with diagonal entries all equal to the $r \times r$ matrix $s(0)$. This implies that

$$\det(B_1 Y_1(T) - C_1) = (\det s(0) s(T)^{-1})^n \det(BY(T) - C).$$

Next consider for $A = A_{B,C}$ in $EDO_{2n;\theta;\alpha;\beta}$ and $\Phi \in C^\infty([0, T], GL_r(\mathbb{C}))$ the generalized ζ-function $\zeta_{\Phi, A;\theta}(s) := \operatorname{tr} \Phi A_\theta^{-s}$. Again this is a function which is holomorphic in $\operatorname{Re} s > \frac{1}{2n}$ and has a meromorphic extension to the whole complex plane. Moreover $s = 0$ is a regular point. Recall that we have introduced $g_\alpha := \frac{1}{2}(\frac{|\alpha|}{n} - n + \frac{1}{2})$, and similarly g_β.

Proposition 2.3.

$$\zeta_{\Phi, A;\theta}(0) = g_\beta \, tr\Phi(0) + g_\alpha \, tr\Phi(T) \tag{2.1}$$

As an immediate consequence we obtain

Corollary 2.4. $\zeta_{A;\theta}(0) = r(g_\alpha + g_\beta) = r(\frac{|\alpha|+|\beta|}{2n} - n + 1)$.

Proof (Proposition 2.3). We first prove that there are numbers $\tilde{g}_\alpha, \tilde{g}_\beta \in \mathbb{C}$ which only depend on α and β respectively such that (2.1) holds. The actual values of $\tilde{g}_\alpha, \tilde{g}_\beta$ are computed at the end of section 3 by considering the case $\Phi(x) \equiv K$ with $K > 1$, $\mathscr{A} = D^n + \lambda$, $\theta = \pi$. In the course of the proof we use a number of results due to Seeley [Se1,2]. For the convenience of the reader we partly keep Seeley's notation. For simplicity, we write $\zeta(s) = \zeta_{\Phi, A;\theta}(s)$. According to [Se2], the value $\zeta(0)$ consists of a sum of two terms, $\zeta(0) = I + II$ where I represents the contribution to $\zeta(0)$ of the resolvent of $\mathscr{A} - \lambda$ and II represents a correction term due to the boundary conditions. According to [BFK1, p. 8],

$$I = -\frac{e^{i\theta}}{4\pi n} \sum_{\tau = \pm 1} \int_0^T dx \int_0^\infty dr \operatorname{tr}\{\Phi(x) c_{-2n-1}(x, \tau, re^{i\theta})\}$$

where $c_{-2n-1}(x, \tau, \lambda)$ comes from the expansion of the symbol

$$r(x, \tau, \lambda) = c_{-2n}(x, \tau, \lambda) + c_{-2n-1}(x, \tau, \lambda) + \cdots$$

of the parametrix for $\mathscr{A} - \lambda = (a_{2n}(x)D^{2n} - \lambda) + \sum_{j=0}^{2n-1} a_j(x)D^j$ and is given by

$$c_{-2n-1}(x, \tau, \lambda) = -\tau^{2n-1}c_{-2n}a_{2n-1}c_{-2n} - i2n\tau^{4n-1}c_{-2n}a_{2n}c_{-2n}\left(\frac{d}{dx}a_{2n}\right)c_{-2n},$$

where $c_{-2n} \equiv c_{-2n}(x, \tau, \lambda) = (a_{2n}(x)\tau^{2n} - \lambda)^{-1}$.

As in [BFK1], Proposition 2.8, in view of the fact that c_{-2n-1} is odd in τ, we conclude $I = 0$. From [Se2], p. 968, it follows that II is of the form

$$II = \operatorname{tr}\{\Delta_0'(0)\Phi(0) + \Delta_T'(0)\Phi(T)\}$$

where $\Delta_0'(s)$ and $\Delta_T'(s)$ are smooth functions described below. Let us first consider the scalar case, $r = 1$. In first approximation the kernel $r(x, y, \lambda)$ of $(A_{B,C} - \lambda)^{-1}$ is given by

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}(a_{2n}(x)\tau^{2n} - \lambda)^{-1}e^{i(x-y)\tau}\,d\tau + r_0(x, y, \lambda) + r_T(x, y, \lambda)$$

where $r_0(x, y, \lambda)$ and $r_T(x, y, \lambda)$ are correction terms so that in first approximation $r(x, y, \lambda)$ satisfies the boundary conditions at $x = 0$ and $x = T$. Let us explain how to obtain $r_0(x, y, \lambda)$; for $r_T(x, y, \lambda)$ one proceeds in a similar fashion. Consider the boundary value problem

$$(aD^{2n} - \lambda)u = 0 \tag{2.2}$$

with the boundary condition

$$\lim_{x\to\infty} u(x) = 0; \quad D^{\beta_j}u(0) = -(a\tau^{2n} - \lambda)^{-1}\tau^{\beta_j}e^{-iy\tau} \tag{2.3}$$

where $a = a_{2n}(0)$ and $D = \frac{1}{i}\frac{d}{dx}$. The solution $u(x) = u(x, \tau, y, \lambda)$ of the boundary value problem (2.2)–(2.3) is given by $u(x) = \sum_{\nu=1}^{n} u_\nu e^{ix(-\lambda/a)^{1/2n}w_\nu}$ where w_ν $(1 \le \nu \le n)$ are the $2n$th roots of -1 with strictly positive imaginary part and where $(-\frac{\lambda}{a})^{1/2n} = (-\frac{\lambda}{a})_\pi^{1/2n} = (\frac{|\lambda|}{|a|})^{1/2n}e^{i(\theta-\pi-\arg a)/2n}$ with $\lambda = |\lambda|e^{i\theta}$ and $\theta - 2\pi < \arg a < \theta$. The coefficients $u_\nu = u_\nu(\tau, y, \lambda)$ are then determined by (2.3)

$$\sum_{\nu=1}^{n} u_\nu\left(-\frac{\lambda}{a}\right)^{\beta_j/2n} w_\nu^{\beta_j} = -\tau^{\beta_j}(a\tau^{2n} - \lambda)^{-1}e^{-iy\tau}.$$

Thus

$$u_\nu = -\sum_{j=1}^{n}\mathscr{H}_{\nu_j}(-\lambda/a)^{-\beta_j/2n}\tau^{\beta_j}(a\tau^{2n} - \lambda)^{-1}e^{-iy\tau}$$

with $\mathscr{H}_{\nu_j}$ defined by

$$\sum_{j=1}^{n}\mathscr{H}_{\nu_j}w_k^{\beta_j} = \delta_{\nu k}. \tag{2.4}$$

The term $r_0(x, y, \lambda)$ is then given by

$$r_0(x, y, \lambda) = \sum_{\nu=1}^{n} e^{ix(-\lambda/a)^{1/2n}w_\nu}\sum_{j=1}^{n}\mathscr{H}_{\nu j}\frac{1}{i}(-\lambda/a)^{-\beta_j/2n}\mathscr{F}$$

where $\mathcal{J}$ is the sum of residues

$$\mathcal{J} = \sum_{k=1}^{n} \operatorname{Res}_{\tau_k=(-\lambda/a)^{1/2n}\overline{w}_k}\{\tau^{\beta_j}(a\tau^{2n}-\lambda)^{-1}e^{-iy\tau}\}$$

of $\tau^{\beta_j}(a\tau^{2n}-\lambda)^{-1}e^{-iy\tau}$ in the lower half plane. One obtains

$$\mathcal{J} = \sum_{k=1}^{n}((-\lambda/a)^{1/2n}\overline{w}_k)^{\beta_j-(2n-1)}\frac{1}{2na}\exp\{-iy(-\lambda/a)^{1/2n}\overline{w}_k\}.$$

Summarizing one obtains

$$r_0(x, y, \lambda) = \frac{i}{2na}(-\lambda/a)^{-(2n-1)/2n}\sum_{\nu, j, k}\mathcal{K}_{\nu j}\overline{w}_k^{\beta+1}\exp\{i(-\lambda/a)^{1/2n}(xw_\nu - y\overline{w}_k)\}.$$

Following Seeley, we now define for $\operatorname{Re} s > 0$

$$\Delta_0'(s) := \int_0^{T/2} dx\frac{1}{2\pi i}\int_{\Gamma_\theta} d\lambda\lambda^{-s}r_0(x, x, \lambda) \tag{2.5}$$

where Γ_0 is the contour that goes from ∞ to 0 along the lower side of ray $\{re^{i\theta} : r > 0\}$, goes around the origin and then returns to ∞ along the upper side of the ray $\{re^{i\theta} : r > 0\}$. By a standard computation,

$$\frac{1}{2\pi i}\int_{\Gamma_\theta} d\lambda\lambda^{-s}r_0(x, x, \lambda)$$
$$= a^{-s}e^{-i\pi ns}\frac{\sin\pi s}{\pi}\Gamma(1-2ns)\sum_{\nu, j, k}\mathcal{K}_{\nu j}\overline{w}_k^{\beta_j+1}((w_\nu - \overline{w}_k)x)^{-1+2ns}$$

and therefore

$$\Delta_0'(0) = \frac{1}{2n}\sum_{\nu, j, k}\mathcal{K}_{\nu j}\frac{\overline{w}_k^{\beta_j+1}}{w_\nu - \overline{w}_k}.$$

In the case $r \geq 2$, we first treat the case where all eigenvalues of $a_{2n}(0)$ are different which can be easily reduced to scalar case $r = 1$. By a continuity argument we then conclude that

$$\tilde{g}_\beta = \frac{1}{2n}\sum_{\nu, j, k}\mathcal{K}_{\nu j}(\beta)\overline{w}_k^{\beta_j+1}(w_\nu - \overline{w}_k)^{-1} \tag{2.6}$$

where $\mathcal{K}_{\nu j} = \mathcal{K}_{\nu j}(\beta)$ are determined by (2.4). Similarly one obtains

$$\tilde{g}_\alpha = \frac{1}{2n}\sum_{\nu j, k}\mathcal{K}_{\nu_j}(\alpha)\overline{w}_k^{\alpha_j+1}(w_\nu - \overline{w}_k)^{-1}.$$

3. Proof of Theorem 1

For the proof of Theorem we need two deformation results. The first one is the analogue of Proposition 3.1 in [BFK1] and proved in a similar way (cf. also [DD] and [Fo1]).

Proposition 3.1. *Suppose* $\mathscr{A} = \sum_{k=0}^{2n} a_k(x)D^k$ *and* $\mathscr{A}' = \sum_{k=0}^{2n} a'_k(x)D^k$ *are in* $EDO_{2n;\theta}$ *with* $a_{2n} = a'_{2n}$ *and* $a_{2n-1} = a'_{2n-1}$. *Then, for* $B \in BDO_\alpha$ *and* $C \in BDO_\beta$

$$\mathrm{Det}_\theta(A_{B,C})\det(BY(T,\mathscr{A}') - C) = \mathrm{Det}_\theta(A'_{B,C})\det(BY(T,\mathscr{A}) - C).$$

The second result concerns a deformation of the boundary conditions. Consider boundary operators $(1 \le j \le n,\ d_x = \frac{d}{dx})$

$$\ell_j = \sum_{k=0}^{\alpha_j} b_{jk} d_x^k, \quad m_j = \sum_{k=0}^{\beta_j} c_{jk} d_x^k; \quad b_{j\alpha_j} = c_{j\beta_j} = \mathrm{Id}$$

and

$$\ell'_j := d_x^{\alpha_j}, \quad m'_j = d_x^{\beta_j}.$$

Form the matrices B, C and B', C' as in Section 1.

Proposition 3.2. *Fix* $\mathscr{A} \in EDO_{2n;\theta}$. *Then*

$$\mathrm{Det}_\theta(A_{B',C'})\det(BY(T) - C) = \mathrm{Det}_\theta(A_{B,C})\det(B'Y(T) - C').$$

Proof. Without loss of generality we may assume that both $A_{B,C}$ and $A_{B',C'}$ are injective. Note that $\{A_{\widetilde{B},\widetilde{C}} : A_{\widetilde{B},\widetilde{C}} \text{ is } 1\text{–}1,\ \widetilde{B} \in BDO_\alpha,\ \widetilde{C} \in BDO_\beta\}$ is arcwise connected in $BDO_\alpha \times BDO_\beta$. Define, for $0 \le t \le 1$,

$$\ell_{tj} = d_x^{\alpha_j} + t\sum_{k=0}^{\alpha_j - 1} b_{jk} d_x^k, \quad c_{tj} = d_x^{\beta_j} + t\sum_{k=0}^{\beta_j - 1} c_{jk} d_x^k$$

such that, with B_t and C_t the corresponding matrices in BDO_α and BDO_β,

$$A_{B_t,C_t} \text{ is } 1\text{–}1 \text{ for } 0 \le t \le 1; \tag{3.1}$$

$$(B_0, C_0) = (B', C'), \quad (B_1, C_1) = (B, C). \tag{3.2}$$

Introduce

$$w(t) := \frac{\frac{d}{dt}\mathrm{Det}_\theta(A_{B_t,C_t})}{\mathrm{Det}_\theta(A_{B_t}, C_t)}, \qquad \delta(t) := \frac{\frac{d}{dt}\det(B_tY(T) - C_t)}{\det(B_tY(T) - C_t)}.$$

The claimed result follows once we show that $w(t) = \delta(t)$ $(0 \le t \le 1)$. Let us first consider $\delta(t)$. Denote by P_t the Poisson operator corresponding to the boundary value problem defined by (B_t, C_t). Then P_t is given by $P_t = Y(x)(B_tY(T) - C_t)^{-1}$ and

$$\begin{aligned}\delta(t) &= \mathrm{tr}\{(\dot{B}_tY(T) - \dot{C}_t)(B_tY(T) - C_t)^{-1}\} \\ &= \mathrm{tr}((\dot{\ell}_{tj}, \dot{m}_{tj})_{1\le j\le n} P_t)\end{aligned} \tag{3.3}$$

when $\dot{} = \frac{d}{dt}$ and $(\dot{\ell}_{tj}, \dot{m}_{tj})_{1\le j\le n}$ is the operator associating to a section u the boundary values $(\dot{\ell}_{tj}u(T), \dot{m}_{tj}u(0))_{1\le j\le n}$.

Next we consider $w(t)$; with the notation $A_t = A_{B_t,C_t}$,

$$w(t) = F.p._{s=0}\,\mathrm{tr}(A_t^{\bullet} A_t^{-1-s})$$

where $F.p._{s=0}$ denotes the finite part at $s = 0$. In order to evaluate $A_t^{-1}A_t^{\bullet} = -(A_t^{-1})^{\bullet}A_t$, consider for a fixed section $u : [0, T] \to \mathbb{C}^r$ the section $v_t := A_t^{-1}u$, i.e. v_t satisfies

$$\mathscr{A} v_t = u, \quad B_tv_t(T) = 0, \quad C_tv_t(0) = 0.$$

Taking derivatives with respect to t we obtain

$$\mathscr{A} v_t^{\cdot} = 0, \quad \ell_{tj} v_t^{\cdot}(T) = -\ell_{tj}^{\cdot} v_t(T), \quad m_{tj} v_t^{\cdot}(0) = -m_{tj}^{\cdot} v_t(0) \quad (1 \le j \le n).$$

Thus $v_t^{\cdot} = -P_t(\ell_{tj}^{\cdot} v_t(T), m_{tj}^{\cdot} v_t(0))_{1\le j\le n}$ where P_t again denotes the Poisson operator. Thus we have proved that $(A_t^{-1})^{\cdot} = -P_t(\ell_{tj}^{\cdot}, m_{tj}^{\cdot})_{1\le j\le n} A_t^{-1}$. Note that $(A_t^{-1})^{\cdot} A_t = -P_t(\ell_{tj}^{\cdot}, m_{tj}^{\cdot})_{1\le j\le n}$ is a singular Green's operator of order ≤ -2 and then of trace class. Thus

$$w(t)^{\cdot} = \operatorname{tr} P_t(\ell_{kj}^{\cdot}, m_{ij}^{\cdot})_{1\le j\le n}. \quad \square$$

Proof of Theorem. We have to prove that

$$f_\theta(A_{B,C}) := \operatorname{Det}_\theta(A_{B,C}) - K_\theta \exp\left\{\frac{i}{2}\int_0^T \operatorname{tr}(a_{2n}(x)^{-1} a_{2n-1}(x))dx\right\} \cdot \det(BY(T) - C)$$

vanishes identically on $\{A_{B,C} \in EDO_{2n;\theta;\alpha;\beta} : A_{B,C} \text{ is } 1\text{–}1\}$. First observe that it suffices to consider the case $\theta = \pi$: For $\mathscr{A}$ in $EDO_{2n;\theta}$, $e^{i(\pi-\theta)}\mathscr{A} \in EDO_{2n;\pi}$ we have $\log \operatorname{Det}_\pi(e^{i(\pi-\theta)} A_{B,C}) = \log \operatorname{Det}_\theta(A_{B,C}) + \zeta_{A,\theta}(0) \log e^{i(\pi-\theta)}$ and $\log K_\theta(e^{i(\pi-\theta)}\mathscr{A}) = \log K_\pi(t) + r(g_\beta + g_\alpha) i(\pi - \theta)$; thus Corollary 2.4 allows to conclude the result as soon as we check it for $\theta = \pi$.

To make writing easier, let $f \equiv f_\pi$, $K \equiv K_\pi$, $\theta \equiv \pi$.

Deformation 1. Consider the factorization $\mathscr{A} = a_{2n}(D^{2n} + \mathscr{H})$ where $\mathscr{H}$ is a differential operator with $\operatorname{ord}\mathscr{H} \le 2n - 1$. Consider the 1-parameter family $(0 \le t \le 1)$

$$\mathscr{A}_t := \alpha_t(D^{2n} + \mathscr{H}), \quad A_t := A_{t,B,C}$$

when $\alpha_t(x) = t a_{2n}(x) + (1 - t)$.

Clearly $\theta = \pi$ is a principal angle for α_t and A_t is $1\text{–}1$ for $0 \le t \le 1$. Moreover $A_t^{\cdot} = (a_{2n}(x) - 1)(t a_{2n}(x) + (1 - t))^{-1} A_t$. Thus, with $w(t) = \log \operatorname{Det}_\pi A_t$ and Proposition 2.3

$$\begin{aligned} w(t)^{\cdot} &= F.p._{s=0} \operatorname{tr}((a_{2n}(x) - 1)(t a_{2n}(x) + (1 - t))^{-1} A(t)^{-s}) \\ &= g_\beta \operatorname{tr}[(a_{2n}(0) - 1)(t a_{2n}(0) + (1 - t))^{-1}] \\ &\quad + g_\alpha \operatorname{tr}[(a_{2n}(T) - 1)(t a_{2n}(T) + (1 - t))^{-1}] \\ &= \frac{d}{dt}\{g_\beta \log\det[t a_{2n}(0) + (1 - t)] \\ &\qquad + g_\alpha \log\det[t a_{2n}(T) + (1 - t)]\}. \end{aligned}$$

Thus

$$\log \operatorname{Det}_\pi A_1 - \log \operatorname{Det}_\pi A_0 = \int_0^1 w(t)^{\cdot} dt = g_\beta \log\det(a_{2n}(0)) + g_\alpha \log\det(a_{2n}(T)).$$

Hence we may and will assume that $a_{2n}(x) \equiv \mathrm{Id}$.

Deformation 2. Define $s \in C^\infty([0, T]; \operatorname{End}\mathbb{C}^r)$ by

$$\frac{d}{dx}s(x) = \frac{i}{2n} a_{2n-1}(x) s(x) \quad (0 \le x \le T); \quad s(0) = \mathrm{Id}.$$

Observe that $\det(s(x)) = \exp\{\frac{i}{2n}\int_0^x \operatorname{tr}(a_{2n-1}(y))dy\} \neq 0$ for $0 \leq x \leq T$ and therefore $s(x) \in GL_r(\mathbb{C})$. Now consider $\mathscr{A}_1 := s(x)^{-1}\mathscr{A}s(x)$ and boundary conditions defined by B_1, C_1 (cf. Proposition 2.2). Then $\operatorname{Det}_\pi(A_1) = \operatorname{Det}_\pi(A)$ as the spectrum of A and the operator A_1, defined by $\mathscr{A}_1$ and boundary conditions (B_1, C_1) do coincide. By Proposition 2.2,

$$\det(B_1Y_1(T) - C_1) = (\det s(T))^{-n}\det(BY(T) - C).$$

As we have noted above, $\det s(T) = \exp\{\frac{i}{2n}\int_0^T \operatorname{tr}(a_{2n-1}(y))\,dy\}$. Finally note that $\mathscr{A}_1$ is of the form

$$\mathscr{A}_1 = D^{2n} + \sum_{k=0}^{2n-2} a_{1k}(x)D^k$$

and then we may and will assume that for $\mathscr{A}$, $a_{2n}(x) \equiv \mathrm{Id}$ and $a_{2n-1}(x) \equiv 0$.

Deformation 3. Applying Proposition 3.1 and Proposition 3.2 we conclude that it remains to prove that $f(A_{B,C}) = 0$ for $\mathscr{A} = D^{2n} + \lambda$ and B, C given by

$$\ell_j = d_x^{\alpha_j}, \quad m_j = d_x^{\beta_j} \quad (1 \leq j \leq n)$$

where λ is chosen positive and sufficiently large so that $A_{B,C}$ is $1-1$. This is verified by an explicit computation. To make writing easier we restrict ourselves to that case $r = 1$. However, to obtain the explicit formulas for g_α and g_β we consider $\mathscr{A} = \rho D^{2n} + \lambda$ with $\rho > 1$. Denote by $Y(x, \lambda)$ the fundamental matrix for $\rho D^{2n} + \lambda$. For $\lambda > 0$, let $\mu = (\frac{\lambda}{\rho})^{1/2n}$. Then, with $w_k := \exp(i\frac{2k-n-1}{2n}\pi)$, $Y(x, \lambda)$ is equal to

$$\begin{pmatrix} e^{\mu w_1 x} & \cdots & e^{\mu w_{2n} x} \\ \mu w_1 e^{\mu w_1 x} & \cdots & \mu w_{2n} e^{\mu w_{2n} x} \\ \vdots & & \vdots \\ (\mu w_1)^{2n-1} e^{\mu w_1 x} & \cdots & (\mu w_{2n})^{2n-1} e^{\mu w_{2n} x} \end{pmatrix} \begin{pmatrix} 1 & \cdots & 1 \\ \mu w_1 & \cdots & \mu w_{2n} \\ \vdots & & \vdots \\ (\mu w_1)^{n-1} & \cdots & (\mu w_{2n})^{2n-1} \end{pmatrix}^{-1}.$$

Further define $B = (B_{jk})$, $C = (C_{jk})$ by

$$B_{jk} = \begin{cases} 1 & \text{if } 1 \leq j \leq n \text{ and } k = \alpha_j, \\ 0 & \text{otherwise;} \end{cases}$$

$$C_{jk} = \begin{cases} 1 & \text{if } n+1 \leq j \leq 2n \text{ and } k = \beta_{j-n}, \\ 0 & \text{otherwise.} \end{cases}$$

We have to show that

$$\operatorname{Det}_\pi((\rho D^{2n} + \lambda)_{B,C}) = (-1)^{|\beta|}(2n)^n(h_\alpha h_\beta)^{-1}\rho^{g_\alpha + g_\beta}\det(BY(T, \lambda) - C). \tag{3.4}$$

For that purpose we introduce

$$w(\lambda) := \log \operatorname{Det}_\pi((\rho D^{2n} + \lambda)_{B,C}),$$
$$\delta(\lambda) := \log\det(BY(T; \lambda) - C).$$

As $n \geq 1$, we know from Proposition 3.1 that $\frac{d}{d\lambda}w(\lambda) = \frac{d}{d\lambda}\delta(\lambda)$. Therefore it suffices to consider the asymptotics of $w(\lambda)$ and $\delta(\lambda)$ as $\lambda \to +\infty$.

First recall from [Fr] (cf. also [Vo]) that $w(\lambda)$ admits an asymptotic expansion of the form $\sum_{k=-1}^{\infty} p_k\lambda^{-k/n} + \sum_{j=0}^{\infty} q_j\lambda^{-j}\log\lambda$ with the property that $p_0 = 0$. To find the asymptotics of $\delta(\lambda)$ as $\lambda \to \infty$, write $Y(T, \lambda)$ in the form

$$Y(T; \lambda) = LWE(LW)^{-1}$$

where $L = \operatorname{diag}(1, \mu, \mu^2, \dots, \mu^{2n-1})$, $E := \operatorname{diag}(e^{\mu w_1 T}, \dots, e^{\mu w_{2n} T})$ and

$$W = \begin{pmatrix} 1 & \cdots & 1 \\ w_1^1 & \cdots & w_{2n}^1 \\ \vdots & & \vdots \\ w_1^{2n-1} & \cdots & w_{2n}^{2n-1} \end{pmatrix}.$$

Thus $\delta(\lambda) = \log(\det W^{-1}L^{-1}) + \log\det(BLWE - CLW)$. Observe that the (j, k)th coefficient of the matrix $BLWE - CLW$ is of the form $e^{\mu w_k T} f_{jk}(\mu) + g_{jk}(\mu)$ where $f_{jk}(\mu)$ and $g_{jk}(\mu)$ are rational functions of μ. We conclude that, with $\Omega = \sum_{j=1}^n w_j = \sum_{j=1}^n \operatorname{Re} w_j$,

$$\begin{aligned} &\log\det(BLWE - CLW) \\ &\quad = \mu\Omega T + \log\det[BLW \begin{pmatrix} \mathrm{Id}_n & 0 \\ 0 & 0 \end{pmatrix} - CLW \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{Id}_n \end{pmatrix}] + e(\lambda) \end{aligned}$$

where $\lim_{\lambda\to\infty} e(\lambda) = 0$. The matrix $BLW \begin{pmatrix} \mathrm{Id}_n & 0 \\ 0 & 0 \end{pmatrix} - CLW \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{Id}_n \end{pmatrix}$ is of the form $\begin{pmatrix} F^{(1)} & 0 \\ 0 & F^{(2)} \end{pmatrix}$ where $F^{(i)}$ are $n \times n$ matrices given by $(1 \le j, k \le n)$

$$F_{jk}^{(1)} := \mu^{\alpha_j} w_k^{\alpha_j}, \quad F_{jk}^{(2)} := -\mu^{\beta_j} w_{n+k}^{\beta_j} = (-1)^{\beta_j+1} \mu^{\beta_j} w_k^{\beta_j}$$

where we used that $w_{n+k} = -w_k$. Therefore, with $|\alpha| = \sum_1^n \alpha_j$, $|\beta| = \sum_1^n \beta_j$

$$\begin{aligned} &\det[BLW \begin{pmatrix} \mathrm{Id}_n & 0 \\ 0 & 0 \end{pmatrix} - CLW \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{Id}_n \end{pmatrix}] \\ &\qquad = \mu^{|\alpha|} \det(w_k^{\alpha_j}) \mu^{|\beta|} (-1)^{|\beta|+n} \det(w_k^{\beta_j}). \end{aligned}$$

In view of the fact that $\det L^{-1}|_{\lambda=1} = \prod_{j=0}^{2n-1} (\frac{1}{\rho})^{-j/2n} = \rho^{\frac{2n-1}{2}}$, this implies that the 0th order coefficient of the asymptotic expansion of $\delta(\lambda)$ for $\lambda \to \infty$ is of the form

$$\begin{aligned} \delta_{+\infty} &:= \det L^{-1}|_{\lambda=1} + \log\{\det(W^{-1}) \det(w_k^{\alpha_j})(-1)^{|\beta|+n} \det(w_k^{\beta_j}) \rho^{-(|\alpha|+|\beta|)/2n}\} \\ &= \log \rho^{\frac{2n-1}{2}} - \log \rho^{(|\alpha|+|\beta|)/2n} + \log((-1)^{|\beta|+n} \det(W^{-1}) h_\alpha h_\beta) \end{aligned}$$

where $h_\alpha = \det(w_k^{\alpha_j})$, $h_\beta \equiv \det(w_k^{\beta_j})$.

By a straightforward computation we have $\det W = (-1)^n (2n)^n$ and therefore

$$(3.5)\quad w(\lambda) = \delta(\lambda) - \delta_{+\infty} = \delta(\lambda) + \log\{(-1)^{|\beta|}(2n)^n h_\alpha^{-1} h_\beta^{-1} \rho^{(\frac{|\alpha|}{2n} - \frac{n}{2} + \frac{1}{4} + \frac{|\beta|}{2n} - \frac{n}{2} + \frac{1}{4})}.$$

The claim (3.4) then follows from the following.

Lemma 3.3. $\tilde{g}_\alpha = \frac{1}{2}\left(\frac{|\alpha|}{n} - n + \frac{1}{2}\right)$.

Proof. In view of Proposition 2.3 we obtain from (3.5) in the case $\alpha = \beta$

$$2\tilde{g}_\alpha = 2\left(\frac{|\alpha|}{2n} - \frac{n}{2} + \frac{1}{4}\right) \quad \text{or} \quad \tilde{g}_\alpha = \frac{1}{2}\left(\frac{|\alpha|}{n} - n + \frac{1}{2}\right). \quad \square$$

References

[BFK1] D. Burghelea, L. Friedlander, and T. Kappeler, *On the determinant of elliptic differential and finite difference operators in vector bundles over* S^1, Comm. Math. Phys. **138** (1991), 1–18.

[BFK2] ——, *Regularized determinants for pseudodifferential operators on vector bundles over* S^1, Integral Equations Operator Theory **16** (1993), 496–513.

[BFK3] ——, *Meyer-Vietoris type formula for determinants of elliptic differential operators*, J. Funct. Anal. **107** (1992), 34–66.

[DD] T. Dreyfus and H. Dym, *Product formulas for the eigenvalues of a class of boundary value problems*, Duke Math. J. **45** (1978), 15–37.

[Fo1] R. Forman, *Functional determinants and geometry*, Invent. Math. **88** (1987), 447–493.

[Fo2] ——, *Determinants, finite difference operators and boundary value problems*, Comm. Math. Phys. **147** (1992), 485–526.

[Fr] L. Friedlander, *The asymptotic of the determinant function for a class of operators*, Proc. Amer. Math. Soc. **107** (1989), 169–178.

[RS] B. Ray and I. Singer, *R-torsion and the Laplacian on Riemann manifolds*, Adv. in Math. **7** (1971), 145–210.

[Se1] R. Seeley, *The resolvent of an elliptic boundary value problem*, Amer. J. Math. **91** (1969), 889–920.

[Se2] ——, *Analytic extension of the trace associated with elliptic boundary value problems*, Amer. J. Math. **91** (1969), 963–983.

[Vo] A. Voros, *Spectral function, special functions and Selberg zeta function*, Comm. Math. Phys. **110** (1987), 439–465.

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