ON THE DETERMINATION OF CRITICAL VALUES FOR BARTLETT'S TEST

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Abstract. The exact critical values for Bartlett's test for homogeneity of variances based on equal sample sizes from several normal populations are tabulated. It is also shown how these values may be used to obtain highly accurate approximations to the critical values for unequal sample sizes.

Key Words: Homogeneity of variances; Bartlett's test; Lognormality of bids.

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On the Determination of Critical Values for Bartlett's Test

1. INTRODUCTION

Testing homogeneity of variances in normal populations is frequently of interest in many statistical analyses. One important use is testing the validity of the assumption of errors with constant variance in a multiple regression model or, from an experimental design point of view, equal "within group" variances in a one-factor analysis of variance model.

Suppose there are k normal populations with unknown means μ_i and variances σ_i^2, $i = 1, \ldots, k$. Independent random samples $\{X_{ij}\}$ of size n_i are taken, $i = 1, \ldots, k$; $j = 1, \ldots, n_i$. We wish to test the null hypothesis $H_0: \sigma_1^2 = \ldots = \sigma_k^2$ against the alternative H_1: $\sigma_s^2 \neq \sigma_t^2$ for some $s \neq t$. The generalized likelihood ratio test of H_0 (the Neyman and Pearson L_1-test 1931) was modified by Bartlett (1937) by replacing the biased maximum likelihood estimators of the variances by the unbiased estimators and substituting $n_i - 1$ for n_i in the weights. Specifically, the Bartlett L_1^*-test of size α has critical region $0 < L_1^* < A$, where

$$L_1^* = \prod_{i=1}^{k} (S_i^2)^{a_i} \Big/ \sum_{i=1}^{k} a_i S_i^2 \quad , \tag{1.1}$$

with $S_i^2 = (1/\gamma_i) \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_i)^2$, $\bar{X}_i = (1/n_i) \sum_{j=1}^{n_i} X_{ij}$, $\gamma_i = n_i - 1$, $\gamma = \sum_{i=1}^{k} \gamma_i$, and $a_i = \gamma_i/\gamma$. The test statistic L_1^* is the ratio of the weighted geometric mean of the sample variances to their weighted arithmetic mean (the weights are relative degrees of freedom). The critical value A is determined by $P_{H_0}(0 < L_1^* < A) = \alpha$. The L_1^*-test is a) consistent against all alternatives (Brown 1939) and b) unbiased (Pitman 1939) and is therefore usually preferred over the L_1-test which is biased except for equal sample sizes $n_1 = \ldots = n_k$, in which case the two tests are identical. It is also a well established fact (Box 1953, Box and Andersen 1955) that the L_1^*-test is rather nonrobust (i.e., sensitive to departures from normality). It is therefore recommended that it be used only when preceded by a preliminary test (e.g., the k-sample W-test, Wilk and Shapiro 1968) which, based on the data, does not reject normality. When normality can be relied on, the L_1^*-test is apparently more powerful than various other tests of H_0 (Gartside 1972). If population normality is suspect, there are a number of robust test procedures which may be used (Keselman, Games, and Clinch 1979).

2. BARTLETT'S DISTRIBUTION

Initially and for a period of about forty years, the size α Bartlett critical value was approximated by $A \cong \exp[-c\chi^2_{k-1}(1-\alpha)/\gamma]$, where $\chi^2_m(\eta)$ is the 100ηth percentile of a chi-square distribution with m degrees of freedom and c is a correction factor given by $c = 1 + [1/3(k-1)](\sum_{i=1}^{k} \gamma_i^{-1} - \gamma^{-1})$. The accuracy of this approximation is somewhat difficult to assess; however, in certain situations (very small sample sizes and/or a large number of populations) it is known to be inadequate (Bishop and Nair 1939, Hartley 1940).

Chao and Glaser (1978) have recently shown the exact null density of L_1^* (hereinafter referred to as Bartlett's distribution) to be

$$g_k(u\,;n_1\,,\ldots,n_k) = K\,u^{\gamma/2-1}\,(-\ell n\,u)^{(k-3)/2}E(u)\;,\;0 < u < 1 \tag{2.1}$$

where

$$K \equiv (2\pi)^{(k-1)/2}\prod_{i=1}^{k} a_i^{(\gamma_i-1)/2}\,\Gamma(\gamma/2)\Big/\prod_{i=1}^{k}\Gamma(\gamma_i/2)\quad ; \tag{2.2}$$

$$E(u) \equiv \sum_{r=0}^{\infty} v_r(-\ell n\,u)^r \quad ; \tag{2.3}$$

$$v_r \equiv \{1/\Gamma[(k-1)/2+r]\}\sum\nolimits_*(y_1^{t_1}y_3^{t_3}y_5^{t_5}\ldots)/(t_1!t_3!t_5!\ldots)\;,\quad r \geq 0\;, \tag{2.4}$$

with $\sum_*$ indicating summation over all distinct sequences $(t_1\,,t_3\,,t_5\,,\ldots)$ of nonnegative integers satisfying $1\cdot t_1 + 3\cdot t_3 + 5\cdot t_5 + \ldots = r$; and $y_r \equiv (-1)^{r+1}[r(r+1)]^{-1}\left(\sum_{i=1}^{k} a_i^{-r} - 1\right)B_{r+1}$, $r \geq 1$, and $B_1\,,B_2\,,\ldots$ are the Bernoulli numbers.

From a mathematical point of view, $g_k(u\,;n_1\,,\ldots,n_k)$ is not well-defined in its left-tail. For although the power series $E(u)$ is known to be convergent whenever $\exp[-2\pi\,\min(a_1\,,\ldots,a_k)] < u < 1$, it is possibly divergent otherwise. The interval of known convergence is no doubt conservative since $\exp[-2\pi\,\min(a_1\,,\ldots,a_k)]$ is actually an upper bound on the left end-point of the true interval of convergence. Fortunately, from a practical point of view this possible divergence is of little consequence in determining size α Bartlett critical values (Chao and Glaser 1978).

Let $b_k(\alpha\,; n_1, \ldots, n_k)$ be the 100αth percentile of the Bartlett's distribution, i.e.,

$$\int_{b_k(\cdot)}^{1} g_k(u\,; n_1, \ldots, n_k)du = 1-\alpha \quad \text{and} \quad b_k(\cdot) \geq \exp[-2\pi \min(a_1, \ldots, a_k] . \tag{2.5}$$

Since

$$\int_z^1 u^p(-\ell n\, u)^q du = (p+1)^{-q-1} \int_0^{(-\ell n\, z)(p+1)} y^q \exp(-y)dy$$

$$= (p+1)^{-q-1}\,\Gamma[q+1,(-\ell n\, z)(p+1)]\ ,\ p,q > -1\ ,\ z > 0$$

where $\Gamma(\phi\,, x) = \int_0^x t^{\phi-1}\exp(-t)dt$ is the incomplete gamma function; the 100αth percentile of the Bartlett distribution (the size α Bartlett critical value) is determined by the nonlinear equation

$$K(\gamma/2)^{-(k-1)/2} \sum_{r=0}^{\infty} (\gamma/2)^{-r}\, \Gamma\{r+(k-1)/2\,, [-\ell n\ b_k(\alpha\,; n_1, \ldots, n_k]\gamma/2\}$$

$$= 1-\alpha\ . \tag{2.6}$$

The solution for $b_k(\cdot)$ in (2.6) may be easily found by a Newton-Raphson procedure. We reject the hypothesis of equal variances, H_0, at the level of significance α if $L_1^* < b_k(\alpha ; n_1, \ldots, n_k)$, where L_1^* is given by (1.1).

3. DISCUSSION

The special case of equal sample sizes is of particular interest. When $n_1 = \ldots = n_k = n$, Table 1 gives the 100αth percentile of the Bartlett's distribution, $b_k(\alpha ; n) \equiv b_k(\alpha ; n_1 = n, \ldots, n_k = n)$, for $\alpha = .01, .05, .10, .25$; $k = 2(1)10$; $n = 3(1)30(10)60(20)100$. An asterisk appears in the table when (2.5) does not hold, that is, the possible divergence of $E(u)$ is a factor. A portion of Table 1 was previously given by Glaser (1976) and, in some instances, Bishop and Nair (1939). There are, however, several worthwhile reasons for presenting a more extensive version.

(1) Very small sample sizes ($n = 3,4$) were not considered by Glaser and were dealt with only in a few isolated cases by Bishop and Nair. Since Bartlett's approximate critical values for these sample sizes are known to be inadequate, the exact percentiles are given (perhaps more for the sake of completeness of the table than for practicality).

(2) When the hypothesis of equal variances is not rejected, the sample variances are often pooled to obtain an estimate of the supposed common variance. However, pooling the sample variances when, in fact, heteroscedasticity is present is likely a more serious error than not pooling when, in fact, homoscedasticity is present. To protect against a Type II error, a level of significance as large as .25 is sometimes used. For this reason, Table 1 includes the 25th percentiles.

(3) There are several sample size "gaps" in the Glaser table. The need for a more complete set of percentiles in the equal sample sizes case is prompted by the rather remarkable fact that they may be used to obtain a highly accurate approximation to the percentiles in the *unequal* sample sizes case. Specifically,

$$b_k(\alpha\,;n_1\,,\,\ldots\,,n_k) \cong (n_1/N)b_k(\alpha\,;n_1) + \ldots + (n_k/N)b_k(\alpha\,;n_k)\,, \quad (3.1)$$

where $N = \sum_{i=1}^{k} n_i$. For specified k, where $k = 2(1)10$, and *any* combination of sample sizes from $5(1)100$, the absolute error of this approximation is less than .005 (the percent relative error is less than ½ of 1%) when $\alpha = .05$, .10, or .25. The larger absolute errors occur in extreme cases such as when, for specified k, $\max(n_1\,,\ldots,n_k)$ is very large, say near 50, while all remaining sample sizes are very small, say near 5. Moreover, for virtually all cases in which $\min(n_1\,,\ldots,n_k) \geq 10$, the absolute error is less than .0005. When $\alpha = .01$, the approximation (3.1) is slightly high--the absolute error is around .015 in extreme cases and less than .005 when $\min(n_1\,,\ldots,n_k) \geq 10$. However, when the correction factors in Table 2 are used, the absolute error of the corrected approximation is as small as for the other α values. The corrected version of (3.1) for $\alpha = .01$ and specified k is

$$b_k(.01\,;n_1\,,\ldots,n_k) \cong (n_1^*/N^*)b_k(.01\,;n_1) + \ldots + (n_k^*/N^*)b_k(.01\,;n_k)\,, \quad (3.2)$$

where

$$n_i^* = n_i - j\,, \text{ if } n_i/N \text{ falls within a coefficient interval with corresponding correction factor } j$$
$$= n_i\,, \text{ otherwise}$$

and $N^* = \sum_{i=1}^{k} n_i^*$. To illustrate, suppose $k = 4$ and $n_1 = 5$, $n_2 = 6$, $n_3 = 10$, $n_4 = 50$. Using (3.1),

$$b_4(.01\,;5\,,6\,,10\,,50) \cong (5/71)(.4607) + (6/71)(.5430) + (10/71)(.7195) + (50/71)(.9433) = .8440 \;.$$

Using (3.2) and Table 2,

$$b_4(.01\,;5\,,6\,,10\,,50) \cong (5/66)(.4607) + (6/66)(.5430) + (10/66)(.7195) + (45/66)(.9433) = .8364 \;.$$

The exact value is $b_4(.01\,;5\,,6\,,10\,,50) = .8359$.

The approximations (3.1)-(3.2) may also be used when $\min(n_1\,,\ldots,n_k) = 3$ or 4; however, the accuracy is not as good as when $\min(n_1\,,\ldots,n_k) \geq 5$. The absolute error is usually less than .02 when $\min(n_1\,,\ldots,n_k) = 3$, and less than .01 when $\min(n_1\,,\ldots,n_k) = 4$. From a practical point of view, this is probably adequate. If, however, the exact value of a percentile is required, it can be obtained from (2.6) using the approximation as a starting value.

The assessment of the accuracy of the approximations was carried out as follows. For each $(k\,,\alpha)$ combination, two groups of sample sizes were considered: $\{n_i : n_i = 3(1)12\}$ and $\{n_i : n_i = 15(5)50\,,75\,,100\}$.

For each possible combination of sample sizes from each of the two groups as well as selected sample sizes from both groups, the exact percentile was obtained from (2.6) using the corresponding approximation as a starting value. In addition, the absolute error of the approximation as well as $\exp[-2\pi \min(a_1, \ldots, a_k)]$ was calculated. Fortunately, it was unnecessary to consider all possible sample sizes because of stable and predictable absolute error patterns. Based on the above computations, it appears that if $b_{k_0}(\alpha_0 ; n_0)$ exists, that is (2.5) is satisfied for a given (k_0, α_0, n_0), then $b_{k_0}(\alpha_0 ; n_1, \ldots, n_k)$ exists for all $(n_1, \ldots, n_k)$ for which $\min(n_1, \ldots, n_k) = n_0$.

(4) The traditional 2-sample test for equal variances is the F-test (Brownlee 1965, pp. 285-288). This test is equivalent to the Neyman and Pearson L_1-test for $k = 2$ and is, therefore, biased when $n_1 \neq n_2$. On the other hand, not only is the Bartlett L_1*-test unbiased for any two sample sizes but also is apparently more powerful than the F-test (Bishop and Nair 1939). Consequently, the 2-sample (equal sample sizes) Bartlett critical values are given in Table 1 from which the critical values for the L_1*-test when $n_1 \neq n_2$ may be easily determined using (3.1) or (3.2).

4. AN EXAMPLE

The data in Table 3 are the sealed bids for each of five Texas offshore oil and gas leases selected from 110 leases issued on May 21, 1968. Using probability plots, Crawford (1970) concluded that the bids for each of these five leases were lognormally distributed. Indeed, lognormality of bids for federal offshore oil and gas leases has often been the rule (Arps 1965; Brown 1969; Pelto 1971; Dougherty and Lohrenz 1976). However, Bruckner and

Johnson (1978) have shown that such a conclusion might be equivocal when the number of bids for a lease is small.

To test simultaneously for lognormality of bids (or normality of log bids), we use the k-sample W-test (Wilk and Shapiro 1968). For the ith sample, let $W(i\,;n_i)$ be the W-test statistic and $F_{n_i}(\cdot)$ is its distribution function:

$$W(i\,;n_i) = \left\{ \sum_{j=1}^{[n_i/2]} a_{n_i-j+1}(X_{i(n_i-j+1)} - X_{i(j)}) \right\}^2 \Big/ \gamma_i S_i^2 \ ,$$

where $\{X_{i(j)}\}$ is the ordered ith sample and the coefficients $\{a_j\}$ are tabled in Shapiro and Wilk (1965). Using the log bids in Table 3, we find:

$$W(1\,;\ 8) = .982 \ , \ F_8(.982) = .971 \ ;$$
$$W(2\,;10) = .970 \ , \ F_{10}(.970) = .875 \ ;$$
$$W(3\,;\ 5) = .982 \ , \ F_5(.982) = .936 \ ;$$
$$W(4\,;12) = .960 \ , \ F_{12}(.960) = .730 \ ;$$
$$W(5\,;13) = .928 \ , \ F_{13}(.928) = .320 \ .$$

Note that for *each* lease, lognormality of bids is not rejected at the .25 level of significance. To combine the results of these independent tests, we determined the standard normal percentile, G_i, corresponding to $W(i\,;n_i)$, $i = 1, \ldots, 5$. In a normal probability plot, all of the G_i's are above the null line, thus jointly indicating that lognormality of bids should not be rejected.

We now test for homogeneity of variances using Bartlett's test. Based on the log bids in Table 3, $L_1{*} = .9141$. Using Table 1, the 25th percentile is approximately

$$b_5(.25\,;\,8\,,\,10\,,\,5\,,\,12\,,\,13) \cong (8/48)(.8499) + (10/48)(.8825)$$

$$+ (5/48)(.7440) + (12/48)(.9035) + (13/48)(.9114) = .8757$$

(the exact value to four digits is .8762); and the hypothesis of equal variances is not rejected at a significance level of .25 or lower. These results are consistent with a feeling among data analysts who deal with bids for federal offshore oil and gas leases that, within a sale, "large bids do not tend to be proportionately more or less precise than small ones" (Brown 1969, p. 37).

1. Percentiles of the Bartlett Distribution

Equal Sample Sizes : $n_1 = \ldots = n_k = n$

1 per cent points

n	Number of populations, k								
	2	3	4	5	6	7	8	9	10
3	.1411	.1672	*	*	*	*	*	*	*
4	.2843	.3165	.3475	.3729	.3937	.4110	*	*	*
5	.3984	.4304	.4607	.4850	.5046	.5207	.5343	.5458	.5558
6	.4850	.5149	.5430	.5653	.5832	.5978	.6100	.6204	.6293
7	.5512	.5787	.6045	.6248	.6410	.6542	.6652	.6744	.6824
8	.6031	.6282	.6518	.6704	.6851	.6970	.7069	.7153	.7225
9	.6445	.6676	.6892	.7062	.7197	.7305	.7395	.7471	.7536
10	.6783	.6996	.7195	.7352	.7475	.7575	.7657	.7726	.7786
11	.7063	.7260	.7445	.7590	.7703	.7795	.7871	.7935	.7990
12	.7299	.7483	.7654	.7789	.7894	.7980	.8050	.8109	.8160
13	.7501	.7672	.7832	.7958	.8056	.8135	.8201	.8256	.8303
14	.7674	.7835	.7985	.8103	.8195	.8269	.8330	.8382	.8426
15	.7825	.7977	.8118	.8229	.8315	.8385	.8443	.8491	.8532
16	.7958	.8101	.8235	.8339	.8421	.8486	.8541	.8586	.8625
17	.8076	.8211	.8338	.8436	.8514	.8576	.8627	.8670	.8707
18	.8181	.8309	.8429	.8523	.8596	.8655	.8704	.8745	.8780
19	.8275	.8397	.8512	.8601	.8670	.8727	.8773	.8811	.8845
20	.8360	.8476	.8586	.8671	.8737	.8791	.8835	.8871	.8903
21	.8437	.8548	.8653	.8734	.8797	.8848	.8890	.8926	.8956
22	.8507	.8614	.8714	.8791	.8852	.8901	.8941	.8975	.9004
23	.8571	.8673	.8769	.8844	.8902	.8949	.8988	.9020	.9047
24	.8630	.8728	.8820	.8892	.8948	.8993	.9030	.9061	.9087
25	.8684	.8779	.8867	.8936	.8990	.9034	.9069	.9099	.9124
26	.8734	.8825	.8911	.8977	.9029	.9071	.9105	.9134	.9158
27	.8781	.8869	.8951	.9015	.9065	.9105	.9138	.9166	.9190
28	.8824	.8909	.8988	.9050	.9099	.9138	.9169	.9196	.9219
29	.8864	.8946	.9023	.9083	.9130	.9167	.9198	.9224	.9246
30	.8902	.8981	.9056	.9114	.9159	.9195	.9225	.9250	.9271
40	.9175	.9235	.9291	.9335	.9370	.9397	.9420	.9439	.9455
50	.9339	.9387	.9433	.9468	.9496	.9518	.9536	.9551	.9564
60	.9449	.9489	.9527	.9557	.9580	.9599	.9614	.9626	.9637
80	.9586	.9617	.9646	.9668	.9685	.9699	.9711	.9720	.9728
100	.9669	.9693	.9716	.9734	.9748	.9759	.9769	.9776	.9783

1. Percentiles of the Bartlett Distribution *(continued)*

Equal Sample Sizes : $n_1 = \ldots = n_k = n$

5 per cent points

n	Number of populations, k								
	2	3	4	5	6	7	8	9	10
3	.3123	.3058	.3173	.3299	*	*	*	*	*
4	.4780	.4699	.4803	.4921	.5028	.5122	.5204	.5277	.5341
5	.5845	.5762	.5850	.5952	.6045	.6126	.6197	.6260	.6315
6	.6563	.6483	.6559	.6646	.6727	.6798	.6860	.6914	.6961
7	.7075	.7000	.7065	.7142	.7213	.7275	.7329	.7376	.7418
8	.7456	.7387	.7444	.7512	.7574	.7629	.7677	.7719	.7757
9	.7751	.7686	.7737	.7798	.7854	.7903	.7946	.7984	.8017
10	.7984	.7924	.7970	.8025	.8076	.8121	.8160	.8194	.8224
11	.8175	.8118	.8160	.8210	.8257	.8298	.8333	.8365	.8392
12	.8332	.8280	.8317	.8364	.8407	.8444	.8477	.8506	.8531
13	.8465	.8415	.8450	.8493	.8533	.8568	.8598	.8625	.8648
14	.8578	.8532	.8564	.8604	.8641	.8673	.8701	.8726	.8748
15	.8676	.8632	.8662	.8699	.8734	.8764	.8790	.8814	.8834
16	.8761	.8719	.8747	.8782	.8815	.8843	.8868	.8890	.8909
17	.8836	.8796	.8823	.8856	.8886	.8913	.8936	.8957	.8975
18	.8902	.8865	.8890	.8921	.8949	.8975	.8997	.9016	.9033
19	.8961	.8926	.8949	.8979	.9006	.9030	.9051	.9069	.9086
20	.9015	.8980	.9003	.9031	.9057	.9080	.9100	.9117	.9132
21	.9063	.9030	.9051	.9078	.9103	.9124	.9143	.9160	.9175
22	.9106	.9075	.9095	.9120	.9144	.9165	.9183	.9199	.9213
23	.9146	.9116	.9135	.9159	.9182	.9202	.9219	.9235	.9248
24	.9182	.9153	.9172	.9195	.9217	.9236	.9253	.9267	.9280
25	.9216	.9187	.9205	.9228	.9249	.9267	.9283	.9297	.9309
26	.9246	.9219	.9236	.9258	.9278	.9296	.9311	.9325	.9336
27	.9275	.9249	.9265	.9286	.9305	.9322	.9337	.9350	.9361
28	.9301	.9276	.9292	.9312	.9330	.9347	.9361	.9374	.9385
29	.9326	.9301	.9316	.9336	.9354	.9370	.9383	.9396	.9406
30	.9348	.9325	.9340	.9358	.9376	.9391	.9404	.9416	.9426
40	.9513	.9495	.9506	.9520	.9533	.9545	.9555	.9564	.9572
50	.9612	.9597	.9606	.9617	.9628	.9637	.9645	.9652	.9658
60	.9677	.9665	.9672	.9681	.9690	.9698	.9705	.9710	.9716
80	.9758	.9749	.9754	.9761	.9768	.9774	.9779	.9783	.9787
100	.9807	.9799	.9804	.9809	.9815	.9819	.9823	.9827	.9830

1. Percentiles of the Bartlett Distribution *(continued)*

Equal Sample Sizes : $n_1 = \ldots = n_k = n$

10 per cent points

	Number of populations, k								
n	2	3	4	5	6	7	8	9	10
3	.4359	.3991	.3966	.4006	.4061	.4116	*	*	*
4	.5928	.5583	.5551	.5582	.5626	.5673	.5717	.5759	.5797
5	.6842	.6539	.6507	.6530	.6566	.6605	.6642	.6676	.6708
6	.7429	.7163	.7133	.7151	.7182	.7214	.7245	.7274	.7301
7	.7834	.7600	.7572	.7587	.7612	.7640	.7667	.7692	.7716
8	.8130	.7921	.7895	.7908	.7930	.7955	.7978	.8000	.8021
9	.8356	.8168	.8143	.8154	.8174	.8196	.8217	.8236	.8254
10	.8533	.8362	.8339	.8349	.8367	.8386	.8405	.8423	.8439
11	.8676	.8519	.8498	.8507	.8523	.8540	.8557	.8574	.8589
12	.8794	.8649	.8629	.8637	.8652	.8668	.8683	.8698	.8712
13	.8892	.8758	.8740	.8746	.8760	.8775	.8789	.8803	.8816
14	.8976	.8851	.8833	.8840	.8852	.8866	.8879	.8892	.8904
15	.9048	.8931	.8914	.8920	.8932	.8944	.8957	.8969	.8980
16	.9110	.9000	.8985	.8990	.9001	.9013	.9025	.9036	.9046
17	.9165	.9061	.9046	.9051	.9062	.9073	.9084	.9094	.9104
18	.9214	.9115	.9101	.9106	.9115	.9126	.9137	.9146	.9156
19	.9257	.9163	.9150	.9154	.9163	.9174	.9183	.9193	.9201
20	.9295	.9206	.9194	.9198	.9207	.9216	.9226	.9234	.9243
21	.9330	.9245	.9233	.9237	.9245	.9255	.9263	.9272	.9280
22	.9362	.9281	.9269	.9273	.9281	.9289	.9298	.9306	.9313
23	.9390	.9313	.9302	.9305	.9313	.9321	.9329	.9337	.9344
24	.9417	.9342	.9332	.9335	.9342	.9350	.9358	.9365	.9372
25	.9441	.9369	.9359	.9362	.9369	.9377	.9384	.9391	.9398
26	.9463	.9394	.9384	.9387	.9394	.9401	.9408	.9415	.9421
27	.9484	.9417	.9408	.9410	.9417	.9424	.9431	.9437	.9443
28	.9503	.9439	.9429	.9432	.9438	.9445	.9452	.9458	.9464
29	.9520	.9458	.9449	.9452	.9458	.9464	.9471	.9477	.9483
30	.9537	.9477	.9468	.9471	.9476	.9483	.9489	.9495	.9500
40	.9655	.9610	.9603	.9605	.9609	.9614	.9619	.9623	.9627
50	.9725	.9689	.9683	.9685	.9688	.9692	.9696	.9699	.9703
60	.9771	.9741	.9737	.9738	.9741	.9744	.9747	.9750	.9753
80	.9829	.9806	.9803	.9804	.9806	.9808	.9811	.9813	.9815
100	.9864	.9845	.9843	.9843	.9845	.9847	.9849	.9851	.9852

1. Percentiles of the Bartlett Distribution (*continued*)

Equal Sample Sizes : $n_1 = \ldots = n_k = n$

25 per cent points

n	Number of populations, k								
	2	3	4	5	6	7	8	9	10
3	.6614	.5711	.5411	.5279	.5212	.5176	.5156	.5146	*
4	.7728	.7025	.6779	.6667	.6609	.6577	.6559	.6549	.6544
5	.8299	.7737	.7534	.7440	.7391	.7363	.7347	.7338	.7333
6	.8644	.8177	.8006	.7926	.7884	.7860	.7846	.7838	.7833
7	.8873	.8475	.8327	.8258	.8221	.8200	.8188	.8181	.8177
8	.9036	.8690	.8560	.8499	.8467	.8448	.8437	.8431	.8427
9	.9158	.8852	.8737	.8682	.8653	.8636	.8626	.8620	.8617
10	.9253	.8978	.8875	.8825	.8799	.8784	.8775	.8769	.8766
11	.9329	.9080	.8985	.8940	.8916	.8902	.8894	.8889	.8887
12	.9390	.9163	.9076	.9035	.9013	.9000	.8993	.8988	.8986
13	.9442	.9232	.9152	.9114	.9094	.9082	.9075	.9071	.9068
14	.9485	.9291	.9217	.9181	.9162	.9151	.9145	.9141	.9139
15	.9522	.9342	.9272	.9239	.9221	.9211	.9205	.9201	.9199
16	.9554	.9385	.9320	.9289	.9273	.9263	.9257	.9254	.9252
17	.9582	.9424	.9363	.9333	.9317	.9308	.9303	.9300	.9298
18	.9607	.9457	.9400	.9372	.9357	.9349	.9343	.9340	.9339
19	.9629	.9487	.9433	.9407	.9393	.9384	.9379	.9377	.9375
20	.9649	.9514	.9463	.9438	.9424	.9416	.9412	.9409	.9407
21	.9667	.9539	.9489	.9466	.9453	.9445	.9441	.9438	.9437
22	.9683	.9561	.9513	.9491	.9479	.9471	.9467	.9465	.9463
23	.9697	.9580	.9535	.9514	.9502	.9495	.9491	.9489	.9487
24	.9710	.9599	.9556	.9535	.9523	.9517	.9513	.9511	.9509
25	.9722	.9615	.9574	.9554	.9543	.9537	.9533	.9531	.9530
26	.9734	.9631	.9591	.9572	.9561	.9555	.9552	.9550	.9548
27	.9744	.9645	.9607	.9588	.9578	.9572	.9569	.9567	.9566
28	.9754	.9658	.9621	.9603	.9594	.9588	.9585	.9583	.9581
29	.9762	.9670	.9635	.9617	.9608	.9603	.9599	.9597	.9596
30	.9771	.9682	.9647	.9630	.9621	.9616	.9613	.9611	.9610
40	.9830	.9763	.9737	.9725	.9718	.9714	.9712	.9710	.9710
50	.9865	.9811	.9791	.9781	.9775	.9772	.9770	.9769	.9769
60	.9888	.9843	.9826	.9818	.9813	.9811	.9809	.9808	.9808
80	.9916	.9883	.9870	.9864	.9861	.9859	.9857	.9857	.9856
100	.9933	.9907	.9896	.9891	.9889	.9887	.9886	.9886	.9885

2. Coefficient Intervals with Corresponding Correction Factors for Approximating $b_k(.01 ; n_1 , \ldots , n_k)$

Correction factors	Number of populations, k: 2	3	4	5	6	7–10
1	[.55,.65)	[.35,.55)	[.30,.50)	[.25,.45)	[.20,.40)	[.15,.35)
2	[.65,.75)	[.55,.65)	[.50,.60)	[.45,.55)	[.40,.50)	[.35,.50)
3	[.75,.80)	[.65,.70)	[.60,.65)	[.55,.60)	[.50,.55)	[.50,.55)
4	[.80,.82)	[.70,.75)	[.65,.70)	[.60,.65)	[.55,.60)	[.55,.60)
5	[.82,.84)	[.75,.80)	[.70,.75)	[.65,.70)	[.60,.65)	[.60,.65)
6	[.84,.86)	[.80,.81)	[.75,.80)	[.70,.75)	[.65,.70)	[.65,.70)
7	[.86,.88)	[.81,.82)	[.80,.81)	[.75,.80)	[.70,.75)	[.70,.75)
8	[.88,.90)	[.82,.83)	[.81,.82)	[.80,.81)	[.75,.80)	[.75,.80)
9	[.90,.91)	[.83,.84)	[.82,.83)	[.81,.82)	[.80,.81)	[.80,.81)
10	[.91,.92)	[.84,.85)	[.83,.84)	[.82,.83)	[.81,.82)	[.81,.82)
11	[.92,.93)	[.85,.86)	[.84,.85)	[.83,.84)	[.82,.83)	[.82,.83)
12	[.93,.94)	[.86,.87)	[.85,.86)	[.84,.85)	[.83,.84)	[.83,.84)
13	[.94,.95)	[.87,.88)	[.86,.87)	[.85,.86)	[.84,.85)	[.84,.85)
14	[.95,.96)	[.88,.89)	[.87,.88)	[.86,.87)	[.85,.86)	[.85,.86)
15	[.96,.97)	[.89,.90)	[.88,.89)	[.87,.88)	[.86,.87)	[.86,.87)

3. Bids for Texas Offshore Oil and Gas Leases

Bidder	Bid	Log Bid
a. Tract 228, Block A-1		
Ashland/Canadian Superior/et al.	$11,628,691	$ 16.269
Shell	6,804,691	15.733
Atlantic Richfield/Continental/Sinclair	4,221,460	15.256
Texaco	3,386,880	15.035
Humble	2,805,120	14.847
Marathon/Tenneco/et al.	1,503,360	14.223
Chevron/Pan American	1,186,560	13.987
Sun	744,710	13.521
(n_1 = 8, $\bar{x}_1$ = 14.859, s_1^2 = .842)		
b. Tract 229, Block 505		
Phillips/American Petrofina/et al.	$11,900,000	$ 16.292
Texaco	4,083,840	15.223
Atlantic Richfield/Continental/Sinclair	3,614,060	15.100
Mobil/Union	3,252,000	14.995
Chevron/Pan American	1,848,960	14.430
Ada	1,634,515	14.307
Sun	744,076	13.520
Humble	702,720	13.463
Shell	503,251	13.129
General Crude/Highland/et al.	295,776	12.597
(n_2 = 10, $\bar{x}_2$ = 14.306, s_2^2 = 1.282)		
c. Tract 286, Block 241, SE/4		
Ashland/Canadian Superior/et al.	$ 1,178,726	$ 13.980
Cabot/Occidental/et al.	581,553	13.273
Atlantic Richfield/Continental/Sinclair	301,460	12.616
Sun	186,105	12.134
Pennzoil/Midwest/et al.	112,320	11.629
(n_3 = 5, $\bar{x}_3$ = 12.726, s_3^2 = .859)		

3. Bids for Texas Offshore Oil and Gas Leases *(continued)*

d. Tract 230, Block 506

Texaco	$43,528,320	$ 17.589
Continental/Phillips/et al.	15,505,000	16.557
Chevron/Pan American	11,566,808	16.264
Champlin/Perry Bass/et al.	8,509,000	15.957
Mobil/Union/Gulf	8,123,000	15.910
Shell	5,606,611	15.539
Skelly/Cities Service/et al.	4,731,006	15.370
Humble	2,805,120	14.847
Ada	2,636,755	14.785
Sun	744,710	13.521
Marathon/Amerada/et al.	731,520	13.503
Ashland/Canadian Superior/et al.	443,635	13.003

$(n_4 = 12 , \bar{x}_4 = 15.237 , s_4^2 = 1.883)$

e. Tract 251, Block A-43

Mobil/Union/Gulf	$29,151,360	$ 17.188
Humble	18,103,680	16.712
Phillips/American Petrofina/et al.	11,515,000	16.259
Skelly/Sunray DX/et al.	10,100,000	16.128
Texaco	5,195,520	15.463
Atlantic Richfield/Continental/Sinclair	3,614,000	15.100
Shell	2,116,051	14.565
Chevron/Pan American	2,021,760	14.519
Sun	744,710	13.521
Ada	448,588	13.014
General Crude/Highland/et al.	443,635	13.003
Cabot/Colorado Oil & Gas/et al.	303,185	12.622
Marathon/Tenneco/Amerada	276,480	12.530

$(n_5 = 13 , \bar{x}_5 = 14.663 , s_5^2 = 2.635)$

Source: Crawford (1970).

REFERENCES

Arps, John J. (1965), "A Strategy of Sealed Bidding", *Journal of Petroleum Technology*, 17, 1033-1039.

Bartlett, M. S. (1937), "Properties of Sufficiency and Statistical Tests", *Proceedings of the Royal Society of London*, Series A, 160, 268-282.

Bishop, D. J., and Nair, U. S. (1939), "A Note on Certain Methods of Testing for the Homogeneity of a Set of Estimated Variances", *Journal of the Royal Statistical Society*, Supplement B, 6, 89-99.

Box, G. E. P. (1953), "Nonnormality and Tests on Variances", *Biometrika*, 40, 318-335.

_______, and Andersen, S. L. (1955), "Permutation Theory in the Derivation of Robust Criteria and the Study of the Departures from Assumption", *Journal of the Royal Statistical Society*, Series B, 17, 1-26.

Brown, George W. (1939), "On the Power of the L_1 Test for Equality of Several Variances", *Annals of Mathematical Statistics*, 10, 119-128.

Brown, Keith C. (1969), "Bidding for Offshore Oil", *Journal of the Graduate Research Center*, 38, 1-71.

Brownlee, Kenneth A. (1965), *Statistical Theory and Methodology*, 2nd ed., New York: John Wiley & Sons.

Bruckner, Lawrence A., and Johnson, Mark M. (1978), "On the Probability Distribution of Bids on Outer Continental Shelf Oil and Gas Leases", Los Alamos Scientific Laboratory Report LA-7190-MS.

Chao, Min-Te, and Glaser, Ronald E. (1978), "The Exact Distribution of Bartlett's Test Statistic for Homogeneity of Variances with Unequal Sample Sizes", *Journal of the American Statistical Association*, 73, 422-426.

Crawford, Paul B. (1970), "Texas Offshore Bidding Patterns", *Journal of Petroleum Technology*, 22, 283-289.

Dougherty, E. L., and Lohrenz, John (1976), "Statistical Analyses of Bids for Federal Offshore Leases", *Journal of Petroleum Technology*, 28, 1377-1390.

Gartside, Peter S. (1972), "A Study of Methods for Comparing Several Variances", *Journal of the American Statistical Association*, 67, 342-346.

Glaser, Ronald E. (1976), "Exact Critical Values for Bartlett's Test for Homogeneity of Variances", *Journal of the American Statistical Association*, 71, 488-490.

Hartley, H. O. (1940), "Testing the Homogeneity of a Set of Variances", *Biometrika*, 31, 249-255.

Keselman, H. J., Games, Paul A., and Clinch, Jennifer J. (1979), "Tests for Homogeneity of Variance", *Communications in Statistics*, B8, 113-129.

Neyman, Jerzy, and Pearson, Egon S. (1931), "On the Problem of k Samples", *Bulletin de l' Academie Polonaise des Sciences et des Lettres*, A, 460-481.

Pelto, Chester R. (1971), "The Statistical Structure of Bidding for Oil and Mineral Rights", *Journal of the American Statistical Association*, 66, 456-460.

Pitman, E. J. G. (1939), "Tests of Hypotheses Concerning Location and Scale Parameters", *Biometrika*, 31, 200-215.

Shapiro, S. S., and Wilk, Martin B. (1965), "An Analysis of Variance Test for Normality (Complete Samples)", *Biometrika*, 52, 591-611.

Wilk, Martin B., and Shapiro, S. S. (1968), "The Joint Assessment of Normality of Several Independent Samples", *Technometrics*, 10, 825-839.