ON THE DETERMINATION OF THE JUMP OF A FUNCTION BY ITS FOURIER COEFFICIENTS

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1. Let $f(x)$ be integrable in the sense of Lebesgue over the interval $(-\pi, \pi)$ and be periodic out side with period 2π. Let the Fourier series of $f(x)$ be

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx). \tag{1. 1}$$

Then the conjugate series of (1. 1) is

$$\sum_{n=1}^{\infty}(b_n \cos nx - a_n \sin nx) = \sum_{n=1}^{\infty} B_n(x). \tag{1. 2}$$

Let

$$\psi(t) \equiv f(x+t) - f(x-t) - D, \qquad D \equiv D(x). \tag{1. 3}$$

Szasz [2] gave the following theorem for the determination of the jump of a function by its Fourier coefficients:

THEOREM A. *If there exists a number* $D \equiv D(x)$ *such that*

$$\int_0^t \psi(t)dt = o(t) \qquad \text{and} \qquad \int_0^t |\psi(t)|dt = O(t) \tag{1. 4}$$

as $t \to 0$, *then*

$$\lim_{n\to\infty} \{\overline{S}_{2n}(x) - \overline{S}_n(x)\} = \frac{1}{\pi}\log 2 \cdot D(x),$$

where $\overline{S}_n(x)$ *is the sequence of arithmetic means of the partial sums of the conjugate series.*

Theorem A was further generalized by Chow [1] in the following form:

THEOREM B. *Under the same hypothesis as in Theorem* A

$$\lim_{n\to\infty} \{\overline{S}^{\alpha}_{2n}(x) - \overline{S}^{\alpha}_n(x)\} = \frac{1}{\pi}\log 2 \cdot D(x)$$

for $\alpha > 0$, *where* $\overline{S}^{\alpha}_n$ *is the nth Cesàro mean of order* α *of the conjugate* *series.*

Again Shin-ichi Izumi proved the following theorem.

THEOREM C. *If*

$$(1.5)\qquad \int_0^t \psi(t)dt = o(t) \ \textit{and} \int_{\pi/n}^{\pi} \frac{|\psi(t+\pi/n)-\psi(t)|}{t}dt = O(\log n),$$

then

$$\lim_{n\to\infty} \{\overline{S}'_{2n}(x) - \overline{S}'_n(x)\} = \frac{1}{\pi}\log 2\cdot D(x).$$

The object of the present paper is to prove the following theorems.

THEOREM 1. *If*

$$(1.61)\qquad \Psi(t) = \int_0^t \psi(t)dt = o(t^{\Delta}) \ \textit{and}$$

$$(1.62)\qquad \Psi^*(t) = \lim_{k\to\infty}\lim_{t\to 0}\int_{(kt)/\Delta}^{\delta} \frac{|\psi(t+\varepsilon)-\psi(t)|}{t}dt = O(\log 1/t)$$

for some $\Delta \geqq 1$, *then*

$$\lim_{n\to\infty} \{\overline{S}^{\alpha}_{2n}(x) - \overline{S}^{\alpha}_n(x)\} = \frac{1}{\pi}\log 2\cdot D(x),$$

for every $\alpha > 0$.

Theorem 1 may further be generalized to the following theorem.

THEOREM 2. *Under the same hypothesis as in Theorem* 1, *if* $\frac{m}{n} \to d$, $m > n$ *and* $n \to \infty$ *then*

$$\lim_{n\to\infty} \{\overline{S}^{\alpha}_m(x) - \overline{S}_n(x)\} = \frac{1}{\pi}\log 2\cdot D(x).$$

2. We shall make use of the following Lemmas.

LEMMA 1. (The following Lemma is due to Kogbetliantz) *If* $\alpha > -1$, *and* $\overline{T}^{\alpha}_n(x)$ *denote the nth Cesàro mean of order* α *of the sequence* $nB_n(x)$, *then*

$$\overline{T}^{\alpha}_n(x) = n\{\overline{S}^{\alpha}_n(x) - \overline{S}^{\alpha}_{n-1}(x)\},$$

$$\overline{T}^{\alpha+1}_n(x) = (\alpha+1)\{\overline{S}^{\alpha}_n(x) - \overline{S}^{\alpha+1}_{n-1}(x)\} = \frac{1}{\pi}\log 2\cdot D(x).$$

LEMMA 2. ([*Chow,* 1]) *If* $g^{\alpha}_n(t)$ *denote the nth Cesàro mean of order*

α of the sequence $g_n(t) = \cos nt \ (n \geqq 1)$, $g_0(t) = \frac{1}{2}$,

we have for $\alpha > 0$, $0 < t < \pi$

$$\left|\left(\frac{d}{dt}\right)^k g_n^\alpha(t)\right| \begin{cases} \leqq A_n^k & (k \geqq 0) \\ \leqq A_n^{-2} t^{-k-2} & (k \leqq \alpha - 2) \\ \geqq A_n^{k-\alpha} t^{-\alpha} & (k > \alpha - 2). \end{cases}$$

LEMMA 3. ([*Chow*, 1]) *If*

$$h_n^\alpha(t) = \sum_{\nu=n+1}^{2n} \frac{1}{\nu} g_\nu^\alpha(t)$$

then for $\alpha > 0$, $0 < t < \pi$, $k = 1, 2, 3, \ldots\ldots$

$$\left|\left(\frac{d}{dt}\right)^k h_n^\alpha(t)\right| \begin{cases} \leqq A_n^k & (k \geqq 0) \\ \leqq A_n^{-2} t^{-k-2} & (k \leqq \alpha - 1) \\ \leqq A_n^{k-\alpha-1} t^{-\alpha-1} & (k > \alpha - 1). \end{cases}$$

By Lemma 1, with $g_n(x)$ in place of $\overline{S}_n(x)$ we easily show that

$$(\alpha + 1)\frac{1}{n} g_n^\alpha(t) = (\alpha + 1)\frac{1}{n} g_n^{\alpha+1}(t) + g_n^{\alpha+1}(t) - g_{n-1}^{\alpha+1}(t).$$

Hence

$$(\alpha + 1)\left|\left(\frac{d}{dt}\right)^k h_n^\alpha(t)\right| \leqq (\alpha + 1) \sum_{\nu=n+1}^{2n} \frac{1}{\nu}\left|\left(\frac{d}{dt}\right)^k g_\nu^{\alpha+1}(t)\right|$$
$$+ \left|\left(\frac{d}{dt}\right)^k g_{2n}^{\alpha+1}(t)\right| + \left|\left(\frac{d}{dt}\right)^k g_n^{\alpha+1}(t)\right|$$

and the result follows from Lemma 2.

LEMMA 4. $$h_n^\alpha\left(t + \frac{\pi}{n}\right) = - h_n^\alpha(t).$$

PROOF. We have

$$g_0(t) = g_0\left(t + \frac{\pi}{n}\right) = \frac{1}{2}$$

by definition and

$$g_n(t) = + \cos nt$$

so
$$g_n\left(t + \frac{\pi}{n}\right) = - \cos nt = - g_n(t)$$

therefore
$$g_n^\alpha\left(t + \frac{\pi}{n}\right) = - g_n^\alpha(t).$$

Consequently

$$h_n^\alpha\left(t+\frac{\pi}{n}\right)=\sum_{\nu=n+1}^{2n}\frac{1}{\nu}g_\nu^\alpha\left(t+\frac{\pi}{n}\right)=\sum_{\nu=n+1}^{2n}-\frac{1}{\nu}g_\nu^\alpha(t)=-h_n^\alpha(t)$$

which proves the Lemma.

LEMMA 5. *If* $\psi(t)$ *satisfies* (1.61) *then*

$$R_1=\int_{\left(\frac{k_n\pi}{n}\right)^{1/\Delta}}^{\left(\frac{k_n\pi}{n}\right)^{1/\Delta}+\frac{\pi}{n}}\psi(t)\frac{d}{dt}h_n^\alpha(t)dt=o(1).$$

PROOF. We have

$$R_1=\left[\psi(t)\frac{d}{dt}h_n^\alpha(t)\right]_{\left(\frac{k_n\pi}{n}\right)^{1/\Delta}}^{\left(\frac{k_n\pi}{n}\right)^{1/\Delta}+\frac{\pi}{n}}-\int_{\left(\frac{k_n\pi}{n}\right)^{1/\Delta}}^{\left(\frac{k_n\pi}{n}\right)^{1/\Delta}+\frac{\pi}{n}}\psi(t)\frac{d^2}{dt^2}h_n^\alpha(t)dt$$

$$=R_{2,1}-R_{2,2}\text{ say.}$$

If $\beta=\min(\alpha,2)$ then

$$R_{1,1}=O(n)^{-\beta}[o(t)^\Delta t^{-\beta-1}]_{\left(\frac{k_n\pi}{n}\right)^{1/\Delta}}^{\left(\frac{k_n\pi}{n}\right)^{1/\Delta}+\frac{\pi}{n}}$$

$$\leqq O(n)^{-\beta}[\{(k_n^{1/\Delta}n^{1-\frac{1}{\Delta}}+\pi)^{-\beta-1+\Delta}\cdot n^{-\beta}\cdot n^{\beta+1-\Delta}\}$$
$$+k_n^{(-\beta-1+\Delta)/\Delta}\cdot n^{(\beta+1-\Delta)/\Delta}\cdot n^{-\Delta}]$$

$$=O\left[\left(\frac{1}{k_n}\right)^{\frac{\beta+1}{\Delta}-1}\cdot n^{1-\Delta\left(\frac{\beta+1}{\Delta}\right)}+\left(\frac{1}{k_n}\right)^{\frac{\beta}{\Delta}+\left(\frac{1}{\Delta}-1\right)}\cdot n^{\frac{\beta}{\Delta}(1-\Delta)}\cdot n^{\frac{1-\Delta}{\Delta}}\right]$$

$$=o(1)$$

$$R_{2,2}=O\left[\left(\frac{1}{k_n}\right)^{\frac{\beta}{\Delta}}\left(\frac{k_n}{n^{\beta+1}}\right)^{1-\frac{1}{\Delta}}+\left(\frac{1}{k_n}\right)^{\frac{\beta}{\Delta}}n^{\frac{\beta}{\Delta}(1-\Delta)}\cdot\left(\frac{k_n}{n}\right)^{\frac{\Delta-1}{\Delta}}\right]$$

$$=o(1).$$

If $\beta'=\min(\alpha,3)$ then

$$|R_{1,2}|\leqq O(n)^{1-\beta'}\int_{\left(\frac{k_n\pi}{n}\right)^{1/\Delta}}^{\left(\frac{k_n\pi}{n}\right)^{1/\Delta}+\pi/n}t^\Delta t^{-\beta-1}dt$$

$$=O[\{k_n^{\frac{1}{\Delta}}n^{1-\frac{1}{\Delta}}+\pi\}^{-\beta'+\Delta}\cdot n^{\beta'-\Delta}+k_n^{-\frac{\beta'}{\Delta}+1}\cdot n^{-1+\frac{\beta'}{\Delta}}\cdot n^{1-\beta'}]$$

$$=O[(k_n)^{-\frac{\beta'}{\Delta}+1}\cdot n^{(1-\Delta)\frac{\beta'}{\Delta}}+k_n^{-\frac{\beta'}{\Delta}+1}\cdot n^{\frac{\beta'}{\Delta}-\beta'}]$$

$$= O\left[\left(\frac{1}{k_n}\right)^{\frac{\beta'}{\Delta}} \cdot \frac{k_n}{n^{(\Delta-1)\beta'/\Delta}}\right]$$

$$= o(1)$$

Thus $R_1 = o(1)$.

LEMMA 6. *If* $\psi(t)$ *satisfies* (1.62) *then*

$$R_2 = \int_{\left(\frac{k_n\pi}{n}\right)^{1/\Delta}}^{\delta} \left|\psi\left(t+\frac{\pi}{n}\right)-\psi(t)\right| \frac{d}{dt} h_n^{\alpha}(t)dt = o(1).$$

PROOF.

$$R_2 = \int_{\left(\frac{k_n\pi}{n}\right)^{1/\Delta}}^{\delta} \frac{\left|\psi\left(t+\frac{\pi}{n}\right)-\psi(t)\right|}{t}\, t\frac{d}{dt} h_n^{\alpha}(t)dt$$

$$|R_2| \leqq O(n)^{-\beta} \int_{\left(\frac{k_n\pi}{n}\right)^{1/\Delta}}^{\delta} \frac{\left|\psi\left(t+\frac{\pi}{n}\right)-\psi(t)\right|}{t} t^{-\beta}dt$$

$$= O(n)^{-\beta}\left[O\left(\log\frac{1}{t}\right)\cdot t^{-\beta}\right]_{\left(\frac{k_n\pi}{n}\right)^{1/\Delta}}^{\delta} + O(n)^{-\beta}\int_{\left(\frac{k_n\pi}{n}\right)^{1/\Delta}}^{\delta} \log\frac{1}{t}\, t^{-\beta-1}\,dt$$

$$= R_{2,1} + R_{2,2} \text{ say.}$$

Now

$$|R_{2,1}| = \left[\left(\frac{1}{k_n}\right)^{\frac{\beta}{\Delta}} n^{\beta\left(\frac{1}{\Delta}-1\right)} O\left(\log\frac{n}{k_n\pi}\right)^{1/\Delta}\right]$$

$$= \left[n^{\beta\left(\frac{1}{\Delta}-1\right)}\left\{\left(\frac{1}{k_n}\right)^{\frac{\beta}{\Delta}} O(\log n)^{\frac{1}{\Delta}}\right\}\right]$$

$$= o(1)\left\{\left(\frac{1}{k_n}\right)^{\beta} O(\log n)\right\}^{\frac{1}{\Delta}}.$$

Now we can make $k_n \to \infty$, in a such a way, that the right hand side above tends to zero, i. e.

$$R_{2,1} = o(1).$$

And

$$|R_{2,2}| \leqq O(n)^{-\beta}\cdot O\left(\log\frac{n}{k_n\pi}\right)^{\frac{1}{\Delta}} \int_{\left(\frac{k_n\pi}{n}\right)^{1/\Delta}}^{\delta} t^{-\beta-1}dt$$

$$= \left[\left(\frac{1}{k_n}\right)^{\beta/\Delta} \cdot n^{\beta\left(\frac{1}{\Delta}-1\right)} \cdot O\left(\log\frac{n}{k_n\pi}\right)^{1/\Delta}\right]$$

$= o(1)$

as in $R_{2,1}$. Therefore $R_2 = o(1)$.

3. PROOF OF THE THEOREM.

We have

$$(3.1)\quad nB_n(x) = \frac{n}{\pi}\int_0^\pi \{f(x+t) - f(x-t)\} \sin nt\, dt$$

$$= -\frac{1}{\pi}\int_0^\pi \{f(x+t) - f(x-t)\} \frac{d}{dt}\cos nt\, dt.$$

If follows that

$$\overline{T}_n^\alpha(x) = -\frac{1}{\pi}\int_0^\pi \{f(x+t) - f(x-t)\} \frac{d}{dt} g_n^\alpha(t)dt,$$

and hence that

$$\overline{S}_{2n}^\alpha(x) - \overline{S}_n^\alpha(x) = \sum_{\nu=n+1}^{2n} \frac{1}{\nu}\overline{T}_\nu^\alpha(x)$$

$$= -\frac{1}{\pi}\int_0^\pi \{f(x+t) - f(x-t)\} \frac{d}{dt} h_n^\alpha(x)dt.$$

Let $\Omega_n = -\frac{1}{\pi}\int_0^\pi \frac{d}{dt} h_n^\alpha(t)dt.$

Then

$$\pi[\overline{S}_{2n}^\alpha(x) - \overline{S}_n^\alpha(x) - \Omega_n D(x)]$$

$$(3.2)\quad = -\left[\int_0^{\left(\frac{k_n\pi}{n}\right)^{1/\Delta}} + \int_{\left(\frac{k_n\pi}{n}\right)^{1/\Delta}}^{\delta} + \int_\delta^\pi\right] \psi(t)\frac{d}{dt} h_n^\alpha(t)dt$$

$$= I_1 + I_2 + I_3 \text{ say.}$$

Now

$$I_1 = -\left[\psi(t)\frac{d}{dt} h_n^\alpha(t)\right]_0^{\left(\frac{k_n\pi}{n}\right)^{1/\Delta}} + \int_0^{\left(\frac{k_n\pi}{n}\right)^{1/\Delta}} \psi(t)\frac{d^2}{dt^2} h_n^\alpha(t)dt.$$

Suppose $\beta = \min(\alpha, 2)$ and $\beta' = \min(\alpha, 3)$.

Then

$$I_1 = O(n)^{-\beta}\left(\frac{k_n+1}{n}\right)^{\frac{-\beta-1}{\Delta}+1} + O(n)^{1-\beta'}\int_0^{\left(\frac{k_n\pi}{n}\right)^{1/\Delta}} o(t)^\Delta t^{-\beta'-1}dt$$

$$(3.3) \qquad = O\left\{\left(\frac{1}{k_n}\right)^{\frac{\beta}{\Delta}} n^{\frac{\beta}{\Delta}(1-\Delta)}\left(\frac{k_n}{n}\right)^{\frac{\Delta-1}{\Delta}} + O(n)^{1-\beta'} \sup_{t\leqq\left(\frac{k_n\pi}{n}\right)^{1/\Delta}} \left(\frac{k_n\pi}{n}\right)^{-\frac{\beta'}{\Delta}+1}\right.$$

$$= o(1) + \sup_{t\leqq\left(\frac{k_n\pi}{n}\right)^{1/\Delta}} O\left[n^{\beta'\left(\frac{1}{\Delta}-1\right)}\left(\frac{1}{k_n}\right)^{\frac{\beta}{\Delta}}\right]\cdot k_n$$

$$= o(1).$$

Also

$$(3.4) \qquad I_2 = -\int_{\left(\frac{k_n\pi}{n}\right)^{1/\Delta}}^{\delta} \psi(t)\frac{d}{dt}h_n^\alpha(t)\,dt.$$

Changing the variable to $\left(t+\dfrac{\pi}{n}\right)$ we get

$$(3.5) \qquad I_2 = -\int_{\left(\frac{k_n\pi}{n}\right)^{1/\Delta}}^{\delta} \psi\left(t+\frac{\pi}{n}\right)\frac{d}{dt}h_n^\alpha\left(t+\frac{\pi}{n}\right)dt$$

$$= +\int_{\left(\frac{k_n\pi}{n}\right)^{1/\Delta}}^{\delta} \psi\left(t+\frac{\pi}{n}\right)\frac{d}{dt}h_n^\alpha(t)dt.$$

From (3. 4) and (3. 5) we get

$$I_2 = \frac{1}{2}\int_{\left(\frac{k_n\pi}{n}\right)^{1/\Delta}}^{\delta} \left\{\psi\left(t+\frac{\pi}{n}\right) - \psi(t)\right\}\frac{d}{dt}h_n^\alpha(t)dt.$$

Hence

$$(3.6) \qquad I_2 \leqq \frac{1}{2}\int_{\left(\frac{k_n\pi}{n}\right)^{1/\Delta}}^{\delta} \left|\psi\left(t+\frac{\pi}{n}\right) - \psi(t)\right|\frac{d}{dt}h_n^\alpha(t)dt$$

$$= o(1).$$

And

$$I_3 = \int_\delta^\pi \psi(t)\frac{d}{dt}h_n^\alpha(t)dt$$

$$(3.7) \qquad |I_3| \leqq \int_\delta^\pi |\psi(t)|\left|\frac{d}{dt}h_n^\alpha(t)\right|dt$$

$$\leqq O(n)^{-\beta}\int_\delta^\pi \frac{|\psi(t)|}{t^{1+\beta}}dt \leqq O(n)^{-\beta}\cdot M$$

$$= o(1).$$

From (3. 2), (3. 5), (3. 6) and (3. 7) we get $I = o(1)$ i. e.

$$\lim_{n\to\infty} [\overline{S}^{\alpha}_{2n}(x) - \overline{S}^{\alpha}_{n}(x) - \Omega_n D(x)] = o(1).$$

Now

$$\Omega_n = -\frac{1}{\pi}[h^{\alpha}_n(\pi) - h^{\alpha}_n(0)]$$

$$= -\frac{1}{\pi}\sum_{\nu=n+1}^{2n}\frac{1}{\nu A^{\alpha}_{\nu}}\sum_{\mu=1}^{\nu}A^{\alpha-1}_{\nu-\mu}\{(-1)^{\mu}-1\},$$

where $$A^{\alpha}_n = \frac{\Gamma(n+1)\Gamma(\alpha+1)}{\Gamma(n+\alpha+1)}.$$

Since the sequence $(-1)^n - 1$ converges (C,α) to -1 as $n\to\infty$ for every $\alpha > 0$, it follows that

$$\frac{1}{A^{\alpha}_{\nu}}\sum_{\mu=1}^{\nu}A^{\alpha-1}_{\nu-\mu}\{(-1)^{\mu}-1\} = -1 + \eta_{\nu}.$$

Where $\eta_{\nu}\to 0$ as $\nu\to\infty$ and hence that

$$\Omega_n = -\frac{1}{\pi}\sum_{\nu=n+1}^{2n}\frac{-1+\eta_{\nu}}{2} = -\frac{1}{\pi}\log 2 \qquad \text{as } n\to\infty.$$

Therefore $$\lim_{n\to\infty}[\overline{S}^{\alpha}_{2n}(x) - \overline{S}^{\alpha}_{n}(x)] = \frac{1}{\pi}\log 2\cdot D(x)$$

which completes the proof.

4. Theorem 1 may further be generalized to Theorem 2, for which we require the following Lemma.

LEMMA 7. *If* $$h^{\alpha}_{m,n}(t) = \sum_{\nu=n+1}^{2m}\frac{1}{\nu}g^{\alpha}_{\nu}(t),$$

then for $\alpha > 0$, $0 < t < \pi$, *and* $k = 1, 2, 3, \ldots\ldots$

we have

$$\left|\left(\frac{d}{dt}\right)^k h^{\alpha}_{m,n}(t)\right| \begin{cases} \leqq A^k_n & (k \geqq 0) \\ \leqq A^{-2}_n t^{-k-2} & (0 < k < \alpha - 1) \\ \leqq A^{k-\alpha-1}_n t^{-\alpha-1} & (k > \alpha - 1). \end{cases}$$

PROOF. From Lemma 3, we have

$$(\alpha+1)\frac{1}{n}g^{\alpha}_n(t) = (\alpha+1)\frac{1}{n}g^{\alpha+1}_n(t) + g^{\alpha+1}_n(t) - g^{\alpha}_{n-1}(t).$$

Hence

$$(\alpha+1)\left|\left(\frac{d}{dt}\right)^k h^{\alpha}_{m,n}(t)\right| \leqq (\alpha+1)\left|\sum_{\nu=n+1}^{m}\frac{1}{\nu}\left(\frac{d}{dt}\right)^k g^{\alpha+1}_{\nu}(t)\right| + \left|\left(\frac{d}{dt}\right)^k g^{\alpha}_{m,n}(t)\right| + \left|\left(\frac{d}{dt}\right)^k g^{\alpha}_{n}(t)\right|$$

and the result follows from Lemma 2.

The analogues of Lemmas 4, 5 and 6 can be proved similarly.

PROOF OF THEOREM 2. As in Theorem 1

$$\overline{T}^{\alpha}_{n}(x) = -\frac{1}{\pi}\int_0^{\pi}\{f(x+t)-f(x-t)\}\frac{d}{dt}g^{\alpha}_{n}(t)dt.$$

Therefore

$$\overline{S}^{\alpha}_{m}(x) - \overline{S}^{\alpha}_{n}(x) = \sum_{\nu=n+}^{m}\frac{\overline{T}^{\alpha+1}_{\nu}(x)}{\nu}$$

$$= -\frac{1}{\pi}\int_0^{\pi}\{f(x+t)-f(x-t)\}\frac{d}{dt}h^{\alpha}_{m,n}(t)dt.$$

Let $$\Omega_{m,n} = -\frac{1}{\pi}\int_0^{\pi}\frac{d}{dt}h^{\alpha}_{m,n}(t)dt.$$

Then

$$\pi[\overline{S}^{\alpha}_{m}(x) - \overline{S}_{n}(x) - \Omega_{m,n}D(x)]$$

$$= \int_0^{\pi}\psi(t)\frac{d}{dt}h^{\alpha}_{m,n}(t)dt$$

$$= I' \text{ say.}$$

I' can be shown to be $o(1)$ on similar lines as in Theorem 1, i. e.

$$\lim_{n\to\infty}[\overline{S}^{\alpha}_{m}(x) - \overline{S}^{\alpha}_{n}(z) - \Omega_{m,n}D(x)] = 0.$$

Now

$$\Omega_{m,n} = -\frac{1}{\pi}[h^{\alpha}_{m,n}(\pi) - h^{\alpha}_{m,n}(0)]$$

$$= -\frac{1}{\pi}\sum_{\nu=n+1}^{m}\left[\frac{1}{\nu A^{\alpha}_{\nu}}\sum_{\mu=1}^{\nu}A^{\alpha}_{\nu-\mu}\{(-1)^{\mu}-1\}\right].$$

And the expression within the squared bracket has been shown to be $\frac{-1+\eta_{\nu}}{2}$

in Theorem 1.

So $$\Omega_{m,n} = -\frac{1}{\pi}\sum_{\nu=n+1}^{m}\frac{-1+\eta_\nu}{n}$$

$$= -\frac{1}{\pi}\log\left(\frac{m}{n}\right) + o(1) \qquad \text{as } n \to \infty.$$

Consequently $$\lim_{u\to\infty}\,[\bar{S}_n^\alpha(x) - \bar{S}_n^\alpha(x)] = \frac{1}{\pi}\log d\cdot D(x)$$

which proves the theorem.

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