On the diameter of Eulerian orientations of graphs

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Abstract

We compare the diameter of a graph with the directed diameter of its Eulerian orientations. We obtain positive results under certain symmetry conditions.

An *Eulerian orientation* of a graph is an orientation such that each vertex has the same indegree and outdegree. A graph is *vertex-transitive* if its vertices are equivalent under automorphisms.

We show that the *directed diameter* of an Eulerian orientation of a finite vertex-transitive graph cannot be much larger than the *undirected diameter*; our bound on the directed diameter is $O(d\Delta \ln n)$ where d is the undirected diameter, Δ is the (out)degree of the vertices, and n is the number of vertices. This implies that for Eulerian orientations of vertex-transitive graphs of bounded degree, the gap between the two diameters is at most quadratic.

As a consequence, we are able to compare the word length and the positive word length of elements of a finite group in terms of a given set of generators; we show that the gap is at most nearly quadratic, where the term "nearly" refers to a factor, polylogarithmic in the order of the group.

It follows that recent polynomial bounds on the diameter of certain large classes of Cayley graphs of the symmetric group and certain linear groups automatically extend to directed Cayley graphs. The result also shows that the directed and undirected versions of long standing conjectures regarding the diameter of Cayley graphs of various classes of groups, including transitive permutation groups and finite simple groups, are equivalent.

We also show that for *edge-transitive* digraphs, the directed diameter is $O(d \ln n)$.

On the other hand, if we weaken the condition of vertex-transitivity to regularity (all vertices have the same degree), then the directed diameter is no longer polynomially bounded in terms of the undirected diameter and the maximum degree (and $\ln n = O(d \ln \Delta)$).

Our upper bounds on the diameter raise the algorithmic challenge to find paths of the length guaranteed by these results. While for undirected graphs, most (but not all) relevant proofs are algorithmic, our bounds for the directed diameter are obtained via a pigeon-hole argument based on expansion and yield existence only.

1 Introduction

Imagine a mean traffic engineer who tries to make motorists' lives hard by adversarially choosing the direction of one-way streets. In other words, given an undirected graph, the engineer will orient the graph such as to make travel difficult. We use the *diameter* to measure the engineer's success.

The *directed diameter* d^+ of a digraph is the maximum directed distance between pairs of vertices. The *undirected diameter* d of a digraph is the diameter of the undirected graph obtained by symmetrizing the edges. Clearly, $d \le d^+$. Note that $d < \infty$ if the digraph is weakly connected; $d^+ < \infty$ if the digraph is strongly connected.

Obviously, we need to impose some rules on the engineer; otherwise travel may become impossible. Requiring that the orientation be strongly connected is clearly necessary. But this is far from sufficient; even if the graph is a bounded degree expander and therefore has logarithmic diameter, the directed diameter can be exponentially worse ($\Theta(n)$).

A natural condition to impose on the engineer is to make the orientation *Eulerian*: each vertex has the same indegree as outdegree. We permit an edge to be oriented both ways, so vertices of odd degree will not preclude a solution.

The *symmetrization* of a digraph X is the undirected graph $\widetilde{X}$ obtained by adding the reverse of each edge to X.

We shall use the term "partial orientation" of the graph Y to describe any digraph X whose symmetrization is $\widetilde{X} = Y$.

If a partial orientation is Eulerian, this guarantees that from any set of vertices, the in-flow is equal to the out-flow, a good rule to avoid cars piling up in a district. It follows that the undirected and the directed

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edge-expansion rates differ in a factor ≤ 2. It seems therefore reasonable to hope that Eulerian orientation will not make the diameter increase by too much.

This hope is false, though. We shall see that even for regular digraphs (all in- and outdegrees equal), the directed diameter d^+ cannot be bounded by a polynomial of d and the maximum degree Δ (Theorem 5.1). (Note that we do not need to include $\ln n$, the logarithm of the number of vertices, among the variables in this statement, since $\ln n = O(d \ln \Delta)$.)

It may then be surprising that there is any natural class of graphs where the engineer's success is polynomially limited. A natural symmetry condition, however, turns out to suffice.

A (di)graph X is *vertex-transitive* if for every pair (v, w) of vertices, there is an automorphism ($X \to X$ isomorphism) which takes v to w.

We note that vertex-transitive graphs include Cayley graphs, an important source of efficient interconnection networks. Cayley graphs also arise in the context of Rubik's cube-type puzzles and card shuffling. In all these contexts, bounds on diameter are critical. The implications of our results to Cayley graphs are explored in the next section.

In this paper we demonstrate that for any Eulerian partial orientation of a vertex-transitive graph, the gap d^+/d is surprisingly small: $d^+/d = O(\Delta \ln n)$ where Δ is the (out)degree and n is the number of vertices.

Theorem 1.1. *Let X be a finite Eulerian digraph with n vertices, undirected diameter d and directed diameter d^+. Let Δ be the maximum outdegree of the vertices of X. Assume the symmetrization $\widetilde{X}$ is vertex-transitive. Then*

$$d^+ = O(d\Delta \ln n). \tag{1.1}$$

The precise inequality we obtain is this:

$$d^+ < 2(d+1)(\Delta+1)\ln n. \tag{1.2}$$

In view of the inequality $n < (2\Delta+1)^d$, the following is immediate.

Corollary 1.2. *Under the conditions of Theorem 1.1,*

$$d^+ = O(d^2 \Delta \ln \Delta). \tag{1.3}$$

Noting that if X is vertex-transitive then so is its symmetrization, we obtain the following corollary.

Corollary 1.3. *Let X be a finite vertex-transitive digraph with n vertices, undirected diameter d and directed diameter d^+. Let Δ be the outdegree of the vertices of X. Then inequalities* (1.2) *and* (1.3) *hold.*

In two important subcases we are able to get rid of the dependence on the degree Δ. The first of these concerns Cayley digraphs (see Definition 2.1) and has direct applications to group theory, to be discussed in Section 2.

Theorem 1.4. *Let X be a finite Cayley digraph with n vertices, undirected diameter d and directed diameter d^+. Then*

$$d^+ = O(d^2 (\ln n)^3). \tag{1.4}$$

(For the proof, see Theorem 2.15.) In applications to groups, the number n (the order of the group) tends to be exponentially large compared to the input size, so reducing the exponent of $\ln G$ in inequality (1.4) would be significant. Such reduction is possible in the context of word length in groups for the worst diameter for hereditary classes of sets of generators (see Corollary 2.4).

Another question arising from Theorem 1.4 is whether it generalizes to vertex-transitive digraphs (see Conjecture 6.1).

The second case in which the dependence on Δ can be eliminated concerns Eulerian orientations of edge-transitive graphs. A (di)graph X is *edge-transitive* if for every pair (e, f) of edges, there is an automorphism which takes e to f.

Many important networks, including the toroidal square grid and the Hamming cube over a finite alphabet are edge-transitive.

Theorem 1.5. *Let X be a finite Eulerian digraph with n vertices, undirected diameter d, and directed diameter d^+. Assume the symmetrization $\widetilde{X}$ is regular and edge-transitive. Then*

$$d^+ < 2(1+8d)\ln n. \tag{1.5}$$

If in addition X is oriented (the reverse of an edge is never an edge) then

$$d^+ < 2(1+4d)\ln n. \tag{1.6}$$

2 Applications to word length in groups

2.1 Cayley digraphs. Generators of a finite group G generate G as a *semigroup*, i. e., using *positive words* only (no inversions). The question arises, what increase in word length does generation as a semigroup require.

Definition 2.1. Let G be a group and $T \subset G$ a set of generators of G. The *Cayley digraph* $\Gamma(G, T)$ has G for its set of vertices; the ordered pair (x, y) is an edge if $yx^{-1} \in T$. We obtain an undirected Cayley graph if the set of generators is symmetric, i. e., $T = T^{-1}$. Note that Cayley digraphs are vertex transitive (G acts on $\Gamma(G,T)$ by right translations).

The positive word length of a group element $g \in G$ (in terms of a given set T of generators) is exactly the *directed distance* from the identity to g in the directed Cayley graph $\Gamma(G,T)$.

We refer to [BHKLS] for additional result and references on the diameter of Cayley graphs.

2.2 Complexity of word length and diameter. A permutation group of *degree n* is a subgroup of the symmetric group S_n. For computational purposes, we represent such a group by a list of generators.

Distance in and diameter of Cayley graphs of permutation groups are computationally hard parameters. We note that the problem of determining the diameter of the Rubik's Cube Cayley graph is still wide open in spite of considerable efforts (see Korf [Ko]). Jerrum [Je] has shown that it is PSPACE-hard to determine the length of the shortest *positive* word representing a given permutation $\sigma \in G \le S_n$ in terms of generators of the permutation group G. Even and Goldreich [EG] have shown that determining the diameter of a Cayley graph of a permutation group is NP-hard even in the case of elementary abelian 2-groups (in this case, all Cayley graphs are automatically undirected). Their proof, in combination with [ABSS], implies that even the approximate word-length is NP-hard to find.

2.3 Comparison of word length and positive word length: worst generators. The following result is an immediate consequence of Corollary 1.3.

Corollary 2.2. *Let G be a finite group and T a set of generators. If every element of G can be represented by words of length $\le d$ over T then every element of G can be represented by* **positive words** *of length less than $2(d+1)(|T|+1)\ln|G|$ over T.*

This result implies that a number of results and conjectures regarding the word length extends to positive word length. We give a selection of these below.

We define the *diameter of a group G*, $\mathrm{diam}_{\max}(G)$, as $\max_T d(\Gamma(G,T))$ where T ranges over all sets of generators of G. Similarly, the *directed diameter of G*, $\mathrm{diam}^+_{\max}(G)$, is defined as $\max_T d^+(\Gamma(G,T))$ where T ranges over all sets of generators of G. Note that these are "worst case" diameters.

Corollary 2.2 implies that the gap between these two quantities is small.

Corollary 2.3. *Let G be a finite group. Then*

$$\mathrm{diam}^+_{\max}(G) \lesssim 3\,\mathrm{diam}_{\max}(G)(\ln|G|)^2. \tag{2.7}$$

The actual constant we obtain is $2/\ln 2 \approx 2.885$.

The $\lesssim$ (less than or asymptotically equal to) relation in Corollary 2.3 refers to a $(1+o(1))$ factor omitted on the right-hand side. This factor approaches 1 as $|G| \to \infty$. An introduction to this notation, extended to not necessarily positive quantities, can be found in [Ba3].

Corollary 2.3 follows from Corollary 2.2. Rather than giving the direct proof, we shall infer the result from a more general statement below (Corollary 2.4).

2.4 Weakly hereditary classes of sets of generators. We prove a more general result about classes of sets of generators satisfying a weak hereditary condition.

Let $\mathcal{C}$ be a class (set) of pairs (G,T) where G is a finite group and T is a set of generators of G. Let $k = k(G)$ be a function which associates positive numbers with finite groups. We say that the class $\mathcal{C}$ is $k(G)$-*hereditary* if for every pair $(G,T) \in \mathcal{C}$ there is a pair $(G,T') \in \mathcal{C}$ such that $T' \subseteq T$ and $|T'| \le k(G)$. Let $\mathcal{G}(\mathcal{C})$ denote the set of those groups G for which $(G,T) \in \mathcal{C}$ for some set T of generators of G. For $G \in \mathcal{G}(\mathcal{C})$, let $\mathrm{diam}_{\max}(G,\mathcal{C})$ denote $\max_T d(\Gamma(G,T) : (G,T) \in \mathcal{T}$. We define $\mathrm{diam}^+_{\max}(G,\mathcal{C})$ analogously.

Corollary 2.4. *Let $\mathcal{C}$ be a k-hereditary class of pairs (G,T). Then, for $G \in \mathcal{G}(\mathcal{C})$,*

$$\mathrm{diam}^+_{\max}(G,\mathcal{C}) \le 2(\mathrm{diam}_{\max}(G,\mathcal{C})+1)(k(g)+1)\ln|G|. \tag{2.8}$$

Proof. Let T be a "worst" set of generators with respect to the directed diameter, i. e., let $(G,T) \in \mathcal{C}$ and $\mathrm{diam}^+_{\max}(G,\mathcal{C}) = d^+(G,T)$. We may assume $|T| \le k(G)$; otherwise we replace T by T' (omitting redundant generators cannot decrease the diameter). Consequently, by Corollary 2.2, $d^+(G,T) \le 2(1 + d(G,T))(1+k(G))\ln|G|$. □

Corollary 2.3 follows by letting $\mathcal{C}$ consist of all pairs (G,T) where G is a finite group and T a set of generators; and setting $k(G) = \log_2|G|$ (because every nonredundant set of generators has $\le \log_2|G|$ elements). Inequality (2.8) yields

$\mathrm{diam}^+_{\max}(G) \le 2(\mathrm{diam}_{\max}(G)+1)(\log_2|G|+1)\ln|G| \lesssim (2/\ln 2)\,\mathrm{diam}_{\max}(G)(\ln|G|)^2$, as claimed.

2.5 Simple groups. The following conjecture was proposed in [BS2].

Conjecture 2.5. ([BS2]) *If G is a nonabelian finite simple group then* $\operatorname{diam}_{\max(G)} = O((\ln|G|)^c$ *for some absolute constant c.*

Corollary 2.3 implies that if this conjecture is true for undirected Cayley graphs then it remains true for directed Cayley graphs; the exponent c increases by ≤ 2.

In major progress announced recently, Conjecture 2.5 has been confirmed by Harald A. Helfgott [He] for 2-dimensional linear groups over fields of prime order:

Theorem 2.6. (Helfgott [He])

$$\operatorname{diam}_{\max}(SL_2(\mathbb{Z}/p\mathbb{Z})) = O((\log p)^c).$$

Corollary 2.3 immediately extends this result to the directed worst-case diameter of the groups $SL_2(\mathbb{Z}/p\mathbb{Z})$, adding only 2 to Helfgott's astronomical exponent c.

We note that Helfgott's proof is based on expansion and therefore is non-algorithmic; this is one of the very few cases this author is aware of when the proof of a diameter bound for a class of *undirected* Cayley graphs does not yield an algorithm.

A major open case in Conjecture 2.5 concerns the alternating (and symmetric) groups which we discuss in the next subsection.

2.6 Permutation groups. Permutation groups of degree n are subgroups of the symmetric group S_n.

Lemma 2.7. ([Ba1]) *Every nonredundant set of generators in a permutation group of degree n has size $< 2n$.*

This follows from the result that S_n has no subgroup chain of length $2n$ [Ba1].

Corollary 2.8. *If $G \le S_n$ is a permutation group of degree n then*

$$\operatorname{diam}^+_{\max}(G) \lesssim 4 \operatorname{diam}_{\max}(G) n \ln|G|. \qquad (2.9)$$

Proof. We apply Corollary 2.4 to the class $\mathcal{C}$ consisting of all pairs (G,T) where G is a permutation group and T a set of generators of G. If $G \le S_n$ then we can set $k(G) = 2n$ by Lemma 2.6. □

For permutation groups $G \le S_n$ we have $\operatorname{diam}_{\max}(G) \le \exp(\sqrt{n \ln n}(1 + o(1)))$ ([BS1, BS2]). Corollary 2.3 shows that this bound extends to Cayley digraphs.

Corollary 2.9. *Let $G \le S_n$ be a permutation group of degree n. Then all Cayley digraphs of G have directed diameter $\le \exp(\sqrt{n \ln n}(1 + o(1)))$.*

Proof. Indeed, by Corollary 2.8, the directed diameter adds at most a factor of $O(n \ln(n!)) = O(n^2 \ln n)$ to the undirected diameter; this factor is is subsumed by the little-oh term in the exponent. □

A *transitive permutation group* is a permutation group $G \le S_n$ such that for every x, y in the permutation domain, some element of G moves x to y.

20 years ago, Kornhauser et al. [KMS] asked the following question:

Problem 2.10. ([KMS]) *Do all Cayley graphs of transitive permutation groups of degree n have diameter, polynomially bounded in n?*

The most important special case was stated as a conjecture in [BS1]:

Conjecture 2.11. *All Cayley graphs of the symmetric groups S_n and the alternating groups A_n have polynomially bounded diameters (as a function of n).*

It is shown in [BS2] that Conjecture 2.11 implies that all transitive permutation groups of degree n have quasipolynomial ($\le \exp((\log n)^C)$) diameters ($C = 3 + o(1)$), linking Conjecture 2.11 to a positive answer to Problem 2.10.

Regarding these problems, Corollary 2.8 implies that the directed version follows from the undirected version. In particular, we obtain the following:

Corollary 2.12. *A positive answer to Problem 2.10, if true, remains valid for the directed diameters of Cayley digraphs of transitive permutation groups.*

The strongest result to date regarding Conjecture 2.11 has been the undirected version of the following.

Theorem 2.13. *If $G = S_n$ or A_n and $T \subseteq G$ contains a permutation that fixes at least 67% of the permutation domain then the diameter of the Cayley digraph $\Gamma(G,T)$ is at most cn^C where c, C are absolute constants.*

The undirected version of this result was proved by Babai, Beals and Seress in [BBS]. The directed version follows by applying Corollary 2.4 to the class $\mathcal{C}$ consisting of all pairs (G,T) where G is a permutation group and T a set of generators of G which includes a permutation that fixes at least 67% of the permutation domain. If $G \le S_n$ we can set $k(G) = 2n$ by Lemma 2.6. The only caveat is that in the course of deleting redundant generators, we must retain at least one generator that fixes at least 67% of the permutation domain. □

We note that the proof in [BBS] strongly relies on commutators and therefore on the undirected nature of the graph and a direct extension of those methods to cover the directed case does not seem evident.

Again using commutators, Babai and Hayes [BH] proved the undirected version of the following result.

THEOREM 2.14. *If σ and τ are two permutations of $\{1, \ldots, n\}$ selected randomly from S_n and G is the group generated by σ and τ then with probability approaching 1 as $n \to \infty$, the diameter of the Cayley digraph $\Gamma(G, \{\sigma, \tau\})$ is at most n^C for some absolute constant C.*

We note that the group G in this result is almost always either S_n or A_n by Dixon's classical result [Dix].

The directed version of Theorem 2.14 follows from the undirected version by applying Corollary 2.4 to the class $\mathcal{C}$ of those pairs (G, T) where $G = S_n$ $(n = 1, 2, \ldots)$, T is a pair of generators of G, and $d(\Gamma(G, T)) \le n^C$. By Theorem 2.14, for every n, almost all pairs $T \subset S_n$ will occur in pairs $(S_n, T) \in \mathcal{C}$.

2.7 Arbitrary Cayley digraphs: a near-quadratic bound. Corollary 2.3 shows that the gap between the directed and undirected worst diameters is nearly linear, and Corollary 2.4 extends this observation to the worst diameters over groups with k-hereditary classes of sets of generators.

In this section we use the term **"nearly"** to indicate factors of $\le (\log |G|)^c$ for some constant c.

We shall show that the gap is at most nearly quadratic for arbitrary sets of generators; this is the main result of this section.

THEOREM 2.15. *Let G be a finite group, $T \subseteq G$ a set of generators, and let d be the undirected diameter of the Cayley digraph $\Gamma(G, T)$. Then the directed diameter of $\Gamma(G, T)$ is $O(d^2 (\log |G|)^3)$.*

(The constant implied by the big-oh notation is absolute.)

All our previous arguments relied on removing redundant generators; we were somehow assured that this step would only moderately increase the diameter. (E. g., this is why we had to take care not to remove a generator that fixes 67% of the permutation domain in the proof of the directed version of Theorem 2.13.) The following result shows that one can always reduce the set of generators to "reasonable" size without a significant penalty in increased undirected diameter.

LEMMA 2.16. *Let G be a finite group of order n, $T \subseteq G$ a set of generators, and let d be the diameter of the Cayley graph $\Gamma(G, T \cup T^{-1})$. Set $\ell = \lfloor \log_2 n + 3 \log_2 \log_2 n \rfloor$. Then T has a subset S such that $|S| \le \ell d$ and the diameter of the Cayley graph $\Gamma(G, S \cup S^{-1})$ is at most ℓd.*

Proof. It is shown in [BE] that G has a sequence of ℓ elements, $g_1, \ldots, g_\ell$, such that all elements of G can be written as subproducts $g_1^{\varepsilon_1}, \ldots, g_\ell^{\varepsilon_\ell}$ $(\varepsilon_i \in \{0, 1\})$. Let us represent each g_i as a word w_i of length $\le d$ over T; let T_i be the set of those elements of T which occur in this word. Set $S = \bigcup_{i=1}^{\ell} T_i$. Note that $|S| \le \ell d$. Since every element of G can be written as a word of length $\le \ell$ in the g_i and each g_i is a word of length $\le d$ over S, we conclude that the (undirected) diameter of the Cayley graph $\Gamma(G, S \cup S^{-1})$ is at most ℓd. □

We can now complete the proof of Theorem 2.15.

Proof. Use Lemma 2.16 to replace T by a subset $S \subseteq T$ of size $O(d \log |G|)$ such that the undirected diameter of $\Gamma(G, S)$ remains $O(d \log |G|)$. Now infer from Corollary 1.3 that the directed diameter of $\Gamma(G, S)$ is $O(d^2 (\log |G|)^3)$. This is then an upper bound on the directed diameter of $\Gamma(G, T)$. □

In this section we made a number of claims of the form that if certain Cayley digraphs have polynomially bounded undirected diameters then they have polynomially bounded directed diameters. Note that all these claims are special cases of Theorem 2.15 as far as polynomiality concerns. However, in all cases except Theorem 2.15, the dependence of the directed diameter on the undirected was in fact linear; only the generic reduction of generators required a quadratic increase. This quadratic increase would be removed if the following conjecture holds.

CONJECTURE 2.17. *Let G be a finite group, $T \subseteq G$ a set of generators, and let d be the diameter of the Cayley graph $\Gamma(G, T \cup T^{-1})$. Then T has a subset S such that $|S| \le (\log |G|)^C$ and the diameter of the Cayley graph $\Gamma(G, S \cup S^{-1})$ is $O(d (\log |G|)^C)$ where C is an absolute constant.*

3 Expansion of vertex-transitive digraphs

For a digraph $X = (V, E)$, let $X^- = (V, E^-)$ be the reverse digraph, defined by $E^- = \{yx \mid xy \in E\}$. Let

$\widetilde{X} = (V, \widetilde{E})$ be the *symmetrized* version of X, defined by $\widetilde{E} = E \cup E^-$.

For a vertex x in a digraph, $\deg^+(x)$ denotes its *outdegree* and $\deg^-(x)$ its *indegree.* A digraph is *Eulerian* if for each vertex x we have $\deg^+(x) = \deg^-(x)$. Note that finite vertex-transitive digraphs are Eulerian.

For a digraph $X = (V, E)$ and a subset $S \subseteq V$, let $\delta^+(S)$ denote the set of edges from S to $V \setminus S$ and $\delta^-(S)$ the set of edges from $V \setminus S$ to S.

Observation 3.1. *If $X = (V, E)$ is an Eulerian digraph and $S \subseteq V$ then $|\delta^+(S)| = |\delta^-(S)|$.*

Proof. Indeed, for all digraphs, $|\delta^+(S)| - |\delta^-(S)| = \sum_{x \in S}(\deg^+(x) - \deg^-(x))$. For Eulerian digraphs, the right-hand side is zero. □

It follows, in particular, that if an Eulerian digraph is weakly connected then it is strongly connected. Therefore if the left-hand side of inequality (1.1) is infinite then so is the right-hand side, so Theorem 1.1 covers this case. Henceforth we assume that X is (strongly) connected and so d and d^+ are finite.

For a graph $X = (V, E)$ and a subset $S \subseteq V$, let $\partial^+(S)$ denote the *out-boundary* of S, i. e.,

$$\partial^+(S) = \{y \in V \setminus S \mid (\exists x \in S)(xy \in E)\}.$$

The *in-boundary* $\partial^-(S)$ is defined analogously:

$$\partial^-(S) = \{y \in V \setminus S \mid (\exists x \in S)(yx \in E)\}.$$

We define the *boundary* $\partial(S)$ as the union of these:

$$\partial(S) = \partial^+(S) \cup \partial^-(S).$$

The following isoperimetric inequality was proved in [BSz].

Theorem 3.2. *Let $X = (V, E)$ be an undirected connected finite vertex-transitive graph of diameter d. Then for any subset $S \subset V$ such that $0 < |S| \le |V|/2$ we have*

$$\frac{|\partial S|}{|S|} \ge \frac{2}{2d+1}. \tag{3.10}$$

This result appears in [BSz] as Corollary 2.3. Previously Aldous [Al] proved a slightly weaker lower bound, $1/2d$, for Cayley graphs; that bound was extended to all vertex-transitive graphs in [Ba2]. The more elegant approach of [BSz] then yielded the stated bound.

We shall need a directed version of this inequality.

Corollary 3.3. *Let $X = (V, E)$ be a connected finite Eulerian digraph with maximum outdegree Δ and undirected diameter d. Assume $\widetilde{X}$, the symmetrization of X, is vertex-transitive. Then for any subset $S \subset V$ such that $0 < |S| \le |V|/2$ we have*

$$\frac{|\partial^+ S|}{|S|} \ge \frac{2}{(\Delta+1)(2d+1)}. \tag{3.11}$$

The proof of Corollary 3.3 is a combination of Theorem 3.2 with the following observation.

Lemma 3.4. *Let $X = (V, E)$ be an Eulerian digraph with maximum indegree Δ. Let $S \subseteq V$. Then*

$$|\partial S| \le (\Delta + 1)|\partial^+(S)|. \tag{3.12}$$

Proof. $|\delta^-(S)| = |\delta^+(S)| \le \Delta|\partial^+(S)|$. Therefore $|\partial(S)| \le |\partial^+(S)| + |\partial^-(S)| \le |\partial^+(S)| + |\delta^-(S)| \le (\Delta+1)|\partial^+(S)|$. □

As a digression, we mention a corollary to this observation which may not have previsouly been pointed out.

The *expansion rate* $\varepsilon(X)$ of a digraph $X = (V, E)$ is the minimum, over all subsets $S \subset V$ such that $0 < |S| \le |V|/2$, of the isoperimetric ratio $|\partial^+(S)|/|S|$. An infinite family of digraphs X_n is a *family of expanders* if $\inf_n \varepsilon(X_n) > 0$. Of particular interest are families of bounded degree expanders.

Corollary 3.5. *A family of Eulerian digraphs X_n of bounded degree is a family of expanders if and only if the symmetrized versions $\widetilde{X}_n$ form a family of expanders.* □

Proposition 3.6. *Let $\varepsilon > 0$. Suppose that for every subset S of the vertex set of the digraph $X = (V, E)$ satisfying $0 < |S| \le |V|/2$ we have $|\partial^+(S)|/|S| \ge \varepsilon$ and $|\partial^-(S)|/|S| \ge \varepsilon$. Then the directed diameter of X is $d^+ < 2 \ln n / \ln(1 + \varepsilon) < 2(1 + 1/\varepsilon) \ln n$, where $n = |V|$.*

Proof. Let u and v be arbitrary vertices. Starting from u, in $\le t$ steps, at least $\min\{(n+1)/2, (1+\varepsilon)^t\}$ vertices can be reached along directed paths. Consequently, more than $n/2$ vertices are reached in $\ln n/\ln(1 + \varepsilon)$ steps. Similarly, from v, more than $n/2$ vertices can be reached in $t \le \ln n/\ln(1 + \varepsilon)$ steps in X^- (the reverse of X). These two sets of more than $n/2$ vertices must overlap, say at a vertex z, so the directed distance from u to z is at most t and from z to v is also at most t. □

Remark 3.7. *A slight improvment of the bound follows for digraphs of large degree noting that in at most one step we can reach* $1+\Delta$ *vertices, where and* Δ *is the minimum of the indegrees and outdegrees of all vertices. The bound becomes* $2 + 2\ln(n/(1+\Delta))/\ln(1+\varepsilon)$.

We are now in the position to complete the proof of Theorem 1.1. We note that if X is a finite connected digraph with vertex-transitive symmetrization, then according to Corollary 3.3, the conditions of Proposition 3.6 hold with $\varepsilon = 2/(\Delta+1)(2d+1)$. Therefore, by Proposition 3.6,

$$d^+ < (1+(\Delta+1)(2d+1))\ln n < 2(\Delta+1)(d+1)\ln n. \tag{3.13}$$ □

4 Edge-transitive digraphs

In this section we prove Theorem 1.5.

For edge-transitive (undirected) graphs, the following isoperimetric inequality appears as Corollary 2.6 in [BSz].

Theorem 4.1. *Let* $X = (V, E)$ *be an undirected connected finite edge-transitive graph of diameter* d. *Then for any subset* $S \subset V$ *such that* $0 < |S| \le |V|/2$ *we have*

$$\frac{|\delta(S)|}{|S|} \ge \frac{r}{2d} \tag{4.14}$$

where r *denotes the harmonic mean of the maximum and minimum degrees:* $r = 2/(1/\deg_{\max} + 1/\deg_{\min})$.

It follows that if the symmetrization $\widetilde{X}$ of an Eulerian digraph X is regular and edge-transitive then

$$\frac{|\delta^+(S)|}{|S|} \ge \frac{\deg}{4d} \tag{4.15}$$

where deg denotes the undirected degree. Since

$$|\delta^+(S)| \le \Delta|\partial^+(S)| \tag{4.16}$$

(where Δ is the maximum indegree), we infer the following vertex-expansion bound.

Corollary 4.2. *If the symmetrization* $\widetilde{X}$ *of an Eulerian digraph* X *is regular and edge-transitive then*

$$\frac{|\partial^+(S)|}{|S|} \ge \frac{\deg/\Delta}{4d} \ge \frac{1}{4d}. \tag{4.17}$$

We also note that for *oriented* digraphs (no out-neighbor is an in-neighbor), $\deg/\Delta = 2$ and therefore

$$\frac{|\partial^+(S)|}{|S|} \ge \frac{1}{2d}. \tag{4.18}$$

To complete the proof of Theorem 1.5, we combine Corollary 4.2 with Proposition 3.6. We use inequality (4.18) for the oriented case. □

5 Regular digraphs: negative results

In this section we address the question whether the main results of this paper extend to regular digraphs, without the vertex-transitivity condition on their symmetrization.

Unless expressly stated otherwise, all digraphs will be *without parallel edges,* so a digraph can be described as a pair $G = (V, E)$ where $E \subseteq V \times V$. While this was a tacit assumption in this entire paper, it was immaterial in the preceding sections; permitting parallel egdes would not change the validity of the positive results discussed so far. The negative results to be discussed in this section would, however, change significantly if parallel edges were permitted (see Proposition 5.6).

An edge of the form (x, x) is called a *loop.* We permit loops (at most one per vertex). A loop adds one to both the indegree and the outdegree of a vertex and has no influence on either the directed or the undirected diameter. If we delete all loops from an Eulerian digraph, it remains Eulerian.

A digraph is *regular* if all indegrees and all outdegrees are equal. For instance, finite vertex-transitive digraphs are regular. Note that every regular digraph is Eulerian.

Corollary 1.2 asserts that for Eulerian digraphs X, if the symmetrization $\widetilde{X}$ is vertex-transitive then the directed diameter d^+ is bounded by a polynomial of the undirected diameter d and the maximum degree Δ. It is natural to ask if such a polynomial bound holds if the condition of vertex-transitivity of $\widetilde{X}$ is relaxed to regularity. Below we give a negative answer to this question.

Theorem 5.1. *For infinitely many values of* n, *there exist regular digraphs with* n *vertices, outdegree 2, and logarithmic undirected diameter* ($d = O(\log n)$) *such that the directed diameter is* $d^+ = \Omega(\sqrt{n})$.

Another natural question is whether the directed diameter d^+ is bounded by some function of the undirected diameter d for vertex-transitive digraphs. We are unable to answer this question (see Problem 6.4). However, if we weaken the condition of vertex-transitivity to regularity then the answer is negative.

Theorem 5.2. *For infinitely many values of* n, *there exist regular digraphs with* n *vertices, undirected diameter* $d = 3$, *and directed diameter* $\Omega(n^{1/3})$.

Both theorems will follow from the following lemma.

LEMMA 5.3. *Let X be an undirected graph with m vertices and e edges. Let G be a regular digraph of outdegree r with e vertices. Then there exists a regular digraph D with ne vertices, outdegree $r+1$, undirected diameter $< (d(X)+1)(d(G)+1)$, and directed diameter $\geq m-1$.*

Proof. Let $[m] = \{1, \ldots, m\}$. For a permutation $\pi : [m] \to [m]$, consider the digraph $P(\pi)$ on the vertex set $[m]$ with edges (x, x^π) for all $x \in [m]$.

For $1 \leq i < j \leq m$, consider the cyclic permutation $\pi(i,j) = (i, i+1, \ldots, j)$. Call the corresponding digraph $P(i,j)$; it consists of a cycle of length $j-i+1$ and $m-j+i-1$ loops. Note that $P(i,j)$ is regular of outdegree 1.

Let now $X = ([m], E)$ be an undirected graph on the vertex set $[m]$.

Consider the digraph $D_1(X)$ with vertex set $[m] \times E$ defined as the disjoint union of the digraphs $P(i,j)$ for all pairs $\{i,j\} \in E$, $i<j$. Here $P(i,j)$ appears on the vertex subset $[m] \times \{\{i,j\}\}$. Note that $D_1(X)$ is regular of outdegree 1.

Let G be a regular digraph of outdegree r on the vertex set E. (So the edges of X will label the vertices of G.) Let $D_2(G)$ denote the union of m disjoint copies of G. Each *layer* $\{i\} \times E$ ($i \in [m]$) will serve as the vertex set of a copy of G; so $V(D_2(G)) = [m] \times E = V(D_1(G))$.

Let, finally, $D(X,G) = D_1(X) \cup D_2(G)$.

Note that $D(X,G)$ is a regular digraph of outdegree $r+1$ with $n = m|E|$ vertices.

CLAIM 5.4. $d(D(X,G)) \leq (d(X)+1)(d(G)+1) - 1$.

Indeed, to get from vertex $(h, \{i,j\})$ to $(h', \{i',j'\})$ in $D(X,G)$, consider a path $h = \ell_0, \ell_1, \ldots, \ell_f = h'$ in X where $f \leq d(X)$. Consider the sequence $v_t = (\ell_t, \{\ell_t, \ell_{t+1}\})$ of vertices in $D(X,G)$ ($t = 0, 1, \ldots, f-1$). By definition, v_t is adjacent to $w_{t+1} := (\ell_{t+1}, \{\ell_t, \ell_{t+1}\})$ in the symmetrization of $D(X,G)$. Set $w_0 = (h, \{i,j)\})$ and $w_f = (h', \{i',j'\})$. Let P_t be a shortest undirected path between w_t and v_t ($t = 0, \ldots, f$). The length of P_t is at most $d(G)$. Now, by alternating between these $f+1$ path and the f edges $\{v_t, w_{t+1}\}$ we obtain a path of length $\leq (f+1)d(G)+f$, proving the Claim. □

CLAIM 5.5. $d^+(D(X,G)) \geq m-1$.

Indeed, $D(X,G)$ has m layers and every edge goes at most one layer up so going for any two edges $z_1, z_2 \in E$, moving from vertex $(1, z_1)$ to (m, z_2) in $D(X,G)$ takes at least $m-1$ directed steps. □

Now to prove the main results of this section, we choose different pairs of graphs (X,G).

If we choose X to be the complete graph K_m and G the complete graph $K_{\binom{m}{2}}$ (viewed as a digraph by orienting every edge in both directions) then $D(X,G)$ is a regular digraph with $n = m^2(m-1)/2$ vertices, outdegree $\Delta = \binom{m}{2} - 1 = \Theta(n^{2/3})$, and from our Claims we obtain $d(D(X,G)) = 3$ and $d^+(X,G) \geq m-1 > (2n)^{1/3} - 1$, verifying Theorem 5.2.

If we choose X to be a regular graph of degree 3 with logarithmic ($O(\log m)$) diameter, and G to be a regular digraph with $3m/2$ vertices, outdegree 2 and logarithmic ($O(\log m)$) directed diameter, then $D(X,G)$ is a regular digraph with $n = 3m^2/2$ vertices, outdegree $\Delta = 3$, and from our Claims we obtain $d(D(X,G)) = O(\log m) = O(\log n)$ and $d^+(D(X,G)) \geq m-1 = \sqrt{2n/3} - 1$. This verifies Theorem 5.1 with outdegree 3 rather than 2. To reduce the outdegree to 2, we replace each vertex by a directed 3-cycle and distribute the 3 incoming and 3 outgoing edges arbitrarily between the 3 new vertices, one incoming and one outgoing edge to each new vertex. This operation triples the number of vertices; it at most doubles the undirected diameter; and it does not decrease the directed diameter. □

We remark that if we permit parallel edges then Theorem 5.2 can be trivially strengthened:

PROPOSITION 5.6. *If we permit parallel edges then for every n, there exists a regular digraph on n vertices, of undirected diameter 1 and directed diameter $n-1$.*

Proof. Take the union of the $\binom{n}{2}$ digraphs $P(i,j)$ ($1 \leq i < j \leq n$) defined in the previous proof (setting $m := n$) viewing the edge set as a multiset. □

Let us now return to our convention that parallel edges are not permitted. Then, in contrast to Proposition 5.6, undirected diameter 1 implies directed diameter ≤ 2 for Eulerian digraphs.

PROPOSITION 5.7. *If an Eulerian digraph X has undirected diameter $d(X) = 1$ then its directed diameter is $d^+(X) \leq 2$.*

Proof. We may assume X has no loops. If X has n vertices and $d(X) = 1$ then for every vertex x we have $\deg^+(x) = \deg^-(x) \geq (n-1)/2$. Take two vertices x, y such that there is no edge from x to y. Consider the set $N^+(x)$ of out-neighbors of x and the set $N^-(y)$ of in-neighbors of y. Each of these sets are subsets of $V(X) \setminus \{x,y\}$; therefore they must overlap, proving that there is a directed path of length 2 from x to y. □

The combination of Theorem 5.2 and Proposition 5.7 leave the case of undirected diameter 2 open. This gap is closed for Eulerian (not necessarily regular) digraphs by the following result.

Theorem 5.8. *For infinitely many values of n, there exist Eulerian digraphs with n vertices, undirected diameter $d = 2$, and directed diameter $\Omega(n^{1/3})$.*

Proof. Let $m \geq 3$, and let p be the smallest prime greater than $\binom{m}{2}$. Consider the following digraph $F_1(m)$ with vertex set $[m] \times \mathbb{Z}/p\mathbb{Z}$. So $F_1(m)$ has $n := mp \sim m^3/2$ vertices. Let $f : E(K_m) \to (\mathbb{Z}/p\mathbb{Z})^*$ be an injection, where the asterisk indicates the multiplicative group (zero omitted). For $1 \leq i < j \leq m$ and $x \in \mathbb{Z}/p\mathbb{Z}$, let $\psi(i,j,x)$ denote the directed cycle of length $j - i + 1$ starting at vertex (i, x) and moving up one layer at a time at slope of $f(i,j)$ until it reaches layer j; then completes the cycle in a single jump back to its start. So the cycle will traverse the vertices $v_t(i,j,x) := (i + t, x + tf(i,j))$ for $t = 0, \dots, j$ in this order. Let $F_1(m)$ be the union of all cycles $\psi(i,j,x)$ for $1 \leq i < j \leq m$ and $x \in \mathbb{Z}/p\mathbb{Z}$.

By a "horizontal shift" by $s \in \mathbb{Z}/p\mathbb{Z}$ of the vertex set of $F_1(m)$ we mean the map $(i,x) \mapsto (i, x+s)$. Clearly, $F_1(m)$ is invariant under horizontal shifts.

It is clear that in this union, no edge is repeated: the slope of an edge determines the entire cycle from which the edge comes, up to a horizontal shift.

Let $F_2(m)$ be the union of m disjoint copies of the complete graph K_p, each copy placed on a layer $\{i\} \times (\mathbb{Z}/p\mathbb{Z})$. So $F_1(m)$ and $F_2(m)$ have the same set of vertices. These two digraphs share no edges since all edges of $F_2(m)$ are horizontal while none of the edges of $F_1(m)$ is.

Let $F(m) = F_1(m) \cup F_2(m)$.

Clearly, $F(m)$ is Eulerian. As in the proof of Lemma 5.3, we have $d^+(F(m)) \geq m - 1 = \Omega(n^{1/3})$ because edges go at most one layer up.

We now have $d(F(m)) = 2$. Indeed, to reach vertex (j,y) from vertex (i,x) $(i < j)$, we first move to $v_{j-i}(i,j,x)$ which is a common neighbor of the two. (If $i = j$, then (i,x) and (j,y) are neighbors.) □

6 Open problems

The main open question is how to make our upper bound proofs algorithmic. For instance, the recent result that almost all undirected Cayley graphs of S_n have polynomially bounded diameter ([BH]) comes with an algorithm that actually constructs paths of length $O(n^C)$ in time $O(n^{C'})$. Our results imply that a polynomial bound extends to almost all *directed* Cayley graphs of S_n; but we have no algorithm to actually find directed paths of polynomially bounded length. Our bound on the directed diameter is obtained via a pigeon-hole argument based on expansion and yields existence only.

There is little hope to make the general result algorithmic but its special consequences such as that about almost all directed Cayley graphs might be provable algorithmically. Such algorithms, however, will require substantial new ideas since the algorithms for undirected diameter heavily rely on commutators which involve orientation reversal.

In Theorem 1.1 it would be desirable to eliminate the dependence on the degree Δ. The following conjecture addresses this problem.

Conjecture 6.1. *There exists a polynomial bound on the directed diameter of vertex-transitive digraphs as a function of the undirected diameter d and $\log n$ where n is the number of vertices. I. e., $d^+ \leq f(d, \log n)$ for some polynomial f.*

We note that such a bound exists for Cayley digraphs ($d^+ = O(d^2(\log n)^3)$, Theorem 2.15) as well as for edge-transitive digraphs ($d^+ = O(d \log n)$, Theorem 1.5).

The related question for Eulerian oriantations of vertex-transitive graphs may have a different answer.

Problem 6.2. *Does there exist a polynomial bound on the directed diameter of Eulerian orientations of vertex-transitive graphs as a function of the undirected diameter d and $\log n$ where n is the number of vertices? I. e., is $d^+ \leq g(d, \log n)$ for some polynomial g?*

It would be desirable to reduce the exponents in the $d^+ = O(d^2(\log n)^3)$ bound for Cayley digraphs (Theorem 1.4). We state a conjecture in this direction.

Conjecture 6.3. *For vertex-transitive digraphs, $d^+ = O(d(\log n)^c)$ for some constant c.*

For Cayley digraphs, Conjecture 6.3 would follow from Conjecture 2.17. Note that Conjecture 6.3 is true for regular edge-transitive digraphs ($d^+ = O(d \log n)$, Theorem 1.5).

Problem 6.4. *(a) Does there exist a bound on the directed diameter of vertex-transitive digraphs which depends on the undirected diameter only? I. e., does there exist a function f such that $d^+ \leq f(d)$ for all vertex-transitive digraphs? (b) Is there such a* polynomial *bound (f is a polynomial)?*

Note that the answer to all questions above will be negative if the condition of vertex-transitivity is relaxed to regularity (Theorems 5.1 and 5.2).

PROBLEM 6.5. *Does there exist a bound on the length of the shortest directed cycle in a vertex-transitive digraph, depending only on the undirected diameter?*

Of course a positive answer to Problem 6.4 (a) implies a positive answer to Problem 6.5.

PROBLEM 6.6. *Is Corollary 1.2 tight for vertex-transitive digraphs in the following sense: is there an infinite family of connected finite vertex-transitive digraphs of bounded degree for which the directed diameter d^+ grows (nearly) quadratically with d?*

PROBLEM 6.7. *What is the answer to Problem 6.5 for connected finite Eulerian digraphs (with no vertex-transitivity assumption on their symmetrization)?*

References

[Al] D. ALDOUS: On the Markov chain simulation method for uniform combinatorial distributions and simulated annealing. *Probability in Engineering and Information Sciences* **1** (1987), 33–46.

[ABSS] S. ARORA, L. BABAI, J. STERN, Z. SWEEDYK: The hardness of approximate optima in lattices, codes, and systems of linear equations. *J. Comput. Sys. Sci.* **54** (1997), 317–331. (Preliminary version: FOCS 1993)

[Ba1] L. BABAI: On the length of subgroup chains in the symmetric group. *Communications in Algebra* **14** (1986), 1729–1736.

[Ba2] L. BABAI: Local expansion of vertex-transitive graphs and random generation in finite groups. In: *Proc. 23rd STOC*, ACM, 1991, pp. 164–174.

[Ba3] L. BABAI: Asymptotic Inequality. Lecture Notes, 2005. `http://people.cs.uchicago.edu/~laci/asymp`

[BHKLS] L. BABAI, G. HETYEI, W.M. KANTOR, A. LUBOTSKY, Á. SERESS: On the diameter of finite groups. in: *Proc. 31st FOCS,* IEEE Computer Society, 1990, pp. 857–865.

[BSz] L. BABAI, M. SZEGEDY: Local Expansion of Symmetrical Graphs. *Combinatorics, Probability and Computing* **1** (1992), 1–11.

[BBS] L. BABAI, R. BEALS, Á. SERESS: On the diameter of the symmetric group: polynomial bounds. In: *Proc. 15th ACM-SIAM Symposium on Discrete Algorithms*, SIAM 2004, pp. 1108–1112

[BE] L. BABAI, P. ERDŐS: Representation of group elements as short products. In: *"Theory and Practice of Combinatorics"* (A. Rosa, G. Sabidussi, J. Turgeon eds.) *Annals of Discrete Math.* **12** (1982), 27–30.

[BH] L. BABAI, T. HAYES: Near-independence of permutations and an almost sure polynomial bound on the diameter of the symmetric group. In: *Proc. 16th ACM-SIAM Symposium on Discrete Algorithms*, SIAM 2005, pp. 1057–1066.

[BS1] L. BABAI, Á. SERESS: On the diameter of Cayley graphs of the symmetric group. *J. Combinatorial Theory-A* **49** (1988), 175–179.

[BS2] L. BABAI, Á. SERESS: On the diameter of permutation groups. *Europ. J. Combinatorics* **13** (1992), 231–243.

[Dix] J. D. DIXON: The probability of generating the symmetric group. *Math. Z.* **110** (1969), 199–205.

[EG] S. EVEN, O. GOLDREICH: The minimum length generator sequence is NP-hard. *J. Algorithms* **2** (1981), 311–313.

[He] H. A. HELFGOTT: Growth and Generation in $SL_2(\mathbb{Z}/p\mathbb{Z})$. `http://arxiv.org/abs/math/050924.`

[HS] A. J. HOFFMAN, R. R. SINGLETON: On Moore graphs with diameters 2 and 3, *IBM J. Res. Devel.* **4** (1960), 497–504.

[Je] M. R. JERRUM: The complexity of finding minimum length generator sequences. *Theoretical Computer Science* **36** (1985), 265–289.

[Ko] R. E. KORF: Finding optimal solutions to Rubik's Cube using pattern databases. In: *Proc. 14th Nat. Conf. on Artificial Intelligence (AAAI-97)*, Amer. Assoc. for Artificial Intelligence, 1997, pp. 700-705.

[KMS] D. KORNHAUSER, G. L. MILLER AND P. SPIRAKIS: Coordinating pebble motion on graphs, the diameter of permutation groups, and applications. In: *Proc. 25th FOCS,* IEEE Computer Society Press, 1986, pp. 292–302.