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ON THE DIFFERENCE CURVATURE OF SURFACES IN EUCLIDEAN SPACE

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We consider a closed oriented surface M^2 and an immersion $x: M^2 \rightarrow E^{2+N}$ into the euclidean space E^{2+N} of dimension $2+N$. Let B_x be the bundle of unit normal vectors of $x(M^2)$, so that a point of B_x is a pair (p, e) , e is a unit normal vector to $x(M^2)$ at $x(p)$. Then to $(p, e) \in B_x$, we put

$$(1) \quad I = dx \cdot dx, \quad II_e = dx \cdot de,$$

where dx, de are vector-valued linear differential forms in B_x and " \cdot " is the scalar product. I and II_e are called the first and the second quadratic forms in B_x . The eigenvalues $k_1(p, e)$ and $k_2(p, e)$ of II_e relative to I are called the principal curvatures at (p, e) . We call

$$(2) \quad S(p, e) = \frac{1}{4}(k_1(p, e) - k_2(p, e))^2$$

the *difference curvature* of the immersion x at (p, e) .

The main purpose of this paper is to get some global theorems for the difference curvature $S(p, e)$.

1. Preliminaries

Let M^2 be an oriented closed surface with an immersion $x: M^2 \rightarrow E^{2+N}$. Let $F(M^2)$ and $F(E^{2+N})$ be the bundles of orthonormal frames of M^2 and E^{2+N} respectively. Let B be the set of elements $b = (p, e_1, e_2, \dots, e_{2+N})$ so that $(p, e_1, e_2) \in F(M^2)$ and $(x(p), e_1, \dots, e_{2+N}) \in F(E^{2+N})$ whose orientation is coherent with the one of E^{2+N} , identifying e_i with $dx(e_i)$, $i=1, 2$. Then $B \rightarrow M^2$ may be considered as a principal bundle with fibre $SO(2) \times SO(N)$, and $\tilde{x}: B \rightarrow F(E^{2+N})$ is naturally defined by $\tilde{x}(b) = (x(p), e_1, \dots, e_{2+N})$.

The structure equations of E^{2+N} are given by

$$(3) \quad \begin{aligned} dx &= \sum_B \theta_A e_A, & de_A &= \sum_B \theta_{AB} e_B \\ d\theta_A &= \sum_B \theta_B \wedge \theta_{B,A}, & d\theta_{AB} &= \sum_C \theta_{AC} \wedge \theta_{CB}, & \theta_{AB} + \theta_{BA} &= 0, \\ & & A, B, C, \dots &= 1, 2, \dots, 2+N, \end{aligned}$$

where θ_A, θ_{AB} are differential 1-forms on $F(E^{2+N})$. Let ω_A, ω_{AB} be the

induced 1-forms on B from θ_A, θ_{AB} by the mapping \bar{x} . Then we have

$$(4) \quad \omega_r = 0, \quad \omega_{tr} = \sum_j A_{rj} \omega_j, \quad A_{rj} = A_{rjt},$$

$$i, j, k, \dots = 1, 2; \quad r, s, t, \dots = 3, \dots, 2+N.$$

The principal curvatures $k_1(p, e)$ and $k_2(p, e)$ are the eigenvalues of the matrix (A_{rj}) where $e = e_r$. The mean curvature $K_1(p, e)$ and the total curvature $K_2(p, e)$ are given by

$$(5) \quad K_1(p, e) = \frac{1}{2}(k_1(p, e) + k_2(p, e)), \quad K_2(p, e) = k_2(p, e)k_1(p, e).$$

Hence we have

$$(6) \quad K_1(p, e) = \frac{1}{2} \text{trace}(A_{rj}), \quad K_2(p, e) = \det(A_{rj}).$$

Let dV be the volume element of M^2 . There is a differential form $d\sigma$ of degree $N-1$ on B_v such that its restriction to a fibre is the volume element of the sphere of unit normal vectors at a point $p \in M^2$; then $d\sigma \wedge dV$ is the volume element of B_v . In fact, we have

$$(7) \quad dV = \omega_1 \wedge \omega_2, \quad d\sigma = \omega_{2+N,3} \wedge \dots \wedge \omega_{2+N,1+N}.$$

We call the integral

$$(8) \quad S^*(p) = \int S(p, e) d\sigma$$

over the sphere of unit normal vectors at $x(p)$ the *difference curvature of M^2 at p* , and define as the *difference curvature of the immersion x itself* $\int_{M^2} S^*(p) dV$, if it converges.

2. Some Global Theorems of the Difference Curvature

In [1] and [2], we have proved the following :

Lemma 1. *Let M^2 be an oriented closed surface immersed in E^{2+N} . If*

$$(9) \quad \int_{B_v} K_1(p, e)^2 dV \wedge d\sigma = 2c_{N+1},$$

then M^2 is imbedded as a sphere in a 3-dimensional linear subspace of E^{2+N} , where c_{N+1} denotes the area of the unit $(N+1)$ -sphere.

The main aim of this section is to prove the following :

Theorem 1. *Let M^2 be an oriented closed surface immersed in E^{2+N} . Its difference curvature satisfies the inequality :*

$$(10) \quad \int_{M^2} S^*(p) dV \geq 2gc_{N+1},$$

where g is the genus of M^2 . The equality sign holds when and only when M^2 is imbedded as a sphere in a 3-dimensional linear subspace of E^{2+N} .

Proof. Let M^2 be an oriented closed surface immersed in E^{2+N} . Let $(p, e_1, e_2, \bar{e}_3, \dots, \bar{e}_{2+N})$ be a local cross-section of $B \rightarrow F(M^2)$ defined on U and for any e in S_p^{N-1} , $p \in U$, put $e = e_{2+N} = \sum_r \xi_r \bar{e}_r(p)$. Denoting the restriction of A_{rj} onto the image of this local cross-section by \bar{A}_{rj} . We may put

$$A_{2+Nj} = \sum_r \xi_r \bar{A}_{rj}.$$

From (6) we get

$$(11) \quad K_2(p, e) = \det(\sum_r \xi_r \bar{A}_{rj}) = (\sum_r \xi_r \bar{A}_{r11})(\sum_s \xi_s \bar{A}_{s22}) - (\sum_t \xi_t \bar{A}_{t12})^2.$$

The right hand side is a quadratic form of ξ_3, \dots, ξ_{2+N} . Hence, by choosing a suitable cross-section, we can write $K_2(p, e)$ as

$$(12) \quad K_2(p, e) = \sum \lambda_{r-2} \xi_r \xi_r, \quad \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N.$$

Thus, by (12), we have

$$(13) \quad \int_{B_0} K_2(p, e) dV \wedge d\sigma = \int_{B_0} (\lambda_1(p) \cos^2 \theta_1 + \dots + \lambda_N(p) \cos^2 \theta_N) dV \wedge d\sigma \\ = \frac{c_{N+1}}{2\pi} \int_{M^2} (\lambda_1(p) + \dots + \lambda_N(p)) dV.$$

On the other hand, we have [4]

$$(14) \quad G(p) = \lambda_1(p) + \dots + \lambda_N(p),$$

where $G(p)$ denotes the Gaussian curvature of M^2 at p . Hence, by the Gauss-Bonnet theorem, we have

$$(15) \quad \int_{B_0} |K_2(p, e)| dV \wedge d\sigma = (2-2g)c_{N+1}.$$

On the other hand, by an inequality of Chern-Lashof [3], we have

$$(16) \quad \int_{B_0} |K_2(p, e)| dV \wedge d\sigma \geq (2+2g)c_{N+1}.$$

Therefore, if we put

$$(17) \quad V+ = \{(p, e) \in B_0 : K_2(p, e) \geq 0\}, \quad V- = \{(p, e) \in B_0 : K_2(p, e) < 0\}.$$

Then, by (15) and (16), we have

$$(18) \quad -\int_{V-} K_2(p, e) dV \wedge d\sigma \geq 2gc_{N+1}.$$

Therefore, by (2), (5) and (18), we get

$$(19) \quad \begin{aligned} \int_M S^*(p) dV &= \int_{B_v} S(p, e) dV \wedge d\sigma \geq \int_{V-} S(p, e) dV \wedge d\sigma \\ &= \int_{V-} (K_1(p, e)^2 - K_2(p, e)) dV \wedge d\sigma \\ &\geq \int_{V-} -K_2(p, e) dV \wedge d\sigma \geq 2gc_{N+1}. \end{aligned}$$

This proves (10). Now suppose the equality of (10) holds, then, by (19), we get

$$(20) \quad -\int_{V-} K_2(p, e) dV \wedge d\sigma = 2gc_{N+1},$$

and

$$(21) \quad S(p, e) = 0 \quad \text{on } V+, \quad \text{and} \quad K_1(p, e) = 0 \quad \text{on } V-.$$

By (15) and (20), we have

$$(22) \quad \int_{V+} K_2(p, e) dV \wedge d\sigma = 2c_{N+1}.$$

Therefore, by (2), (5), (21) and (22), we have

$$(23) \quad \begin{aligned} \int_{B_v} K_1(p, e)^2 dV \wedge d\sigma &= \int_{V+} K_1(p, e)^2 dV \wedge d\sigma = \int_{V+} K_2(p, e) dV \wedge d\sigma \\ &= 2c_{N+1}. \end{aligned}$$

Thus, by Lemma 1 and (23), we know that M^2 is imbedded as a sphere in a 3-dimensional linear subspace of E^{2+N} . Conversely, if M^2 is imbedded as a sphere in a 3-dimensional linear subspace of E^{2+N} , then it is easy to verify that the equality of (10) holds. This completes the proof of the theorem.

From theorem 1 we can easily get the following :

Theorem 2. *Let M^2 be an oriented closed surface immersed in E^{2+N} . If the difference curvature satisfies the inequality :*

$$(24) \quad \int_{M^2} S^*(p) dV \leq 2c_{N+1},$$

then M^2 is diffeomorphic to a sphere.

Theorem 3. *Let M^2 be an oriented closed surface immersed in E^{2+N} . If the difference curvature satisfies the inequality :*

$$(25) \quad \int_{M^2} S^*(p) dV \leq 4c_{N+1},$$

then M^2 is either diffeomorphic to a sphere or a torus.

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