If $l > 4$ our $l-1$ numbers $Y_j^{(l-1)}(0)$ may be approximated better than almost all $(l-1)$-tuples in R^{l-1}. It is not difficult to show, by the methods of these papers, that one can never approximate much better in the above case, i.e. with a somewhat larger exponent on the $\log|q_N|$ the last inequality could only be satisfied finitely often for any choice of q_N and $P_{N,j}$.

References

[1] Frances Throndike Cope, *Formal solutions of irregular linear differential equations*, Part I, Amer. J. Math. 56 (1934), pp. 411–437.
[2] Charles F. Osgood, *A method in diophantine approximation*, Acta Arith. 12 (1966), pp. 111–128.
[3] — *A method in diophantine approximation (II)*, Acta Arith. 13 (1967), pp. 383–393.
[4] — *A method in diophantine approximation (III)*, Acta Arith. 16 (1969), pp. 5–22.
[5] — *A method in diophantine approximation (IV)*, Acta Arith. 16 (1969), pp. 23–38.
[6] — *A method in diophantine approximation V*, Acta Arith. 22 (1973), pp. 353–369.
[7] J. Popken, *Über arithmetische Eigenschafter analytischer Funktionen*, Dissert, Groningen 1935, Amsterdam, Nord-Hollandische Uitgerversmy (1935), pp. 1–121.

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On the difference of consecutive terms of sequences defined by divisibility properties, II

by

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In a paper of the same title P. Erdös proved the following theorem: Let $b_1 < b_2 < \ldots$ be an infinite sequence of integers satisfying

$$\sum \frac{1}{b_i} < \infty \qquad (b_i, b_j) = 1.$$

Denote by $a_1, a_2, \ldots$ the sequence of integers not divisible by any b. Then there is an absolute constant c, independent of our sequence $b_1 < b_2 < \ldots$ so that for all sufficiently large x the interval $(x, x+x^{1-c})$ contains a's.

P. Erdös conjectured that perhaps $a_{i+1}-a_i = o(a_i)^\varepsilon$ holds for every $\varepsilon > 0$. We are unable to prove this at present, but we are going to prove the following sharpening of the result of P. Erdös.

THEOREM. *Let $B = \{b_1 < b_2 < \ldots\}$ be an increasing sequence of positive integers such that*

$$\text{(i)} \qquad \sum_{i=1}^{\infty} \frac{1}{b_i} < \infty$$

and

$$\text{(ii)} \qquad (b_i, b_j) = 1 \quad \text{if} \quad i \neq j.$$

Then for every $\varepsilon > 0$, if x is large enough, the interval $(x, x+x^{1/2+\varepsilon})$ contains a number a which is divisible by no b_j.

Proof. We can assume $b_1 > 1$. Let us define ε_1 and a so that

$$\text{(1)} \qquad \varepsilon_1 = \min\left\{\prod_{j=1}^{\infty}\left(1-\frac{1}{b_j}\right), \varepsilon^2\right\}$$

and

$$\text{(2)} \qquad \sum_{j=a}^{\infty} \frac{1}{b_j} < \varepsilon_1^2 < \varepsilon/8.$$

We shall assume that x is greater than a suitable function of ε, ε_1 and a.

Denote by P the set of all primes $p \notin B$. Put

$$H(x) = \{pt \in (x, x+x^{1/2+\varepsilon}) \colon p \in P \cap (2x^{1/2}, x^{1/2+\varepsilon})\},$$
$$\tilde{H}(x) = \{y \in H(x),\ b_j \nmid y \text{ for } j \leqslant a\}$$

and then for $p \in P \cap (2x^{1/2}, x^{1/2+\varepsilon})$

$$H(x, p) = \{y \in H(x) \colon p \mid y\},$$
$$\tilde{H}(x, p) = H(x, p) \cap \tilde{H}(x).$$

Let

$$B(x) = \{b_i \in B,\ b_i = pt \geqslant x^{1/2+\varepsilon},\ 2x^{1/2} \leqslant p < b_i\},$$
$$\tilde{B}(x) = \{y \in (x, x+x^{1/2+\varepsilon});\ \exists_{b_j \in B(x)} b_j \mid y\}$$

and

$$L(x) = \{y \in (x, x+x^{1/2+\varepsilon});\ \exists_{b_i} b_i \in (b_a, x^{1/2+\varepsilon}),\ b_i \mid y\},$$
$$T(x) = \tilde{H}(x) - L(x) - \tilde{B}(x).$$

If $y \in T(x)$ and $b_j \in B$, then $b_j \nmid y$. Indeed

$$T(x) = \{y \in H(x);\ \forall_{j \leqslant a} b_j \nmid y,\ \forall_{b_a < b_i < x^{1/2+\varepsilon}} b_i \nmid y\ \forall_{\substack{b_j = qt \geqslant x^{1/2+\varepsilon} \\ 2x^{1/2} \leqslant q < b_j}} b_j \nmid y\}$$
$$= \{y \in H(x);\ b_i \mid y \Rightarrow b_i \geqslant x^{1/2+\varepsilon},\ b_i \neq qt,\ 2x^{1/2} \leqslant q < b_i\}$$
$$\subseteq \{y = pt \in (x, x+x^{1/2+\varepsilon});\ 2x^{1/2} < p < x^{1/2+\varepsilon},\ p \notin B,\ b_i \mid y \Rightarrow b_i \geqslant x^{1/2+\varepsilon},\ b_i \mid t\}$$
$$\subseteq \{y \in H(x)\ \forall_{b_i \in B} b_i,\ b_i \nmid y\}.$$

Therefore, it suffices to show $T(x) \neq \emptyset$. Since

$$(3) \qquad \sum_{p \in (2x^{1/2}, x^{1/2+\varepsilon})} \frac{1}{p} \geqslant \frac{\varepsilon}{4}$$

it follows by (2) that

$$(4) \qquad \sum_{p \in P \cap (2x^{1/2}, x^{1/2+\varepsilon})} \frac{1}{p} \geqslant \frac{\varepsilon}{8}.$$

Using the sieve of Erathostenes for all $p \in P \cap (2x^{1/2}, x^{1/2+\varepsilon})$ we obtain

$$(5) \qquad |\tilde{H}(x, p)| \geqslant H(x, p) \prod_{i=1}^{a} \left(1 - \frac{1}{b_i}\right) - 2^a \geqslant \left(\frac{x^{1/2+\varepsilon}}{p} - 1\right)\varepsilon_1 - 2^a.$$

For different p's in question the sets $H(x, p)$ are disjoint. Since

$$\tilde{H}(x) \supset \bigcup_{p \in P \cap (2x^{1/2}, x^{1/2+\varepsilon})} H(x, p) \cap \tilde{H}(x) = \bigcup_{p \in P \cap (2x^{1/2}, x^{1/2+\varepsilon})} \tilde{H}(x, p)$$

we get from (4) and (5)

$$(6) \qquad |\tilde{H}(x)| \geqslant \sum_{p \in P \cap (2x^{1/2}, x^{1/2+\varepsilon})} |\tilde{H}(x, p)|$$
$$\geqslant x^{1/2+\varepsilon} \varepsilon_1 \frac{\varepsilon}{8} - (2^a + \varepsilon_1)\pi(x^{1/2+\varepsilon}) > \frac{\varepsilon_1 \cdot \varepsilon}{16} x^{1/2+\varepsilon}.$$

On the other hand, we obtain from (2)

$$(7) \qquad |L(x)| \leqslant 2x^{1/2+\varepsilon} \sum_{b_i \in (b_a, x^{1/2+\varepsilon}) \cap B} \frac{1}{b_i} \leqslant 2\varepsilon_1^2 x^{1/2} + \varepsilon.$$

Since $(b_i, b_j) = 1$ for $i \neq j$ we have by the definition of $B(x)$

$$(8) \qquad |B(x) \cap (0, 2x)| = \sum_{\substack{b_i \in [x^{1/2}, x^{1/2+\varepsilon}] \cap B \\ b_i = p_i t_i,\ 2x^{1/2} < p_i < b_i}} 1 \leqslant \sum_{1 < t_i < x^{1/2}} \sum_{\substack{b \in B \\ t_i \mid b}} 1 \leqslant \sum_{1 < t < x^{1/2}} 1 < x^{1/2}.$$

Hence

$$(9) \qquad |\tilde{B}(x)| \leqslant \sum_{b \in B(x)} \Big(\sum_{y \in (x, x+x^{1/2+\varepsilon})} 1 \Big) \leqslant \sum_{\substack{b \in B(x) \\ b < 2x}} 1 < x^{1/2}.$$

Finally, the estimates (6), (7) and (9) give

$$|T(x)| \geqslant |\tilde{H}(x)| - |L(x)| - |\tilde{B}(x)| > 0$$

which completes the proof of the theorem.

I express my thanks to the referee for the helpful criticism.

Reference

[1] P. Erdös, *On the difference of consecutive terms of sequences defined by divisibility properties*, Acta Arith. 12 (1966), pp. 175–182.

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