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On the differentiability and the representation of one-parameter semi-group of linear operators.

By Kôsaku YOSIDA.

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1. *The theorems.* Let $\{U_t\}$, $0 \leqq t < \infty$, be a one-parameter semi-group of linear (=additive, continuous) operators from a complex Banach space E to E:

$$U_t U_s = U_{t+s},\ U_0 = I \quad (=\text{the identity operator}), \tag{1.1}$$

such that

$$\sup_t \| U_t \| \leqq 1, \tag{1.2}$$

$$\lim_{t \to t_0} U_t x = U_{t_0} x \quad (\lim = \text{strong limit}),\ 0 \leqq t_0 < \infty,\ x \in E. \tag{1.3}$$

The purpose of the present note is to prove the following two theorems[1].

Theorem 1. If we denote by D the totality of x for which

$$\text{weak} \lim_{h \downarrow 0} h^{-1}(U_h - I)x = Ax \tag{1.4}$$

exists, then D is dense in E. Moreover A is a closed additive operator from D to E with the properties:

$$\textit{for any } x \in D,\ \lim_{h \to 0} h^{-1}(U_{t+h} - U_t)x = AU_t x = U_t Ax, \tag{1.5}$$

there exists a sequence $\{I_n\}$ of linear operators each commutative with every U_t and A such that i) the range $R(I_n) = \{I_n x;\ x \in E\} \subseteqq D$, $AI_n = n(I_n - I)$, ii) $\| I_n \| \leqq 1$, $\lim_{n\to\infty} I_n x = x$, iii) $U_t x = \lim_{n\to\infty} exp(tAI_n)x = \lim_{n\to\infty} \sum_{m=0}^{\infty} (m!)^{-1}(tAI_n)^m x$ uniformly for t in any finite interval[2], (1.6)

$\| (A - nI)x \| \geqq n \| x \|$ $(n = 1, 2, \ldots\ldots)$ for $x \in D$ and the range $R(A - nI)$ coinsides with E $(n = 1, 2, \ldots\ldots)$, (1.7)

let, by (1.7), y_n be the unique solution of $(A - nI)y_n = y$ $(n = 1, 2, \ldots\ldots)$, then $\lim_{n\to\infty} A(-ny_n) = \lim_{n\to\infty} (-n(y + ny_n)) = Ay$ for $y \in D$. (1.8)

Theorem 2. Let, conversely, A be an additive operator from a dense *linear subset D of E to E such that* (1·7) *and* (1·8) *are satisfied. Then there exists a one-parameter semi-group U_t which satisfies* (1·1)—(1·3) *and* (1·5).

We have, thus, a characterisation of the differential quotient A of the one-parameter semi-group of linear operators. This may be applied to an operator-theoretical treatment of temporally homogeneous stochastic process[3]. As an application of the theorem 1, we give a new proof of Stone's theorem[4] (see 4 below).

2. *Proof of the theorem 1. The differentiability.* We may define the integral

$$C_\varphi \cdot x = \int_0^\infty \varphi(s) U_s x ds \tag{2·1}$$

for complex-valued continuous function $\varphi(s)$ following after S. Bochner, G. Birkhoff, I. Gelfand, B. J. Pettis and other authors[5]. Let $\varphi(s)$ satisfy

$$\lim_{h \downarrow 0} \int_h^\infty \left| \frac{\varphi(s-h)-\varphi(s)}{h} + \varphi'(s) \right| ds = 0. \tag{2·2}$$

We have, by (1·1)

$$\frac{1}{h}(U_h - I) C_\varphi x = \frac{1}{h}\int_0^\infty \varphi(s) U_h U_s x ds - \frac{1}{h}\int_0^\infty \varphi(s) U_s x ds$$

$$= \int_h^\infty \frac{\varphi(s-h)-\varphi(s)}{h} U_s x ds - \frac{1}{h}\int_0^h \varphi(s) U_s x ds.$$

Thus, by (2·2) and $U_0 = I$,

$$\lim_{h \downarrow 0} \frac{1}{h}(U_h - I) C_\varphi x \quad \text{exists and} \quad = C_{-\varphi'} \cdot x - \varphi(0) x. \tag{2·3}$$

Such $C_\varphi x$ is dense in E, since for any $\delta > 0$

$$\varphi_\delta(s) = \delta \exp(-\delta s) \tag{2·4}$$

satisfies (2·2) and moreover, for any $x \in E$, $\lim\limits_{\delta \to \infty} C_{\varphi_\delta} \cdot x = x$.

We have, by (1·4), for $x \in D$,

$$\text{weak} \lim_{h \downarrow 0} \frac{1}{h}(U_h - I) U_t x = \text{weak} \lim_{h \downarrow 0} \frac{1}{h}(U_{t+h} - U_t) x$$

$$= U_t \cdot \text{weak} \lim_{h \downarrow 0} \frac{1}{h}(U_h - I) x.$$

Hence $U_t \cdot D \subseteq D$ and $AU_t x = U_t Ax$ for any $x \in D$, viz. A is commutative with every U_t, and

$$\begin{cases} \text{the right weak derivative } D^+ U_t x \text{ exists and} \\ = AU_t x = U_t Ax \text{ for any } x \in D. \end{cases} \tag{2·5}$$

Hence, by the continuity (1·3) we have, for any $f \in E^*$ (=the conjugate space of E)

$$f(U_t x) - f(x) = \int_0^t D^+ f(U_s x) ds = \int_0^t f(U_s Ax) ds = f\left(\int_0^t U_s Ax ds\right)$$

and therefore

$$U_t x - x = \int_0^t U_s Ax ds \quad \text{for} \quad x \in D. \tag{2·6}$$

Thus we have the strong differentiability (1·5).

The closedness of A. Let $x_n \in D$ ($n=1, 2, \ldots\ldots$) and let $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} Ax_n = z$. Then, by (2·6),

$$U_t x - x = \int_0^t U_s z ds.$$

The representation (1·6). Put

$$I_n = C\varphi_n \quad (n=1, 2, \ldots\ldots), \tag{2·7}$$

then, by (2·3) and (2·4), i) and ii) are surely satisfied. In fact, we have by (1·2) and (2·4)

$$\| I_n \| \leqq \int_0^\infty n \exp(-ns) ds = 1. \tag{2·8}$$

Thus, since

$$AI_n = n(I_n - I) \tag{2·9}$$

by (2·3), we have

$$\exp(tAI_n) = \sum_{m=0}^{\infty} \frac{(tAI_n)^m}{m!} = \exp(tn(I_n - I)), \; 0 \leqq t < \infty. \tag{2·10}$$

Thus

$$\| \exp(tAI_n) \| = \| \exp(tnI_n) \exp(-tnI) \| \leqq \exp(tn) \exp(-tn) = 1. \tag{2·11}$$

Since AI_n is commutative with each U_t, we have, for $x \in D$,

$$\| U_t x - \exp(tAI_n)x \| = \left\| \int_0^t \frac{d}{ds}\Big((\exp(t-s)AI_n)U_s x\Big) ds \right\|$$

$$= \left\| \int_0^t \Big(\exp(t-s)AI_n\Big) U_s(A-AI_n)\, x ds \right\|$$

$$\leq \int_0^t \| (A-AI_n)x \| ds \quad \text{by (1·2) and} \quad (2·11)$$

$$= t \| (A-AI_n)x \| . \qquad (2·12)$$

We have

$$AI_n = I_n Ax \quad \text{for} \quad x \in D, \qquad (2·13)$$

for A is a closed operator commutative with each U_t. Thus, by (2·12),

$$\| U_t x - \exp(tAI_n)x \| \leq t \| (I-I_n)Ax \| \quad \text{for} \quad x \in D. \qquad (2·14)$$

Since U_t and $\exp(tAI_n)$ are both of norm ≤ 1, we have iii) by the fact that D is dense in E.

The proof of (1·7). We first show that $R(A-nI)$ is dense in E. If otherwise, there exists $f \in E^*$, $f \neq 0$, such that $f(Ax-nx)=0$ on D. Thus, by $U_t D \subseteq D$ we have $f(AU_t x) = nf(U_t x)$ and hence

$$\frac{d}{dt} f(U_t x) = nf(U_t x).$$

Therefore, by $f(U_0 x) = f(x)$, we obtain $f(U_t x) = f(x)\exp(nt)$. This is a contradiction. Proof. By $f \neq 0$ and by the fact D is dense in E there exists $x \in D$ such that $f(x) \neq 0$. Then $f(x)\exp(nt)$, is unbounded in t when $t \to \infty$, contrary to $|f(U_t x)| \leq \|f\| \cdot \|U_t\| \cdot \|x\| \leq \|f\| \cdot \|x\|$. Next we show that

$$\| (A-nI)x \| \geq n \| x \| \quad \text{for} \quad x \in D.$$

Assume the contrary and let $\| (A-nI)x \| = a < n$ for a certain $x \in D$ with $\| x \| = 1$. Let $f \in E^*$ be such that $f(x)=1$, $\|f\|=1$. Then, by

$$\frac{d}{dt} U_t x = U_t Ax = nU_t x + U_t(A-nI)x,$$

we obtain

$$\begin{cases} \frac{d}{dt}\varphi(t) = n\varphi(t) + \psi(t), & \text{where} \\ \varphi(t) = f(U_t x), & \psi(t) = f(U_t(A-nI)x). \end{cases}$$

Since $\varphi(0)=1$ we have

$$\varphi(t)=\exp(nt)\left(\int_0^t \exp(-nt)\psi(t)dt+1\right)$$

and hence, by $|\psi(t)|\leqq\|f\|\cdot\|U_t\|\cdot\|(A-nI)x\|\leqq a$,

$$|\varphi(t)|\geqq\exp(nt)(1-an^{-1}(1-\exp(-nt))).$$

Thus $\varphi(t)$ is unbounded in t when $t\to\infty$, contrary to $|\varphi(t)|\leqq\|f\|\cdot\|U_t\|\cdot\|x\|\leqq 1$. Therefore, for any $y\in E$, there exists a sequence $\{x_h\}\subseteq D$ such that $\lim_{h\to\infty}(A-nI)x_h=y$. Because of $\|(A-nI)(x_h-x_k)\|\geqq n\|x_h-x_k\|$, $\{x_h\}$ is a Cauchy sequence. Let $\lim_{h\to\infty}x_h=x$, then $\lim_{h\to\infty}(A-nI)x_h=y$ and by the closedness of A we have $y=(A-nI)x$.

The proof of (1·8). We have, by (2·9), $AI_ny-nI_ny=-ny$ and hence

$$-ny_n=I_ny. \tag{2·15}$$

Thus (1·6) and (2·13) imply (1·8).

3. *Proof of the theorem 2.* By (1·7), the operator J_n defined by

$$J_ny=-ny_n \quad (n=1,\ 2,\ \ldots\ldots) \tag{3·1}$$

satisfies

$$\|J_n\|\leqq 1. \tag{3·2}$$

Since

$$AJ_ny=A(-ny_n)=-nAy_n=-n(y+ny_n)=n(J_n-I)y, \tag{3·3}$$

we have, by (3·2),

$$\|\exp(tAJ_n)y\|=\|\exp(ntJ_n)\exp(-ntI)y\|\leqq\exp(nt)\exp(-nt)\cdot\|y\|,$$

hence the linear operator defined by

$$U_t^{(n)}=\exp(tAJ_n) \tag{3·4}$$

satisfies

$$\|U_t^{(n)}\|\leqq 1, \tag{3·5}$$

$$U_t^{(n)}x-x=\int_0^t U_s^{(n)}AJ_nxds, \tag{3·6}$$

$$\lim_{h\to 0}h^{-1}(U_{t+h}^{(n)}-U_t^{(n)})x=U_t^{(n)}AJ_nx. \tag{3·7}$$

We have

$$J_nJ_m=J_mJ_n, \tag{3·8}$$

for, by (3·1),

$$J_n = -n^{-1}(A-nI)^{-1}. \tag{3·9}$$

Thus AJ_n is commutative with $U_t^{(m)}$ and hence

$$\begin{aligned}\|(U_t^{(m)}-U_t^{(n)})x\| &= \left\| \int_0^t \frac{d}{ds}\Big((\exp(t-s)AJ_n)U_s^{(m)}x\Big)ds \right\| \\ &= \left\| \int_0^t \Big(\exp(t-s)AJ_n\Big)U_s^{(m)}(AJ_m-AJ_n)x\,ds \right\| \\ &= \int_0^t \|(AJ_m-AJ_n)x\|\,ds \quad \text{by (3·5) and (3·7)} \\ &= t\,\|(AJ_m-AJ_n)x\|.\end{aligned}$$

Therefore, by (1·8),

$$U_t y = \lim_{n\to\infty} U_t^{(n)}y \quad (y \in D) \tag{3·10}$$

exists uniformly for t in any finite interval. Since D is dense in E and since we have (3·5), we see that the limit $U_t^{(n)}$ exists for all $y \in E$ and that U_t satisfies (1·1)—(1·3). Hence, by letting $n\to\infty$ in (3·6), we obtain

$$U_t y - y = \int_0^t U_s Ay\,ds, \quad y \in D. \tag{3·11}$$

4. *Stone's theorem.* If E is a Hilbert space and if U_t is, for any $t \geqq 0$, a unitary operator, then $H=-iA$ $(i=\sqrt{-1})$ is self-adjoint and

$$U_t = \int_{-\infty}^{\infty} \exp(i\lambda t)dE(\lambda), \quad \text{where} \quad H=\int_{-\infty}^{\infty}\lambda dE(\lambda). \tag{4·1}$$

This theorem due to M. H. Stone may be obtained as follows[6]. Put $U_{-t} = U_t^{-1} = U_t^*$ for $t \geqq 0$, then $\{U_t\}$, $-\infty < t < \infty$, is a one-parameter group of unitary operators strongly continuous in t. Thus it is easy to see, by (1·5),

$$\frac{dU_t x}{dt} = iHU_t = iU_t Hx \quad \text{for} \quad x \in D, \; -\infty < t < \infty. \tag{1·5)'$$

Hence, if $x, y \in D$,

$$(Hx, y) = \frac{1}{i}\Big(\frac{d}{dt}(U_t x, y)\Big)_{t=0} = \frac{1}{i}\Big(\frac{d}{dt}(x, U_{-t}y)\Big)_{t=0} = (x, Hy), \tag{4·2}$$

which shows that H is a symmetric operator. We proved in 2 that $R(H+iI)=E$ ((1·7)) by letting $t\to\infty$. Letting $t\to-\infty$, similar argument shows that $R(H-iI)=E$ also. Therefore the Cayley transform of

H is unitary and hence H is self-adjoint. Let $H=\int_{-\infty}^{\infty}\lambda dE(\lambda)$ be its spectral resolution, then, as in (3·9), $I_n=(I-n^{-1}A)^{-1}$ and hence

$$AI_n=n(I_n-I)=n\Big((I-in^{-1}H)^{-1}-I\Big)=\int_{-\infty}^{\infty}n\Big((1-i\lambda n^{-1})^{-1}-1\Big)dE(\lambda)$$

$$=\int_{-\infty}^{\infty}i\lambda(1-i\lambda n^{-1})^{-1}dE(\lambda).$$

Hence

$$U_t x=\lim_{n\to\infty}\exp(tAI_n)x=\lim_{n\to\infty}\int_{-\infty}^{\infty}\exp\Big(ti\lambda(1-i\lambda n^{-1})^{-1}\Big)dE(\lambda)x$$

$$=\int_{-\infty}^{\infty}\exp(i\lambda t)dE(\lambda)x.$$

Mathematical Institute,
Nagoya University.

References.

1) Cf. I. Gelfand: C. R. URSS, **25** (1939), 713–718 and N. Dunford—I. E. Segal: Bullet. Amer. Math. Soc, **52** (1946), 911–914.

2) According to N. Dunford and I. E. Segal's paper referred to in 1), E. Hille (Proc. Nat. Acad. Sci., **28** (1942), 175–178, 421–424 obtained the representation $U_t x=\lim_{n\to\infty}\exp(tn(U_{\frac{1}{n}}-I))x$. Hille's parer is not available to the author.

3) The question of the characterization of A together with differentiability of U_t is proposed to the author by Dr. K. Itô in connection with his theory of stochastic differential equations. See his forthcoming paper in Jap. J. of Math.

4) Proc. Nat. Acad. Sci., **16** (1930), 172–175. See also J. von Neumann: Ann. of Math., **33** (1932), 567–573. Proofs of Stone's theorem are given by many authors: F. Riesz, B. von Sz. Nagy and H. Nakano.

5) S. Bochner: Fund. Math., **20** (1933), 262–276. G. Birkhoff: Trans. Amer. Math. Soc. **38** (1935), 357–378. I. Gelfand: Commun. Inst. Sci. Math. et Mech. Univ. Kharkoff, **13** (1936), 35–40. B. J. Pettis: Trans. Amer. Meth. Soc., **44** (1938), 277–304.

6) In the case of Stone's theorem, we may replace (1·3) by the separability of $\{U_t x;\ -\infty<t<\infty\}$ and the weak measurability of U_t. This may be carried out by virtue of N. Dunford's theorem in Ann. of Math., **39** (1938), 567–573. This fact is, however, already proved by J. von Neumann in another way. See his paper referred to in 4).

Added during the proof. Meanwhile, E. Hille kindly communicated me that he also obtained essentially the same results as above by a different method. See his paper in C. R., 8 september (1947) and PNAS notes referred to in 2).