78. On the Differentiability of Semi-groups of Linear Operators

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§1. The purpose of the present note is to prove the two theorems given below which may be applied to the "abstract Cauchy problem" connected with the semi-group theory.[1] We recall, for the sake of exposition, some of the basic results from the theory. Let a one-parameter family T_t, $0\leqq t<\infty$, of bounded linear operators on a complex Banach space X into X satisfy the semi-group conditions:

(1.1) $$T_tT_s=T_{t+s},\ T_0=I \text{ (the identity operator)},$$

(1.2) $$s-\lim_{t\to t_0} T_tx=T_{t_0}x,\ t_0\geqq 0,\ x\in X,$$

(1.3)[2] $$\|T_t\|\leqq 1,\quad t\geqq 0.$$

Then the infinitesimal generator A of T_t defined by

(1.4) $$Ax=s-\lim_{h\downarrow 0} h^{-1}(T_h-I)x$$

is a closed linear operator with its domain $D(A)$ dense in X such that

(1.5) $$R(\lambda, A)x=(\lambda I-A)^{-1}x=\int_0^\infty e^{-\lambda t}T_txdt,\ Re(\lambda)>0$$

exists as a bounded linear operator on X into X satisfying

(1.6) $$\|R(\lambda, A)\|\leqq(Re(\lambda))^{-1},\quad Re(\lambda)>0.$$

Conversely, let a linear operator A with its domain $D(A)$ dense in X and taking the values in X admit the resolvent $(\lambda I-A)^{-1}=R(\lambda, A)$ which satisfies (1.6). Then A is the infinitesimal generator of a semi-group T_t which enjoys (1.1)–(1.5).

We have the representations

(1.7) $$T_tx=s-\lim_{n\to\infty}(I-n^{-1}tA)^{-n}x,\ x\in X,$$

(1.7)′ $$T_tx=s-\lim_{n\to\infty}\exp(tA(I-n^{-1}A)^{-1})x,\ x\in X,$$

uniformly in t in any compact interval of t. We have moreover,

(1.8) $$T_tx=s-\lim_{|\tau|\uparrow\infty}(2\pi)^{-1}\int_{\sigma-i\tau}^{\sigma+i\tau}e^{\lambda t}R(\lambda, A)xd\lambda,\ x\in D(A),\ t>0,\ \sigma=Re(\lambda)>0.$$

In these senses, we may write $T_t=\exp(tA)$.

(1.1), (1.2) and (1.4) imply that $T_t=\exp(tA)$ solves, for given A,

1) E. Hille and R. S. Phillips: Functional Analysis and Semi-groups, New York (1957). K. Yosida: On the differentiability and the representation of one-parameter semi-group of linear operators, J. Math. Soc. Japan, **1**, 15–21 (1948).

2) (1.1) and (1.2) imply that $\lim_{t\to\infty} t\log\|T_t\|=\beta<\infty$. Thus, by taking $e^{-\beta t}T_t$ in place of the original T_t, we may assume that the condition (1.3) is satisfied.

"the abstract Cauchy problem" (ACP):

(1.9) for $x \in D(A)$ and $t \geqq 0$, we have

$$T_t' x = s-\lim_{h\to 0} h^{-1}(T_{t+h}-T_t)x = AT_t x = T_t Ax,\ s-\lim_{t\downarrow 0} T_t x = x.$$

However, there are semi-groups satisfying the condition:

(1.10) for every $x \in X$ and every $t>0$, we have $T_t x \in D(A)$.

In such case, T_t solves the $(ACP)_0$:

(1.9)′ for every $x \in X$ and every $t>0$, we have

$$T_t' x = AT_t x,\ s-\lim_{t\downarrow 0} T_t x = x.$$

There are also semi-groups T_t which can be extended to T_λ which are analytic in a sector of the complex λ-plane of the form

(1.11) $$|\operatorname{Arg}\lambda| < \alpha < \pi/2.$$

(1.10) implies that $T_t' = AT_t$ is, as a closed linear operator defined on the Banach space X, a bounded linear operator and hence we have, by (1.1) and (1.10),

$$T_t'' = (T_t')' = (T_{t-s}AT_s)' = (AT_{t/2})^2 = (T_{t/2}')^2$$

or more generally

(1.12) $$T_t^{(n)} = (AT_{t/n})^n = (T_{t/n}')^n.$$

E. Hille and R. S. Phillips have discussed such differentiability properties of semi-groups and obtained conditions upon the behaviour of the resolvent $R(\lambda, A)$ which imply the differentiability for $t>0$ of $T_t = \exp(tA)$. The following two theorems are suggested by their researches and may be applied to discuss the behaviour near $t=0$ of solutions $T_t x = \exp(tA)x$ of $(ACP)_0$.

Theorem 1. *T_t' exists for $t>0$ and*

(1.13) $$\overline{\lim_{t\downarrow 0}}\, t\,\|T_t'\| < \infty$$

if and only if

(1.14) $$\overline{\lim_{|\tau|\uparrow\infty}}\, |\tau|\cdot\|R(1+i\tau, A)\| < \infty.$$

Moreover, the semi-group T_t of this class can be extended to T_λ which is analytic in a sector of the form

(1.11)′ $$|\lambda - t| < t/eC, \text{ where } C = \sup_{t>0} \|(e^{-t/2}T_t)'\|.$$

Theorem 2. *T_t' exists for $t>0$ and*

(1.13)′ $$\overline{\lim_{t\downarrow 0}}\, t\cdot\log\|T_t'\| = 0$$

if and only if

(1.14)′ $$\overline{\lim_{|\tau|\uparrow\infty}}\, \log|\tau|\cdot\|R(1+i\tau, A)\| = 0.$$

Remark. The semi-groups T_t are assumed to satisfy (1.1)–(1.3) so that the theorems are exhibited under the assumption (1.6).

§ 2. *Proof of Theorem* 1. Let $\overline{\lim}_{t\downarrow 0}\, t\,\|T_t'\| < \infty$. Then, since $\|T_t\| \leqq 1$, we have $\sup_{t>0} t\,\|(e^{-t/2}T_t)'\| = C < \infty$. Hence, by applying (1.12) to the semi-group $S_t = e^{-t/2}T_t$, we see that

$$(t/n)^n \| S_t^{(n)} \| = (t/n)^n \| (S'_{t/n})^n \| \leqq C^n \text{ for } t>0,\ n \geqq 1.$$

Thus

$$(n!)^{-1} |\lambda - t|^n \cdot \| S_t^{(n)} \| \leqq (n!)^{-1} (nCt^{-1} |\lambda - t|)^n, \tag{2.1}$$

and the analytical continuation

$$S_\lambda = e^{-\lambda/2} T_\lambda = e^{-1/2} T_t + \sum_{n=1}^{\infty} (n!)^{-1} (\lambda - t)^n (e^{-t/2} T_t)^{(n)} \tag{2.2}$$

is possible for λ satisfying (1.11)′. Moreover, we have, from (2.1)–(2.2),

$$\| S_\lambda \| = \| e^{-\lambda/2} T_\lambda \| \leqq (1 - eCt^{-1} |\lambda - t|)^{-1} \text{ when } eCt^{-1} |\lambda - t| < 1. \tag{2.3}$$

Let $\tau > 0$. Then, because of the analyticity of S_λ in λ, we can deform the path of integration $0 \leqq t < \infty$ of the integral

$$R(1+i\tau, A) = \int_0^\infty e^{-(1+i\tau)t} T_t dt = \int_0^\infty e^{-i\tau t} e^{-t/2} S_t dt$$

to the path $te^{-i\varphi}$, $0 \leqq t < \infty$, with a fixed φ such that $0 < \tan \varphi < 1/eC$:

$$R(1+i\tau, A) = \int_0^\infty \exp(-i\tau t e^{-i\varphi}) \exp(-te^{-i\varphi}/2) S_{te^{-i\varphi}} e^{-i\varphi} dt.$$

Such deformation is possible thanks to the convergence factor $\exp(-te^{-i\varphi}/2)$ and the estimate (2.3). Thus it is easy to obtain (1.14).

Let conversely $\| R(1+i\tau, A) \| \leqq C/|\tau|$ for large $|\tau|$. Then, by virtue of the resolvent equation

$$R(\lambda, A) = \sum_{n=0}^{\infty} R(1+i\tau, A)^{n+1} (1+i\tau - \lambda)^n \tag{2.4}$$

valid for $|1+i\tau-\lambda| \cdot \| R(1+i\tau, A) \| < 1$, we see that $R(\lambda, A)$ exists and is analytic in λ for large value of $|\tau| = |Im(\lambda)|$ outside a sector of the left half λ-plane defined by the boundary curve $\lambda(s) = \sigma(s) + i\tau(s)$ which satisfies

$$\lim_{\tau(s)\uparrow\infty} (-\sigma(s)/\tau(s)) = \tan \varepsilon = \lim_{\tau(s)\downarrow-\infty} \sigma(s)/\tau(s),\ \varepsilon > 0.$$

Moreover, (2.4) shows that $R(\lambda, A)$ is of order $|\tau^{-1}|$ when $|\tau| = |Im(\lambda)|$ tends to ∞ outside the above sector and lying in the left half λ-plane. Therefore, by deforming the path of integration $-\infty < \tau < \infty$ of the integral

$$T_t x = (2\pi)^{-1} i \int_{-\infty}^{\infty} e^{(1+i\tau)t} R(1+i\tau, A) x d\tau,\ x \in D(A),\ t > 0,$$

to the path $\tilde{\lambda}(s) = 2^{-1}\sigma(s) + i\tau(s)$, we see that $T'_t x$ exists for every $x \in X$ and every $t > 0$. Hence we easily see that T'_t satisfies (1.13).

Remark. The above proof shows that T_t is differentiable even at $t=0$ if T'_t exists for $t>0$ and $\overline{\lim}_{t\downarrow 0}\, t \| T'_t \| < e^{-1}$. This remarkable result was proved by E. Hille[3] in another way.

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3) E. Hille: On the differentiability of semi-group operators, Acta Sci. Math. Szeged, **12**, 19–24 (1950).

Proof of Theorem 2.[4] We have, by partially integrating (1.5) and making use of the existence of T_t' for $t>0$,

$$R(1+i\tau, A)=\int_0^\delta e^{-(1+i\tau)t}T_t dt+(1+i\tau)^{-1}e^{-(1+i\tau)\delta}T_\delta$$
$$+(1+i\tau)^{-1}\int_\delta^\infty e^{-(1+i\tau)t}T_t' dt.$$

$\|T_t'\|$ is, by $T_t'=AT_t$, (1.1) and (1.2), monotone decreasing in t. Thus

$$\|R(1+i\tau, A)\|\leqq\delta+(1+\tau^2)^{-1/2}(1+\|T_\delta'\|). \tag{2.5}$$

Now $t^{-1}\|T_t'\|$ is monotone decreasing in t with $\|T_t'\|$ and so there are two cases: $\lim_{t\downarrow 0} t^{-1}\|T_t'\|=\infty$ and $\lim_{t\uparrow 0} t^{-1}\|T_t'\|<\infty$. In the latter case, we have, by Theorem 1, $\lim_{|\tau|\uparrow\infty} \tau\cdot\|R(1+i\tau, A)\|<\infty$ and hence (1.14)′. In the former case, we take $|\tau|$ as equal to $\delta^{-1}\|T_\delta'\|$. Then, since $\lim_{t\downarrow 0} t\cdot\log\|T_t'\|=0$ by the assumption, we have $\lim_{\delta\downarrow 0}\delta(\log\delta+\log\tau)=0$, viz. $\lim_{\delta\downarrow 0}\delta\log\tau=0$. This proves (1.14)′ by (2.5).

Let, conversely, (1.14)′ hold good. By virtue of the resolvent equation (2.4) and the assumption (1.14)′, we see that $R(\lambda, A)$ is analytic in λ for large value of $|\tau|=Im(\lambda)$ outside a curved sector of the left half λ-plane with boundary curve $\lambda(s)=\sigma(s)+i\tau(s)$ which satisfies the condition

$$|\sigma(s)|=\varepsilon(s)^{-1}\log|\tau(s)|,\ 0<\varepsilon(s)\ \text{ and } \lim_{|s|\uparrow\infty}\varepsilon(s)=0,$$
$$\lim_{s\uparrow\infty}\tau(s)=\infty,\quad \lim_{s\downarrow-\infty}\tau(s)=-\infty.$$

Moreover, by (2.4), $\|R(\lambda, A)\|$ is of order $o(1/\log|\tau|)$ when $|\tau|=|Im(\lambda)|$ tends to ∞ outside the above sector and lying in the left half λ-plane. Thus, by deforming the path of integration $-\infty<\tau<\infty$ of the integral

$$T_t x=(2\pi)^{-1}i\int_{-\infty}^{\infty} e^{-(1+i\tau)t}R(1+i\tau, A)x d\tau,\ x\in D(A),\ t>0,$$

to the path $\tilde{\lambda}(s)=2^{-1}\sigma(s)+i\tau(s)$, we see that $T_t'x$ exists for every $x\in X$ and every $t>0$. Hence we easily see that (1.13)′ holds good.

4) The proof of the "only if part" is suggested by E. Hille, loc. cit. in 3). That the condition (1.14)′ implies the existence of T_t' for $t>0$ has already been proved in Hille and Phillips, loc. cit. in 1).