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On the dimension of divergence sets of dispersive equations

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Abstract We refine results of Carleson, Sjögren and Sjölin regarding the pointwise convergence to the initial data of solutions to the Schrödinger equation. We bound the Hausdorff dimension of the sets on which convergence fails. For example, with initial data in $H^1(\mathbb{R}^3)$, the sets of divergence have dimension at most one.

1 Introduction

We consider the Schrödinger equation

$$i\partial_t u + \Delta u = 0 \tag{1}$$

with initial data in $H^s(\mathbb{R}^n)$. A classical problem is to identify the exponents s for which

$$\lim_{t\to 0} u(x,t) = u(x,0), \qquad \text{a.e. } x \in \mathbb{R}^n.$$

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It is natural to consider a refinement of this question regarding the Hausdorff dimension of the sets on which convergence fails. For previous results in this direction, see Sjögren and Sjölin [**?**].

For example, in one spatial dimension, with $1/4 \leqslant s \leqslant 1/2$, we will prove that

$$\dim_H \{ \ x \in \mathbb{R} \ : \ u(x,t) \not\to u(x,0) \ \text{ as } \ t \to 0 \ \} \leq 1 - 2s,$$

where $\dim_H$ denotes the Hausdorff dimension. This refines the result of Carleson [**?**], who proved that, for data in $H^{1/4}(\mathbb{R})$, convergence takes place almost everywhere with respect to Lesbesgue measure. Dahlberg and Kenig [**?**] proved that divergence can occur on a set of nonzero Lebesgue measure when $s < 1/4$, so one may have expected the existence of sets of divergence with full Hausdorff dimension when $s = 1/4$. We see that this is not the case and that the sets of divergence can have Hausdorff dimension at most $1/2$ at the critical exponent.

2 Set-up and statement of results

For initial data u_0 belonging to the Schwartz class $\mathcal{S}(\mathbb{R}^n)$, the solution of (1) can be written as

$$u(x,t) = \frac{1}{(2\pi)^n} \int \widehat{u}_0(\xi)\, e^{i(x\cdot\xi - t|\xi|^m)} d\xi, \tag{2}$$

where $m = 2$. A number of our conclusions will hold for general $m > 1$, and we shall also consider the case $m = 1$, corresponding to the wave equation.

For $u_0 \in H^s(\mathbb{R}^n)$ defined by

$$\Big\{ f \in L^2(\mathbb{R}^n) \ : \ \int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2 (1 + |\xi|^2)^s d\xi < \infty \Big\},$$

the integral in (2) does not in general exist in the sense of Lebesgue. In this broader setting we may define u as the pointwise limit

$$u(\cdot, t) = \lim_{N\to\infty} S_t^N u_0 \tag{3}$$

whenever the limit exists, where the operator S_t^N is defined by

$$S_t^N f(x) = \frac{1}{(2\pi)^n} \int \psi(N^{-1}|\xi|) \widehat{f}(\xi)\, e^{i(x\cdot\xi - t|\xi|^m)} d\xi.$$

Here, for convenience, we take ψ to be the Gaussian $\psi(r) = e^{-r^2}$. By standard arguments, $u(\cdot, t)$ coincides with the traditional L^2-limit, almost everywhere with respect to Lebesgue measure. However, $u(\cdot, t)$ is also well-defined with respect to fractal measures when $s > 0$.

We say that a positive Borel measure μ is α-dimensional if

$$c_\alpha(\mu) := \sup_{x\in\mathbb{R}^n,\, r>0} \frac{\mu\big(B(x,r)\big)}{r^\alpha} < \infty, \qquad 0 \leq \alpha \leq n,$$

and denote by $\mathcal{M}^\alpha(\mathbb{B}^n)$ the α-dimensional probability measures which are supported in the unit ball $\mathbb{B}^n$. Now, $u(\cdot,t)$ is well-defined with respect to $\mu \in \mathcal{M}^\alpha(\mathbb{B}^n)$ for $\alpha > n-2s$, due to the inequality[1]

$$\Big\| \sup_{N\geqslant 1} |S_t^N f| \Big\|_{L^1(d\mu)} \lesssim \sqrt{c_\alpha(\mu)}\, \|f\|_{H^s(\mathbb{R}^n)}, \tag{4}$$

which holds for these exponents. We give a simple proof of (4) in Appendix A.

It is necessary to choose a preferred representation of the equivalence class u_0, as different representations can differ on sets of full Hausdorff dimension. A reasonable choice is $u(\,\cdot\,,0)$, as defined in (3), as it coincides with u_0, almost everywhere with respect to Lebesgue measure. It also coincides, up to a set of Hausdorff dimension at most $n-2s$, with the usual choice given in terms of the Bessel potential. The number $n-2s$ is a natural threshold for the problem, as the Bessel potential representation can be singular on sets with dimension smaller than this (see [?]). We elaborate further on such matters in Appendix A.

As usual in such contexts, our results will follow from appropriate maximal estimates. We denote by $\alpha_{m,n}(s)$ the infimum of the numbers α for which

$$\Big\| \sup_{k\geqslant 1} \sup_{N\geqslant 1} |S_{t_k}^N f| \Big\|_{L^1(d\mu)} \lesssim \sqrt{c_\alpha(\mu)}\, \|f\|_{H^s(\mathbb{R}^n)} \tag{5}$$

whenever $\mu \in \mathcal{M}^\alpha(\mathbb{B}^n)$, $f \in H^s(\mathbb{R}^n)$ and $(t_k) \in \mathbb{R}^{\mathbb{N}}$. If there is no such α, we say that $\alpha_{m,n}(s)$ does not exist.

By standard arguments combining (4), (5) and Frostman's lemma,

$$\dim_H \big\{\ x \in \mathbb{R}^n \ :\ u(x,t_k) \not\to u(x,0) \ \text{ as } \ k\to\infty \ \big\} \le \alpha_{m,n}(s), \tag{6}$$

for all $u_0 \in H^s(\mathbb{R}^n)$ and every sequence $t_k \to 0$ (see Appendix B).

We will also show in Appendix C how to "strengthen" the L^1-estimate (5) to an L^2-estimate.

Our positive results (i.e. upper bounds for the exponents $\alpha_{m,n}(s)$) are the forthcoming Proposition 1, Proposition 2 and Corollary 1, and form the content of Section 2. These are followed in Section 3 by some negative results (lower bounds). For the convenience of the reader we now summarise these results alongside those which follow from the literature.

It is straightforward to calculate (see (21)) that $\alpha_{m,n}(s) = 0$ when $s > n/2$, and so we restrict our attention to $s \leqslant n/2$.

For $m = 2$, the work of Dahlberg and Kenig [?] shows that (5) cannot hold for any $\alpha \leqslant n$ when $s < 1/4$, so that $\alpha_{2,n}(s)$ can only exist when $s \geqslant 1/4$. This is in fact the case for $\alpha_{m,n}(s)$ for any $m > 1$ (see Section 4). In one dimension we observe that

$$\alpha_{m,1}(s) = 1-2s, \qquad 1/4 \leqslant s \leqslant 1/2, \quad m > 1,$$

[1] The expression $A \lesssim B$ denotes $A \leqslant CB$, where the value of the positive constant C may depend on m, n and s, but never on f or μ, and will vary from line to line.

refining the work of Carleson [?], as mentioned in the introduction.

In higher dimensions, Sjögren and Sjölin [?] proved that for $m \geqslant 2$,

$$\alpha_{m,n}(s) \leqslant n + 1 - 2s, \qquad \frac{1}{2} < s \leqslant \frac{n}{2}, \tag{7}$$

although their set-up was slightly different. We refer the reader to their paper for a precise statement of their results. We extend (7) so that it holds for all $m \geqslant 1$, and improve it when $n = 2$ or 3. For $n \geqslant 4$, we lower their bound in the range $s > (n-1)/4$. In particular we will see that

$$\alpha_{2,n}(s) = n - 2s, \qquad \frac{n}{4} \leqslant s \leqslant \frac{n}{2}.$$

For $m = 1$, work of Walther [?] shows that (5) cannot hold when $s \leqslant 1/2$, so that $\alpha_{1,n}(s)$ can only exist when $s > 1/2$. Thus there is no issue for the one-dimensional wave equation. In two dimensions, we prove that

$$\alpha_{1,2}(s) = \begin{cases} 4(1-s), & 1/2 < s < 3/4, \\ 2(1-s), & 3/4 < s \leqslant 1. \end{cases}$$

This refines a theorem due to Cowling [?] (see also [?]), who proved the convergence with respect to Lebesgue measure when $s > 1/2$. We note a discontinuity when $s = 3/4$. In higher dimensions, we prove that

$$\alpha_{1,n}(s) = n - 2s, \qquad \frac{n+1}{4} < s \leqslant \frac{n}{2}.$$

For explicit bounds on $\alpha_{m,n}(s)$ for small values of s, we refer the reader to Sections 3 and 4. A complete resolution would of course be difficult as it includes the outstanding problem of Lebesgue measure convergence for the Schrödinger equation.

The general set-up we describe here has a number of further antecedents in Fourier analysis; in particular the work of Beurling [?], Salem–Zygmund [?], Carbery–Soria [?], and Montini [?] on the sets of divergence of Fourier series and integrals.

3 Positive results

In this section we obtain upper bounds on the exponents $\alpha_{m,n}(s)$ using three different approaches.

3.1 Via the Kolmogorov–Seliverstov–Plessner method

Here we employ the Kolmogorov–Seliverstov–Plessner method, as used by Carleson [**?**], and the following lemma due to Sjölin [**?**].[2]

Lemma 1 *[?] Let $m > 1$, $x, t \in \mathbb{R}$, $\gamma \in [1/2, 1)$ and $N \geq 1$. Then*

$$\left| \int_{\mathbb{R}} \frac{\eta(N^{-1}\xi)\, e^{i(x\xi - t\xi^m)}}{|\xi|^\gamma}\, d\xi \right| \lesssim \frac{1}{|x|^{1-\gamma}},$$

where the implicit constant depends on m, γ and the Schwartz function η.

The higher dimensional part of the following proposition follows by iteration of the one dimensional part, which is only possible when $m = 2$. This was first noted for Lebesgue measure by Dahlberg and Kenig [**?**].

Proposition 1 *Let $m > 1$ if $n = 1$, or $m = 2$ if $n \geqslant 2$ and $\frac{n}{4} \leqslant s \leqslant \frac{n}{2}$. Then for $\alpha > n - 2s$,*

$$\| \sup_{k \geqslant 1} \sup_{N \geqslant 1} |S_{t_k}^N f| \|_{L^1(d\mu)} \lesssim \sqrt{c_\alpha(\mu)}\, \|f\|_{H^s(\mathbb{R}^n)}$$

whenever $\mu \in \mathcal{M}^\alpha(\mathbb{B}^n)$, $f \in H^s(\mathbb{R}^n)$ and $(t_k) \in \mathbb{R}^{\mathbb{N}}$; i.e. $\alpha_{m,n}(s) \leq n - 2s$.

Proof We suppose that $n/4 \leqslant s < n/2$, as the case $s = n/2$ follows as a consequence. Recall the α-energy of μ, denoted by $I_\alpha(\mu)$, and defined by

$$I_\alpha(\mu) = \iint \frac{d\mu(x) d\mu(y)}{|x - y|^\alpha}.$$

By an appropriate dyadic decomposition,

$$\begin{aligned} \iint \frac{d\mu(x) d\mu(y)}{|x - y|^{n-2s}} &\lesssim \int \sum_{j=0}^{\infty} c_\alpha(\mu) 2^{-j\alpha} 2^{j(n-2s)} d\mu(y) \\ &\lesssim c_\alpha(\mu), \quad \alpha > n - 2s \end{aligned}$$

for $\mu \in \mathcal{M}^\alpha(\mathbb{B}^n)$, so it will suffice to prove the somewhat sharper

$$\| \sup_{k \geqslant 1} \sup_{N \geqslant 1} |S_{t_k}^N f| \|_{L^1(d\mu)} \lesssim \sqrt{I_{n-2s}(\mu)}\, \|f\|_{\dot{H}^s(\mathbb{R})}. \tag{8}$$

Here, as usual, $\dot{H}^s(\mathbb{R}^n)$ denotes the homogeneous L^2-Sobolev space

$$\Big\{ f \in L^2(\mathbb{R}^n) \, : \, \int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2 |\xi|^{2s} d\xi < \infty \Big\}.$$

[2] An n-dimensional version of this lemma for $m = 2$ may be obtained by scaling Lemma 1.A of [**?**] as in [**?**].

On linearising, (8) follows from the estimate

$$\left|\int_{\mathbb{B}^n} S_{t(x)}^{N(x)} f(x)\, w(x) d\mu(x)\right|^2 \lesssim I_{n-2s}(\mu)\, \|f\|_{\dot{H}^s}^2, \tag{9}$$

uniformly in the measurable functions $t : \mathbb{B}^n \to \mathbb{R}$, $N : \mathbb{B}^n \to [1,\infty)$ and $w : \mathbb{B}^n \to \mathbb{S}^1$.

By Fubini's theorem and the Cauchy–Schwarz inequality, the left hand side of (9) is bounded by

$$\int |\widehat{f}(\xi)|^2 |\xi|^{2s} d\xi \int \left|\int \psi(N(x)^{-1}|\xi|)\, e^{i(x\cdot\xi - t(x)|\xi|^m)} w(x)\, d\mu(x)\right|^2 \frac{d\xi}{|\xi|^{2s}}.$$

Writing the squared integral as a double integral, and applying Fubini's theorem again, in order to prove (8) it will suffice to show that

$$\iiint \psi(N(x)^{-1}|\xi|)\, \psi(N(y)^{-1}|\xi|)\, e^{i((x-y)\cdot\xi - (t(x)-t(y))|\xi|^m)} \frac{d\xi}{|\xi|^{2s}}\, w(x)w(y)\, d\mu(x)d\mu(y) \lesssim I_{n-2s}(\mu) \tag{10}$$

uniformly in the functions t, N and w.

From now on we assume that $m = 2$, $n \geqslant 2$; the argument for $m > 1$, $n = 1$ being simpler. Fixing $x, y \in \mathbb{B}^n$, we choose our orthogonal coordinate axes in frequency space $\xi = (\xi_1, \ldots, \xi_n)$ so that the associated unit vectors $e_1, \ldots, e_n$ satisfy

$$e_1 \cdot (x-y) = \ldots = e_n \cdot (x-y).$$

With respect to these axes we have

$$|x-y| = \sqrt{n}|x_1 - y_1| = \ldots = \sqrt{n}|x_n - y_n|. \tag{11}$$

Now, as ψ is a Gaussian and $|\xi|^{2s} \geqslant |\xi_1|^{\frac{2s}{n}} \ldots |\xi_n|^{\frac{2s}{n}}$, the inner integral of (10) is bounded by

$$\prod_{j=1}^n \left|\int \psi(N(x)^{-1}\xi_j)\, \psi(N(y)^{-1}\xi_j)\, e^{i\left((x_j-y_j)\xi_j - (t(x)-t(y))\xi_j^2\right)} \frac{d\xi_j}{|\xi_j|^{\frac{2s}{n}}}\right|,$$

and by Lemma 1, for $j = 1, \ldots, n$,

$$\left|\int \frac{\psi(N(x)^{-1}\xi_j)\, \psi(N(y)^{-1}\xi_j)\, e^{i\left((x_j-y_j)\xi_j - (t(x)-t(y))\xi_j^2\right)}}{|\xi_j|^{\frac{2s}{n}}} d\xi_j\right| \lesssim \frac{1}{|x_j - y_j|^{1-\frac{2s}{n}}}.$$

Substituting in, we see that the left hand side of (10) is bounded by a constant multiple of

$$\begin{aligned}\iint \frac{|w(x)w(y)|}{|x_1-y_1|^{1-\frac{2s}{n}} \ldots |x_n-y_n|^{1-\frac{2s}{n}}} d\mu(x)d\mu(y) &\lesssim \iint \frac{d\mu(x)d\mu(y)}{|x-y|^{n-2s}} \\ &= I_{n-2s}(\mu),\end{aligned}$$

where in the inequality we use (11). $\square$

Replacing the multiplier $e^{-it|\xi|^2}$ by $e^{-it(\xi_1^2-\xi_2^2)}$, we note that the previous argument works equally well for the two dimensional nonelliptic Schrödinger equation. Given that in [?] it was proven that divergence can occur on a set of nonzero Lebesgue measure when $s < 1/2$, it is noteworthy that at the critical exponent $s = 1/2$, the sets of divergence have Hausdorff dimension at most one.

3.2 Via bilinear Fourier extension estimates

Here we obtain upper bounds on $\alpha_{m,n}(s)$ as a consequence of certain bilinear Fourier extension estimates of Tao [?, Theorem 1.1 and Section 9]. Although the resulting relation between the parameters α and s will not be sharp, it will allow us to relax the restriction on s that was present in Section 2.1. To this end, define the extension operators

$$\widehat{T_j\, g}(\xi) = e^{-it\phi_j(\xi)}\widehat{g}(\xi),$$

where

$$\phi_j(\xi) = 2^{2j}|2^{-j}\xi + \xi_0|^m - m2^j|\xi_0|^{m-2}\xi_0 \cdot \xi - 2^{2j}|\xi_0|^m.$$

The support of $\widehat{g}$ will be restricted to $[-5,5]^n$ and so the domain of definition of ϕ_j is similarly restricted. We also only consider $\xi_0 \in \{5n \leqslant |\xi| \leqslant 10n\}$. After a finite splitting (depending on m), one can decompose the ϕ_j so that they are elliptic (as defined in [?]) on their restricted domains, with constants which depend on $|\xi_0|$ and m, and which are independent of $j \in \mathbb{N}$. Thus, the following estimate holds uniformly for all $\xi_0 \in \{5n \leqslant |\xi| \leqslant 10n\}$ and $j \in \mathbb{N}$, with the implicit constant depending only on m, n and q.

Theorem 3.1 *[?, Section 9] Let $m > 1,\ n \geqslant 2$ and $q > \frac{2(n+3)}{n+1}$. Then*

$$\|T_j\, g\, T_j\, h\|_{L^{q/2}(\mathbb{R}^{n+1})} \lesssim \|g\|_{L^2(\mathbb{R}^n)}\|h\|_{L^2(\mathbb{R}^n)},$$

whenever $\operatorname{supp}\widehat{g}$, $\operatorname{supp}\widehat{h} \subset [-5,5]^n$ *and* $\operatorname{dist}(\operatorname{supp}\widehat{g}, \operatorname{supp}\widehat{h}\,) \geqslant 1$.

The following argument is similar to one employed by Tao and Vargas [?] (see also [?] and [?] for refinements).

Proposition 2 *Let $m > 1$, $n \geqslant 2$ and $\alpha > \frac{n+3}{n+1}(n-2s)$. Then*

$$\big\|\sup_{k\geqslant 1}\sup_{N\geqslant 1}|S^N_{t_k} f|\,\big\|_{L^1(d\mu)} \lesssim \sqrt{c_\alpha(\mu)}\,\|f\|_{H^s(\mathbb{R}^n)}$$

whenever $\mu \in \mathcal{M}^\alpha(\mathbb{B}^n)$, $f \in H^s(\mathbb{R}^n)$ and $(t_k) \in \mathbb{R}^{\mathbb{N}}$; i.e. $\alpha_{m,n}(s) \leq \frac{n+3}{n+1}(n-2s)$.

Proof It will suffice to prove that for $\alpha > \frac{n+3}{n+1}(n-2s)$,

$$\big\| \sup_{t\in\mathbb{R}} |S_t f| \,\big\|_{L^1(d\mu)} \lesssim \sqrt{c_\alpha(\mu)}\, \|f\|_{H^s(\mathbb{R}^n)}, \tag{12}$$

whenever $\mu \in \mathcal{M}^\alpha(\mathbb{B}^n)$ and $\widehat{f} \in L^1(\mathbb{R}^n)$ has compact support. Here S_t is defined by

$$S_t f(x,t) = \frac{1}{(2\pi)^n} \int \widehat{f}(\xi)\, e^{i(x\cdot\xi - t|\xi|^m)} d\xi.$$

To see this, we note that by the Fundamental Theorem of Calculus,

$$\sup_{N\geqslant 1} |S_t^N f| \leqslant |S_t^1 f| + \int_1^\infty \Big| \frac{d}{dN} S_t^N f \Big| dN.$$

Now, letting $^\vee$ denote the inverse Fourier transform,

$$\Big|\frac{d}{dN} S_t^N f\Big| = \Big| S_t \Big(\frac{d}{dN}\psi(N^{-1}|\cdot|)\widehat{f}\Big)^\vee \Big| = \Big| S_t \Big(\frac{\psi'(N^{-1}|\cdot|)|\cdot|\widehat{f}}{N^2}\Big)^\vee \Big|,$$

so that by the triangle inequality and Minkowski's integral inequality, it will suffice to prove

$$\big\| \sup_{k\geqslant 1} |S_{t_k}\big(\psi(|\cdot|)\widehat{f}\big)^\vee| \,\big\|_{L^1(d\mu)} \lesssim \sqrt{c_\alpha(\mu)}\, \|f\|_{H^s(\mathbb{R}^n)},$$

which follows from (12) as $\big\|\big(\psi(|\cdot|)\widehat{f}\big)^\vee\big\|_{H^s(\mathbb{R}^n)} \leqslant \|f\|_{H^s(\mathbb{R}^n)}$, and

$$\int_1^\infty \Big\| \sup_{k\geqslant 1} \Big| S_{t_k}\Big(\frac{\psi'(N^{-1}|\cdot|)|\cdot|\widehat{f}}{N^2}\Big)^\vee \Big| \Big\|_{L^1(d\mu)} dN \lesssim \sqrt{c_\alpha(\mu)}\, \|f\|_{H^s(\mathbb{R}^n)}.$$

Now as $\psi'(N^{-1}|\cdot|) \lesssim \sum_{k\geqslant 0} 2^{-2nk}\chi_{B(0,2^kN)}$, for all $\varepsilon > 0$,

$$\begin{aligned}\Big\|\Big(\frac{\psi'(N^{-1}|\cdot|)|\cdot|\widehat{f}}{N^2}\Big)^\vee\Big\|_{H^s(\mathbb{R}^n)} &\lesssim \sum_{k\geqslant 0} \frac{2^{-2nk}}{N^{1+\varepsilon}} \Big\|\Big(\frac{\chi_{B(0,2^kN)}|\cdot|\widehat{f}}{N^{1-\varepsilon}}\Big)^\vee\Big\|_{H^s(\mathbb{R}^n)} \\ &\lesssim \frac{1}{N^{1+\varepsilon}} \|f\|_{H^{s+\varepsilon}(\mathbb{R}^n)},\end{aligned}$$

so that if (12) were true,

$$\begin{aligned}\int_1^\infty \Big\| \sup_{k\geqslant 1} \Big| S_{t_k}\Big(\frac{\psi(N^{-1}|\cdot|)|\cdot|\widehat{f}}{N^2}\Big)^\vee \Big| \Big\|_{L^1(d\mu)} dN &\lesssim \int_1^\infty \sqrt{c_\alpha(\mu)} \frac{1}{N^{1+\varepsilon}} \|f\|_{H^{s+\varepsilon}(\mathbb{R}^n)} dN \\ &\lesssim \sqrt{c_\alpha(\mu)}\, \|f\|_{H^{s+\varepsilon}(\mathbb{R}^n)},\end{aligned}$$

from which the proposition would follow. Thus, it remains to prove (12).

Using the fact that $c_\alpha(\mu) \geqslant 1$ when $\mu \in \mathcal{M}^\alpha(\mathbb{B}^n)$, by Hölder's inequality, (12) would follow from

$$\big\| \sup_{t\in\mathbb{R}} |S_t f| \,\big\|_{L^q(d\mu)} \lesssim c_\alpha(\mu)^{\frac{1}{q}}\, \|f\|_{H^s(\mathbb{R}^n)},$$

with $s > \frac{n}{2} - \frac{\alpha}{q}$ and q arbitrarily close to $\frac{2(n+3)}{n+1}$. By summing a Littlewood–Paley geometric series it will be enough to prove

$$\big\| \sup_{t\in\mathbb{R}} |S_t f| \,\big\|_{L^q(d\mu)} \lesssim c_\alpha(\mu)^{\frac{1}{q}} R^{\frac{n}{2}-\frac{\alpha}{q}} \|f\|_2,$$

whenever $\operatorname{supp}\widehat{f}\subset\{5nR\leqslant|\xi|\leqslant 10nR\}$, and by scaling, this problem reduces to

$$\big\|\sup_{t\in\mathbb{R}}|S_tf(R\,\cdot\,)|\,\big\|_{L^q(d\mu)}\lesssim c_\alpha(\mu)^{\frac{1}{q}}R^{-\frac{\alpha}{q}}\|f\|_2,$$

whenever $\operatorname{supp}\widehat{f}\subset\{5n\leqslant|\xi|\leqslant 10n\}$.

In order to take advantage of bilinear estimates we square the inequality;

$$\big\|\sup_{t\in\mathbb{R}}|S_tf(R\,\cdot\,)\,S_tf(R\,\cdot\,)|\,\big\|_{L^{q/2}(d\mu)}\lesssim c_\alpha(\mu)^{\frac{2}{q}}R^{-\frac{2\alpha}{q}}\|f\|_2^2.$$

After a further finite splitting, for each $j\in\mathbb{N}$ we break up the support of $\widehat{f}$ into dyadic cubes τ_k^j of side 2^{-j}. We write $\tau_k^j\sim\tau_{k'}^j$ if τ_k^j and $\tau_{k'}^j$ have adjacent parents, but are not adjacent. Writing $\widehat{f}=\sum_k\widehat{f}_{jk}$, where $\widehat{f}_{jk}=\widehat{f}\chi_{\tau_k^j}$, we have the Whitney decomposition (see for example [?,?]),

$$\begin{aligned}S_tf(Rx)\,S_tf(Rx)&=\frac{1}{(2\pi)^{2n}}\iint\widehat{f}(\xi)\widehat{f}(y)\,e^{i(Rx\cdot(\xi+y)-t(|\xi|^m+|y|^m))}d\xi dy\\&=\sum_{j,k,k':\tau_k^j\sim\tau_{k'}^j}\frac{1}{(2\pi)^{2n}}\iint\widehat{f}_{jk}(\xi)\widehat{f}_{jk'}(y)\,e^{i(Rx\cdot(\xi+y)-t(|\xi|^m+|y|^m))}d\xi dy\\&=\sum_{j,k,k':\tau_k^j\sim\tau_{k'}^j}S_tf_{jk}(Rx)S_tf_{jk'}(Rx).\end{aligned}$$

Calculating the temporal Fourier transform of $S_tf_{jk}(R\,\cdot\,)$, it is easy to see that the support is contained in an interval of length $\lesssim 2^{-j}$. This is also true of $S_tf_{jk}(R\,\cdot\,)\,S_tf_{jk'}(R\,\cdot\,)$, and so by Bernstein's inequality,

$$\sup_{t\in\mathbb{R}}|S_tf_{jk}(Rx)\,S_tf_{jk'}(Rx)|\lesssim 2^{\frac{-2j}{q}}\|S_tf_{jk}(Rx)\,S_tf_{jk'}(Rx)\|_{L_t^{q/2}(\mathbb{R})}.$$

Thus, by the triangle inequality and Fubini's theorem,

$$\big\|\sup_{t\in\mathbb{R}}|S_tf(R\,\cdot\,)|\,\big\|_{L^q(d\mu)}^2\lesssim\sum_{j,k,k':\tau_k^j\sim\tau_{k'}^j}2^{\frac{-2j}{q}}\|S_tf_{jk}(R\,\cdot\,)\,S_tf_{jk'}(R\,\cdot\,)\|_{L^{q/2}(d\mu dt)}.$$

On the other hand, the spatial Fourier transform of $S_tf_{jk}(R\,\cdot\,)\,S_tf_{jk'}(R\,\cdot\,)$ is contained in a ball of radius $10nR^{-1}$, so that

$$\begin{aligned}\|S_tf_{jk}(R\,\cdot\,)\,S_tf_{jk'}(R\,\cdot\,)\|_{L^{q/2}(d\mu)}&=\big\|\big(S_tf_{jk}(R\,\cdot\,)\,S_tf_{jk'}(R\,\cdot\,)\big)*\eta_R^\vee\big\|_{L^{q/2}(d\mu)}\\&\leqslant\|S_tf_{jk}(R\,\cdot\,)\,S_tf_{jk'}(R\,\cdot\,)\|_{L^{q/2}(|\eta_R^\vee|*\mu(x)dx)}\|\eta_R^\vee\|_{L^1}^{1-\frac{2}{q}},\\&\lesssim\|S_tf_{jk}(R\,\cdot\,)\,S_tf_{jk'}(R\,\cdot\,)\|_{L^{q/2}(|\eta_R^\vee|*\mu(x)dx)},\end{aligned}$$

where $\eta_R = \eta(R^{-1}\cdot)$ and η is a Schwartz function, equal to one on the ball of radius $10n$. In the first inequality we use that

$$|F * G(x)| \leqslant \int |F(y)|\, |G(x-y)|^{2/q} |G(x-y)|^{1-2/q} dy$$
$$\leqslant \Big(|F|^{q/2} * |G|(x)\Big)^{2/q} \|G\|_{L^1}^{1-2/q},$$

and then move the convolution onto the measure by Fubini's theorem.

Now, letting ξ_0 denote the midpoint between the supports of $\widehat{f}_{jk}$ and $\widehat{f}_{jk'}$, by the affine change of variables $x \to R^{-1}(x + tm|\xi_0|^{m-2}\xi_0)$, we have

$$\|S_t f_{jk}(R\,\cdot\,)\, S_t f_{jk'}(R\,\cdot\,)\|_{L^{q/2}(|\eta_R^\vee| * \mu(x) dxdt)} = R^{-\frac{2n}{q}} \times$$
$$\|S_t f_{jk}(\cdot + tm|\xi_0|^{m-2}\xi_0)\, S_t f_{jk'}(\cdot + tm|\xi_0|^{m-2}\xi_0)\|_{L^{q/2}\left(|\eta_R^\vee| * \mu(R^{-1}(x+tm|\xi_0|^{m-2}\xi_0)) dxdt\right)},$$

and by the changes of variables $\xi \to 2^{-j}\xi + \xi_0$, $x \to 2^j x$, and $t \to 2^{2j} t$, this is in turn equal to

$$R^{-\frac{2n}{q}} 2^{-j\left(2n - \frac{2(n+2)}{q}\right)} \|T_j\, g\, T_j\, h\|_{L^{q/2}(w(x,t) dxdt)},$$

where

$$w(x,t) = |\eta_R^\vee| * \mu(R^{-1}2^j(x + 2^j tm|\xi_0|^{m-2}\xi_0)),$$
$$\widehat{g}(\xi) = \widehat{f}_{jk}(2^{-j}\xi + \xi_0) \qquad \text{and} \qquad \widehat{h} = \widehat{f}_{jk'}(2^{-j}\xi + \xi_0),$$

Note that $\operatorname{supp} \widehat{g}$, $\operatorname{supp} \widehat{h} \subset [-5,5]^n$ and $\operatorname{dist}(\operatorname{supp} \widehat{g}, \operatorname{supp} \widehat{h}) \geqslant 1$.

Now, $\eta^\vee$ is a Schwartz function, and so satisfies $|\eta^\vee| \lesssim \sum_{k \geqslant 0} 2^{-2nk} \chi_{B(0,2^k)}$. This enables us to calculate an upper bound for w in terms of $c_\alpha(\mu)$;

$$w(x,t) = R^n \int |\eta^\vee|\Big(R\big(R^{-1}2^j(x + 2^j tm|\xi_0|^{m-2}\xi_0) - y\big)\Big) d\mu(y)$$
$$\lesssim R^n \sum_{k \geqslant 0} 2^{-2nk} \mu\Big(B\big(R^{-1}2^j(x + 2^j tm|\xi_0|^{m-2}\xi_0), R^{-1}2^k\big)\Big)$$
$$\lesssim R^n \sum_{k \geqslant 0} 2^{-2nk} c_\alpha(\mu)\, R^{-\alpha} 2^{k\alpha} \lesssim R^{n-\alpha} c_\alpha(\mu).$$

Putting things together, we see that

$$\big\| \sup_{t \in \mathbb{R}} |S_t f(R\,\cdot\,)| \big\|_{L^q(d\mu)}^2 \lesssim c_\alpha(\mu)^{\frac{2}{q}} \sum_{j,k,k' : \tau_k^j \sim \tau_{k'}^j} R^{-\frac{2\alpha}{q}} 2^{-j(2n - \frac{2(n+1)}{q})} \|T_j\, g\, T_j\, h\|_{L^{q/2}(dxdt)},$$

and we are in a position to apply Theorem 3.1. Applying the theorem and rescaling yields

$$\big\| \sup_{t \in \mathbb{R}} |S_t f(R\,\cdot\,)| \big\|_{L^q(d\mu)}^2 \lesssim c_\alpha(\mu)^{\frac{2}{q}} R^{-\frac{2\alpha}{q}} \sum_{j,k,k' : \tau_k^j \sim \tau_{k'}^j} 2^{-j(n - \frac{2(n+1)}{q})} \|f_{jk}\|_2 \|f_{jk'}\|_2.$$

Now as the sum in k' is finite and the supports of both $\widehat{f}_{jk}$ and $\widehat{f}_{jk'}$ are contained in $7\tau_k^j$, by almost orthogonality,

$$\sum_{k,k':\tau_k^j\sim\tau_{k'}^j} \|f_{jk}\|_2\|f_{jk'}\|_2 \lesssim \|f\|_2^2.$$

Finally, the sum in j converges as $q > \frac{2(n+1)}{n}$, so that

$$\big\| \sup_{t\in\mathbb{R}} |S_t f(R\,\cdot)|\,\big\|_{L^q(d\mu)}^2 \lesssim c_\alpha(\mu)^{\frac{2}{q}} R^{-\frac{2\alpha}{q}} \|f\|_2^2,$$

as required. □

3.3 Via weighted Fourier extension estimates

The exponents $\alpha_{m,n}(s)$ are closely related to certain other exponents arising in geometric measure theory. Let $\beta_n(\alpha)$ denote the supremum of the numbers β for which

$$\|\widehat{\mu}(R\,\cdot)\|_{L^2(\mathbb{S}^{n-1})} \lesssim R^{-\beta}\sqrt{c_\alpha(\mu)\|\mu\|} \tag{13}$$

holds, for all $R \geq 1$ and $\mu \in \mathcal{M}^\alpha(\mathbb{B}^n)$, where we include measures which have not been normalised for the moment. The problem of identifying the precise value of $\beta_n(\alpha)$ has been considered by several authors, beginning with Mattila [?]. Further references will follow.

Writing $h = h_1 - h_2 + i(h_3 - h_4)$, where the components are positive, we have

$$\sqrt{c_\alpha(h_i\mu)\|h_i\mu\|} \leqslant \sqrt{c_\alpha(\mu)\|\mu\|}\,\|h_i\|_{L^\infty(d\mu)}, \quad i = 1, \dots, 4,$$

so that by the triangle inequality, (13) yields

$$\|\widehat{h\mu}(R\,\cdot)\|_{L^2(\mathbb{S}^{n-1})} \lesssim R^{-\beta}\sqrt{c_\alpha(\mu)\|\mu\|}\,\|h\|_{L^\infty(d\mu)}.$$

Thus, by duality, $\beta_n(\alpha)$ is the supremum of the numbers β for which

$$\|\widehat{gd\sigma}(R\,\cdot)\|_{L^1(d\mu)} \lesssim R^{-\beta}\sqrt{c_\alpha(\mu)\|\mu\|}\,\|g\|_{L^2(\mathbb{S}^{n-1})} \tag{14}$$

holds, for all $R \geq 1$, $g \in L^2(\mathbb{S}^{n-1})$ and $\mu \in \mathcal{M}^\alpha(\mathbb{B}^n)$. Here $d\sigma$ denotes the Lebesgue measure on the unit sphere $\mathbb{S}^{n-1}$. By similar arguments to those contained in Appendix C, one can show that $\beta_n(\alpha)$ is also the supremum of the numbers β for which

$$\|\widehat{gd\sigma}(R\,\cdot)\|_{L^2(d\mu)} \lesssim R^{-\beta}\sqrt{c_\alpha(\mu)}\,\|g\|_{L^2(\mathbb{S}^{n-1})}$$

holds, however (14) will suffice for our purposes.

Weighted Fourier extension estimates of this type played a central role in [**?**] and [**?**], and their use in the context of almost everywhere convergence problems for dispersive equations goes back to Vega [**?**].

In two dimensions, combining results of Mattila [?] and Wolff [?], the sharp decay rates are known;

$$\beta_2(\alpha) = \begin{cases} \alpha/2,\ 0<\alpha<1/2, \\ 1/4,\ 1/2 \leqslant \alpha \leqslant 1, \\ \alpha/4,\ 1<\alpha \leqslant 2. \end{cases}$$

In higher dimensions, combining the partial results due to Mattila [?], Sjölin [?,?], and Erdoğan [?], it is known that

$$\beta_n(\alpha) \geqslant \begin{cases} \frac{\alpha}{2}, & \alpha \in (0, \frac{n-1}{2}], \\ \frac{n-1}{4}, & \alpha \in [\frac{n-1}{2}, \frac{n}{2}], \\ \frac{n+2\alpha-2}{8}, & \alpha \in [\frac{n}{2}, \frac{n+2}{2}], \\ \frac{\alpha-1}{2}, & \alpha \in [\frac{n+2}{2}, n]. \end{cases}$$

On the other hand, using the so-called 'Knapp examples' one can calculate that

$$\beta_n(\alpha) \leqslant \begin{cases} \frac{\alpha}{2}, & \alpha \in (0, n-2], \\ \frac{n+\alpha-2}{4}, & \alpha \in [n-2, n]. \end{cases}$$

Worse counterexamples have been constructed when the averages are taken over a piece of paraboloid rather than the sphere [?], or for signed measures with finite Fourier energy [?].

Note that the upper and lower bounds coincide when $\alpha \in (0, \frac{n-1}{2}]$, so that $\beta_n(\alpha) = \alpha/2$ in that range.

The following is a simple generalisation of a result of Sjölin [?] and Vega [?], and, unlike the previous propositions, also holds for the wave equation.

Proposition 3 *Let $m \geqslant 1$, $n \geqslant 2$ and $s > \frac{n}{2} - \beta_n(\alpha)$. Then*

$$\big\| \sup_{k\geqslant 1} \sup_{N\geqslant 1} |S^N_{t_k} f| \,\big\|_{L^1(d\mu)} \lesssim \sqrt{c_\alpha(\mu)}\, \|f\|_{H^s(\mathbb{R}^n)}$$

whenever $\mu \in \mathcal{M}^\alpha(\mathbb{B}^n)$, $f \in H^s(\mathbb{R}^n)$ and $(t_k) \in \mathbb{R}^{\mathbb{N}}$.

Proof Using polar coordinates we may write

$$\begin{aligned} |S^N_t f(x)| &= \left| \int_{\mathbb{R}^n} \psi(N^{-1}|\xi|)\, \widehat{f}(\xi)\, e^{i(x\cdot\xi - t|\xi|^m)}\, d\xi \right| \\ &= \left| \int_0^\infty \psi(N^{-1} r)\, r^{n-1} e^{-itr^m} \int_{\mathbb{S}^{n-1}} \widehat{f}(r\omega)\, e^{irx\cdot\omega} d\sigma(\omega)\, dr \right| \\ &\leqslant \int_0^\infty r^{n-1} \left| \int_{\mathbb{S}^{n-1}} \widehat{f}(r\omega)\, e^{irx\cdot\omega} d\sigma(\omega) \right| dr. \end{aligned}$$

Thus, by Fubini's theorem,

$$\big\| \sup_{k\geqslant 1} \sup_{N\geqslant 1} |S^N_{t_k} f| \,\big\|_{L^1(d\mu)} \leqslant \int_0^\infty r^{n-1} \big\| \big(\widehat{f}(r\,\cdot) d\sigma\big)^\vee (r\,\cdot) \big\|_{L^1(d\mu)} dr. \tag{15}$$

Now, by (14) we have

$$\big\| \big(\widehat{f}(r\,\cdot)d\sigma\big)^{\vee}(r\,\cdot)\big\|_{L^1(d\mu)} \lesssim \sqrt{c_\alpha(\mu)}\, r^{-\beta} \|\widehat{f}(r\,\cdot\,)\|_{L^2(\mathbb{S}^{n-1})}$$

for all $\beta < \beta_n(\alpha)$, so that (15) is bounded by

$$\lesssim \sqrt{c_\alpha(\mu)} \int_0^\infty r^{n-1-\beta} \|\widehat{f}(r\,\cdot\,)\|_{L^2(\mathbb{S}^{n-1})} dr.$$

Finally, by an application of the Cauchy–Schwarz inequality, this in turn is bounded by

$$\begin{aligned}
&\lesssim \sqrt{c_\alpha(\mu)} \left(\int_0^\infty \frac{r^{n-1-2\beta}}{(1+r^2)^s} \right)^{1/2} \left(\int_0^\infty \|\widehat{f}(r\,\cdot\,)\|^2_{L^2(\mathbb{S}^{n-1})} (1+r^2)^s r^{n-1} dr \right)^{1/2} \\
&\lesssim \sqrt{c_\alpha(\mu)}\, \|f\|_{H^s(\mathbb{R}^n)},
\end{aligned}$$

where the first integral converges by choosing β sufficiently close to $\beta_n(\alpha)$. □

Combining the previous proposition with the best known lower bounds for $\beta_n(\alpha)$, we obtain the following corollary.

Corollary 1 *Let $m \geqslant 1$, $n \geqslant 2$ and*

$$\alpha > \begin{cases} n+1-2s, & s \in (\frac{1}{2}, \frac{n}{4}], \\ \frac{3n}{2}+1-4s, & s \in (\frac{n}{4}, \frac{n+1}{4}], \\ n-2s, & s \in (\frac{n+1}{4}, \frac{n}{2}]. \end{cases}$$

Then

$$\big\| \sup_{k\geqslant 1} \sup_{N\geqslant 1} |S^N_{t_k} f| \,\big\|_{L^1(d\mu)} \lesssim \sqrt{c_\alpha(\mu)}\, \|f\|_{H^s(\mathbb{R}^n)}$$

whenever $\mu \in \mathcal{M}^\alpha(\mathbb{B}^n)$, $f \in H^s(\mathbb{R}^n)$ and $(t_k) \in \mathbb{R}^{\mathbb{N}}$; i.e.

$$\alpha_{m,n}(s) \leq \begin{cases} n+1-2s, & s \in (\frac{1}{2}, \frac{n}{4}], \\ \frac{3n}{2}+1-4s, & s \in (\frac{n}{4}, \frac{n+1}{4}], \\ n-2s, & s \in (\frac{n+1}{4}, \frac{n}{2}]. \end{cases}$$

In the next section we will see that this is sharp for the wave equation when $n = 2$. It is also an improvement on the previous results for the Schrödinger equation when $n \geqslant 4$ and $s \in (\frac{1}{2}, \frac{n-1}{4})$.

4 Negative results

In this section we obtain some lower bounds on the exponents $\alpha_{m,n}$ via some examples.

Example 1

Consider $\widehat{f} = \chi_A$ and $d\mu(x) = N^n \chi_E(x)dx$, where

$$A = B(0,N), \quad E = B(0,N^{-1}),$$

so that

$$S_t^N f(x) = \frac{1}{(2\pi)^n} \int_{B(0,N)} \psi(N^{-1}|\xi|)\, e^{i(x\cdot\xi - t|\xi|^m)} d\xi.$$

Taking $t = N^{-m}$, we see that the phase is close to zero for all $x \in E$, so that

$$\Big\| \sup_{0<t<1} |S_t^N f| \Big\|_{L^1(d\mu)} \gtrsim N^n |A||E| = N^n.$$

On the other hand,

$$\sqrt{c_\alpha(\mu)}\, \|f\|_{H^s} \lesssim N^{\frac{\alpha}{2}} N^{s+\frac{n}{2}},$$

which, letting $N \to \infty$, yields the necessary condition

$$\alpha_{m,n}(s) \geqslant n - 2s.$$

Example 2

Consider $\widehat{f} = \chi_A$ and $d\mu(x) = N^{\frac{n-1}{2}} \chi_E(x)dx$, where

$$A = [N, N + n^{-1}N^{1/2}] \times [0, n^{-1}N^{1/2}]^{n-1},$$

$$E = [0, (2(m+1))^{-1}] \times [0, (2N)^{-1/2}]^{n-1}.$$

By a change of variables, we see that

$$|S_t^N f(x)| = \frac{1}{(2\pi)^n} \left| \int_{A-Ne_1} \psi\big(N^{-1}|Ne_1 + \xi|\big)\, e^{i(x\cdot\xi - t|Ne_1+\xi|^m)} d\xi \right|$$

$$= \frac{1}{(2\pi)^n} \left| \int_{A-Ne_1} \psi\big(|e_1 + N^{-1}\xi|\big)\, e^{i\left((x - mN^{m-1}te_1)\cdot\xi - t(|Ne_1+\xi|^m - mN^{m-1}\xi_1)\right)} d\xi \right|,$$

where $e_1 = (1, 0, \dots, 0)$. Now

$$\begin{aligned} &\nabla(|Ne_1 + \xi|^m - mN^{m-1}\xi_1) \\ &\qquad = m|Ne_1 + \xi|^{m-2}(Ne_1 + \xi) - mN^{m-1}e_1 \\ &\qquad = mN(|Ne_1 + \xi|^{m-2} - N^{m-2})e_1 + m|Ne_1 + \xi|^{m-2}\xi, \end{aligned}$$

so that, by the mean value theorem, for $|\xi| \leqslant N^{1/2}$,

$$|\nabla(|Ne_1 + \xi|^m - mN^{m-1}\xi_1)| \leqslant m(m+1)N^{m-3/2}.$$

Taking $t = m^{-1}N^{1-m}x_1$ we see that

$$|t(|Ne_1 + \xi|^m - mN^{m-1}\xi_1) - m^{-1}Nx_1| \leqslant 1/2$$

when $|\xi| \leqslant N^{1/2}$ and $|x_1| \leqslant (2(m+1))^{-1}$. Moreover, when $x \in E$,

$$|(x - mN^{m-1}te_1) \cdot \xi| = |x_2\xi_2 + \ldots + x_n\xi_n| \leqslant 1/2,$$

so, for each $x \in E$, the phase is effectively constant, so that

$$\Big\| \sup_{0<t<1} |S_t^N f| \Big\|_{L^1(d\mu)} \gtrsim N^{\frac{n-1}{2}} |A||E| = N^{\frac{n}{2}}.$$

On the other hand, by calculating,

$$\sqrt{c_\alpha(\mu)}\, \|f\|_{H^s} \lesssim N^{\frac{\max\{\alpha-1,0\}}{4}} N^{s+\frac{n}{4}}.$$

We see that $s \geqslant 1/4$ is necessary for all values of α. Furthermore, we see that when $s < n/4$, it is necessary that

$$\alpha_{m,n}(s) \geqslant n + 1 - 4s.$$

Example 3

Finally, with $m = 1$, we consider $\widehat{f} = \chi_A$ and $d\mu(x) = N^{\frac{n-1}{2}} \chi_E(x)dx$, where

$$A = \Big\{ \xi \in \mathbb{R}^n \,:\, |\theta_{\xi,e_1}| < N^{-1/2} \text{ and } |\xi| < N \Big\},$$

$$E = \Big\{ x \in \mathbb{R}^n \,:\, |\theta_{x,e_1}| < N^{-1/2} \text{ and } |x| < 1/10 \Big\}.$$

Here θ_{ξ,e_1} denotes the angle between ξ and e_1. Taking $t = |x| \cos\theta_{x,e_1}$,

$$\begin{aligned} |x \cdot \xi - t|\xi|| &= |\xi||x||\cos\theta_{x,\xi} - \cos\theta_{x,e_1}| \\ &\leqslant \frac{N}{10} \left|\theta_{x,e_1} - \theta_{x,\xi}\right| \left|\theta_{x,e_1} + \theta_{x,\xi}\right| \\ &\leqslant \frac{N}{2} N^{-1/2} N^{-1/2} = \frac{1}{2}, \end{aligned}$$

so that

$$\Big\| \sup_{0<t<1} |S_t^N f| \Big\|_{L^1(d\mu)} \gtrsim N^{\frac{n-1}{2}} |A||E| = N^{\frac{n+1}{2}}.$$

On the other hand,

$$\sqrt{c_\alpha(\mu)}\, \|f\|_{H^s} \lesssim N^{\frac{\max\{\alpha-1,0\}}{4}} N^{s+\frac{n+1}{4}},$$

which, when $s < \frac{n+1}{4}$, yields the necessary condition

$$\alpha_{1,n}(s) \geqslant n + 2 - 4s.$$

5 Concluding remarks

The results as stated in the introduction are obtained by comparing the upper and lower bounds for $\alpha_{m,n}$. In particular, the upper bounds for $\alpha_{1,2}$ are contained in Corollary 1. In higher dimensions, the precise decay rates for the Fourier transform averaged over spheres are not known. The possibility remains that a positive resolution of this question would yield sharp estimates for the wave equation in higher dimensions.

Theorem 5.1 *Let $n \geqslant 3$. If it is true that $\beta_n(\alpha) = \min\left\{\frac{\alpha}{2}, \frac{n+\alpha-2}{4}\right\}$, then*

$$\alpha_{1,n}(s) = \begin{cases} n+2-4s, & 1/2 < s < 1, \\ n-2s, & 1 \leqslant s \leqslant n/2. \end{cases}$$

As mentioned in Section 3, evidence was provided in [**?**] to suggest that

$$\beta_n(\alpha) < \min\left\{\frac{\alpha}{2}, \frac{n+\alpha-2}{4}\right\}, \quad \text{when} \quad \frac{n-1}{2} < \alpha < n, \tag{16}$$

so perhaps the previous theorem is somewhat optimistic. For the pessimists, (16) would follow if one could improve the lower bounds on $\alpha_{1,n}(s)$ for $s \in (\frac{1}{2}, \frac{n+1}{4})$.

Finally, we note that if $\alpha_{2,n}(s)$ existed in the range $1/4 \leqslant s \leqslant 1/2$, then by Example 2 of the previous section, it would satisfy

$$\alpha_{2,n}(s) \geqslant n+1-4s, \quad n \geqslant 2.$$

Thus, $\alpha_{2,n}(1/4)$ would have to equal n. This contrasts with the case $n = 1$ since $\alpha_{2,1}(1/4) = 1/2$ (see Proposition 1 and Example 1).

Appendix A: Proof of (4)

Here we consider the maximal operator M defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \Big| \int_{B(x,r)} f(y)\, dy \Big|.$$

This is dominated by the Hardy–Littlewood maximal operator. The following lemma is well-known (see for example [**?**, pp. 159]). We include a proof, avoiding interpolation of weak estimates, from which we will also deduce (4).

Lemma 2 *Let $0 < s \leqslant n/2$ and $\alpha > n-2s$. Then*

$$\|Mf\|_{L^1(d\mu)} \lesssim \sqrt{c_\alpha(\mu)}\, \|f\|_{H^s(\mathbb{R}^n)}$$

whenever $\mu \in \mathcal{M}^\alpha(\mathbb{B}^n)$ and $f \in H^s(\mathbb{R}^n)$.

We remark that by the proof of Lemma 3 below, the $L^1(d\mu)$-norm in the previous lemma can be replaced by the $L^2(d\mu)$-norm.

We note in passing that Lemma 2 implies that for all $f \in H^s(\mathbb{R}^n)$,

$$\dim_H \left\{ x \in \mathbb{R}^n \ : \ \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)\, dy \ \text{ diverges as } \ r \to 0 \right\} \leq n - 2s.$$

This refines (but does not recover) the L^2-Lebesgue differentiation theorem.

Proof of Lemma 2 We suppose that $0 < s < n/2$, as the case $s = n/2$ follows as a consequence. As in the proof of Proposition 1, it will suffice to prove the somewhat sharper

$$\|Mf\|_{L^1(d\mu)} \lesssim \sqrt{I_{n-2s}(\mu)}\, \|f\|_{\dot{H}^s(\mathbb{R}^n)}.$$

Defining η to be the Fourier transform of $\frac{1}{|B(0,1)|}\chi_{B(0,1)}$, by linearising the operator, it will suffice to prove

$$\left| \iint \eta(r(x)\xi)\, \widehat{f}(\xi)\, e^{i\xi\cdot x} d\xi\, w(x)\, d\mu(x) \right|^2 \lesssim I_{n-2s}(\mu)\, \|f\|^2_{\dot{H}^s},$$

whenever $r : \mathbb{B}^n \to (0,\infty)$ and $w : \mathbb{B}^n \to \mathbb{S}^1$ are measurable functions. Now, by Fubini's theorem and the Cauchy–Schwarz inequality, the left hand side is bounded by

$$\int |\widehat{f}(\xi)|^2 |\xi|^{2s} d\xi \int \left| \int \eta\big(r(x)\xi\big)\, e^{ix\cdot\xi} w(x)\, d\mu(x) \right|^2 \frac{d\xi}{|\xi|^{2s}}.$$

Writing the squared integral as a double integral, and applying Fubini's theorem again, it will suffice to show that

$$\iiint \eta\big(r(x)\xi\big)\, \eta\big(r(y)\xi\big)\, e^{i(x-y)\cdot\xi} \frac{d\xi}{|\xi|^{2s}} w(x)w(y)\, d\mu(x) d\mu(y) \lesssim I_{n-2s}(\mu).$$

Thus, it remains to prove that for $0 < s < n/2$,

$$\left| \int \eta\big(r(x)\xi\big)\eta\big(r(y)\xi\big)\, e^{i(x-y)\cdot\xi} \frac{d\xi}{|\xi|^{2s}} \right| \lesssim \frac{1}{|x-y|^{n-2s}}$$

uniformly for all choices of $r(x), r(y) > 0$. Now the Fourier transform of $|\cdot|^{-2s}$ is equal to a constant multiple of $|\cdot|^{2s-n}$, so, by the change of variables $z = x - y$, this would follow from the inequality

$$\sup_{r_1,r_2>0} \left| \frac{1}{|B(z,r_2)|} \int_{B(z,r_2)} \frac{1}{|B(x,r_1)|} \int_{B(x,r_1)} \frac{dy}{|y|^{n-2s}} dx \right| \lesssim \frac{1}{|z|^{n-2s}}.$$

This in turn would follow from the inequality

$$\frac{1}{|B(x,r)|} \int_{B(x,r)} \frac{dy}{|y|^{n-2s}} \lesssim \frac{1}{|x|^{n-2s}} \tag{17}$$

uniformly for $x \in \mathbb{R}^n$ and $r > 0$. By scaling this reduces to proving

$$\int_{B(x,1)} \frac{dy}{|y|^{n-2s}} \lesssim \frac{1}{|x|^{n-2s}}, \tag{18}$$

which can be shown by direct calculation. □

Now (4) follows from the previous proof. Using the fact that $|\widehat{f}(\xi)\, e^{-i|\xi|^m t}| = |\widehat{f}(\xi)|$, the only other change comes in the last line where instead of proving (18) we are required to show that

$$\int \psi(x-y)\,\frac{dy}{|y|^{n-2s}} \lesssim \frac{1}{|x|^{n-2s}}.$$

However this follows using (17), as

$$\int \psi(x-y)\,\frac{dy}{|y|^{n-2s}} \leqslant \sum_{k\geqslant 0} 2^{-2nk} \int \chi_{B(0,2^k)}(x-y)\,\frac{dy}{|y|^{n-2s}} \lesssim \frac{1}{|x|^{n-2s}},$$

and so we are done.

By standard arguments, similar to those in the next section, (4) implies that for all $g \in L^2(\mathbb{R}^n)$,

$$\dim_H \Big\{\ x \in \mathbb{R}^n\ :\ S_0^N(G_s * g)(x) \not\to G_s * g(x)\ \text{ as }\ N \to \infty\ \Big\} \le n - 2s,$$

where G_s denotes the Bessel potential of order s defined via the Fourier transform by $\widehat{G}_s(\xi) = (1+|\xi|^2)^{-s/2}$. The proof of this requires the additional ingredient

$$\|G_s * g\|_{L^1(d\mu)} \lesssim \sqrt{c_\alpha(\mu)}\, \|g\|_{L^2(\mathbb{R}^n)},$$

which holds under the hypotheses of Lemma 2.

Appendix B: Proof of (6)

Consider $g \in \mathcal{S}(\mathbb{R}^n)$ such that $\|u_0 - g\|_{H^s(\mathbb{R}^n)} < \varepsilon$, and note that

$$|S_t^N u_0 - S_0^N u_0| \leqslant |S_t^N u_0 - S_t^N g| + |S_t^N g - S_0^N g| + |S_0^N g - S_0^N u_0|.$$

We have,

$$\begin{aligned}\mu\{\, x\ :\ \limsup_{k\to\infty}\limsup_{N\to\infty} |S_{t_k}^N u_0 - S_0^N u_0| > \lambda\,\} &\leqslant \mu\{\, x\ :\ \sup_{k\geqslant 1}\sup_{N\geqslant 1} |S_{t_k}^N (u_0 - g)| > \lambda/3\,\} \\ &+ \mu\{\, x\ :\ \lim_{k\to\infty}\lim_{N\to\infty} |S_{t_k}^N g - S_0^N g| > \lambda/3\,\} + \mu\{\, x\ :\ \sup_{N\geqslant 1} |S_0^N (g - u_0)| > \lambda/3\,\}.\end{aligned}$$

Now, if $t_k \to 0$, the second term on the right hand side of the inequality is zero, so by the maximal inequalities (4) and (5),

$$\begin{aligned}\mu\{\, x\ :\ \lim_{k\to\infty}\lim_{N\to\infty} |S_{t_k}^N u_0 - S_0^N u_0| > \lambda\,\} &\lesssim \lambda^{-1}\sqrt{c_\alpha(\mu)}\, \|u_0 - g\|_{H^s(\mathbb{R}^n)} \\ &\lesssim \lambda^{-1}\sqrt{c_\alpha(\mu)}\ \varepsilon.\end{aligned}$$

Letting ε tend to zero, then λ tend to zero, we see that

$$\mu\{\, x\ :\ u(x, t_k) \not\to u(x,0)\ \text{ as }\ k \to \infty\,\} = 0$$

whenever $\mu \in \mathcal{M}^\alpha(\mathbb{B}^n)$ with $\alpha > \alpha_{m,n}(s)$. Thus by Frostman's lemma (see for example [**?**]),

$$\mathcal{H}^\alpha\{\, x \in \mathbb{B}^n\ :\ u(x,t_k) \not\to u(x,0)\ \text{ as }\ k \to \infty\,\} = 0, \quad \alpha > \alpha_{m,n}(s),$$

where $\mathcal{H}^\alpha$ denotes the α-Hausdorff measure. By translation invariance and the countable additivity of Hausdorff measure, we obtain (6). ◻

Appendix C: L^2-estimates

Consider the estimate

$$\big\|\sup_{k\geqslant 1}\sup_{N\geqslant 1} |S_{t_k}^N f|\,\big\|_{L^1(d\mu)} \lesssim \sqrt{c_\alpha(\mu)}\, \|f\|_{H^s(\mathbb{R}^n)} \qquad (L_1)$$

whenever $\mu \in \mathcal{M}^\alpha(\mathbb{B}^n)$ and $f \in H^s(\mathbb{R}^n)$, and the estimate

$$\big\| \sup_{k\geqslant 1} \sup_{N\geqslant 1} |S_{t_k}^N f| \big\|_{L^2(d\mu)} \lesssim \sqrt{c_\alpha(\mu)}\, \|f\|_{H^s(\mathbb{R}^n)} \tag{L_2}$$

whenever $\mu \in \mathcal{M}^\alpha(\mathbb{B}^n)$ and $f \in H^s(\mathbb{R}^n)$.

In the following lemma we see that these estimates are essentially equivalent. This is reminiscent of the Nikisin–Stein maximal principle which allows one to deduce a weak L^2-estimate for the maximal operator from an L^1-estimate via the almost everywhere convergence that the L^1-estimate implies (see [?]).

Lemma 3 (L_1) *holds for all* $s > s_0$ $\Leftrightarrow$ (L_2) *holds for all* $s > s_0$.

Proof By Hölder's inequality, (L_1) follows from (L_2), so one direction of the equivalence is clear. For the reverse direction, we suppose that (L_1) holds for all $s > s_0$, so that by the Fourier inversion formula,

$$\big\| \sup_{k\geqslant 1} \sup_{N\geqslant 1} |S_{t_k}^N \widehat{f}\,| \big\|_{L^1(d\mu)} \lesssim \sqrt{c_\alpha(\mu)}\, R^s \|f\|_{L^2}, \quad s > s_0, \tag{19}$$

whenever $f \in L^2(B_R)$ and $\mu \in \mathcal{M}^\alpha(\mathbb{B}^n)$.

Considering $\mu \in \mathcal{M}^\alpha(\mathbb{B}^n)$ defined by

$$d\mu(x) = \frac{\chi_E(x) d\nu(x)}{\nu(E)},$$

where ν is any α-dimensional measure supported in the unit ball $\mathbb{B}^n$, this yields

$$\int \sup_{k\geqslant 1} \sup_{N\geqslant 1} |S_{t_k}^N \widehat{f}(x)|\, \chi_E(x)\, \frac{d\nu(x)}{\sqrt{\nu(E)}} \lesssim \sqrt{c_\alpha(\nu)} R^s\, \|f\|_{L^2}.$$

Setting $E = \{x : \sup_{k\geqslant 1} \sup_{N\geqslant 1} |S_{t_k}^N \widehat{f}(x)| > \lambda\}$, we deduce the weak type (2,2) estimate

$$\nu\big(\{x : \sup_{k\geqslant 1} \sup_{N\geqslant 1} |S_{t_k}^N \widehat{f}(x)| > \lambda\}\big) \lesssim c_\alpha(\nu) R^{2s} \lambda^{-2} \|f\|_{L^2}^2. \tag{20}$$

On the other hand, it is trivial to observe that

$$\big\| \sup_{k\geqslant 1} \sup_{N\geqslant 1} |S_{t_k}^N \widehat{f}\,| \big\|_{L^\infty(d\nu)} \leqslant \|f\|_{L^1(\mathbb{R}^n)} \lesssim R^{n/2} \|f\|_{L^2} \tag{21}$$

whenever $f \in L^2(B_R)$. Using real interpolation between (20) and (21), we obtain

$$\big\| \sup_{k\geqslant 1} \sup_{N\geqslant 1} |S_{t_k}^N \widehat{f}\,| \big\|_{L^p(d\nu)} \lesssim c_\alpha(\nu)^{1/p} R^s \|f\|_{L^2}, \quad s > \frac{2}{p} s_0 + \Big(1 - \frac{2}{p}\Big)\frac{n}{2}, \tag{22}$$

whenever $f \in L^2(B_R)$ and $\mu \in \mathcal{M}^\alpha(\mathbb{B}^n)$, for all $p > 2$.

Using complex interpolation between (19) and (22), we obtain

$$\big\| \sup_{k\geqslant 1} \sup_{N\geqslant 1} |S_{t_k}^N \widehat{f}\,| \big\|_{L^2(d\mu)} \lesssim \sqrt{c_\alpha(\mu)}\, R^s \|f\|_{L^2}, \quad s > s_0,$$

whenever $f \in L^2(B_R)$ and $\mu \in \mathcal{M}^\alpha(\mathbb{B}^n)$. Here we have used the fact that $c_\alpha(\mu) \geqslant 1$. Finally, we apply the Fourier inversion formula and sum a geometric series to obtain (L_2) for all $s > s_0$. □

We thus see that the positive results of Section 3 can be strengthened to L^2-estimates. In Proposition 1 however, we are obliged to restrict attention to the range $n/4 < s \leqslant n/2$. Thus, we do not refine, or even recover, the result of Kenig and Ruiz [**?**] which gave a strong $L^2([-1,1])$-version of Carleson's estimate directly.