On the dimension of increments of Tychonoff spaces

by

A. Ch. Chigogidze (Tbilisi)

Dedicated to Professor G. S. Chogoshvili on the occasion of his 65-th birthday

Abstract. All spaces are to be completely regular and Hausdorff. The present paper is an investigation of the following problem of extensions of spaces: find a characterization of those spaces X which possess the Wallman realcompact extension [12] Y whose increment $Y-X$ has dimension not exceeding n (n be an integer $\geqslant -1$).

The present paper is motivated by problems which relate extensions Y of a space X and the dimension of the increment $Y-X$. Let $\mathscr{P}$ be a topologically closed class of spaces and n be an integer. Then find a characterization of those spaces X for which there is an extension $Y \in \mathscr{P}$ whose increment $Y-X$ has dimension not exceeding n. Classes of spaces which have been considered are the class of bicompact spaces (see for example [9], [10], [11]) and the class of topologically complete spaces [1]. For these classes characterizations have been found. Two general approaches to this problem (for metric spaces) are given in [2], [3].

To the best of our knowledge, the class of realcompact spaces have not been considered earlier. In the present paper are given characterizations of those spaces X which possess the Wallman realcompact extension Y whose remainder $Y-X$ has dimension not exceeding n. But it should be observed that we consider only part (in general, proper part) of the class of all realcompact extensions of Tychonoff spaces. On the other hand, this part is sufficiently rich in elements. In acknowledgment of this recall one fact from [12], which confirms that if X is not Lindelöf and not pseudocompact then X has infinitely many Wallman realcompactifications.

1. Notations and basic definitions. All given spaces are assumed to be completely regular and Hausdorff, that is Tychonoff spaces.

Collection of all zero-sets of a space X is denoted by $\mathscr{Z}(X)$. If $X \subseteq Y$, then $\mathscr{Z}(X, Y)$ denotes the trace of $\mathscr{Z}(Y)$ on X. Let $\mathscr{L}(X)$ denote the family of all nest generated intersection rings in X [12]. It is easy to see that $\mathscr{L}(X)$ is precisely the family of all collections $\mathscr{Z}(X, Y)$ (X is fixed). In the present paper we consider the Wallman bicompactification $w(X, \mathscr{F})$ and the Wallman realcompactification $v(X, \mathscr{F})$ associated with a given element $\mathcal{F}$ of $\mathcal{L}(X)$ [12]. When there is no question as to the space X, we will simply write $w(\mathcal{F})$, $v(\mathcal{F})$. Collection of complements (in X) of elements of collection $\mathcal{Z}(X, Y)$ is denoted by $\mathcal{CZ}(X, Y)$. R denotes the space of real numbers.

1.1. Definition. Let $X \subseteq Y$. We shall say that a space X is *realcompact with respect to* Y if $X = v(X, \mathcal{Z}(X, Y))$.

It is easy to see that the notion of relative realcompactness generalized the notion of usual realcompactness. Really, if a space X is z-embedded in Y then $\mathcal{Z}(X, Y) = \mathcal{Z}(X)$ [4], and hence, the Wallman realcompact extension, which is constructed by means of collection $\mathcal{Z}(X, Y)$, coincides with the Hewitt realcompactification of a space X. This shows that Definition 1.1 transforms, in this particular case, into the definition of the notion of usual realcompactness.

The following proposition directly follows from the construction of the Wallman realcompactification [12].

1.2. Proposition. *Let Y be any bicompactification of a space X. Then X is realcompact with respect to Y if and only if for each point $p \in Y - X$ there is a continuous f on Y with $f(p) = 0$ but $f(x) \neq 0$ for all $x \in X$. In other words, if and only if each $p \in Y - X$ is contained in a zero-set of Y which does not meet X.*

The following proposition is to be contrasted with the absolute case, where the union of two realcompact subspaces may fail to be realcompact.

1.3. Proposition. *Let $X_1, X_2, \ldots \subseteq X$ all be realcompact with respect to X. Then $\bigcup_{n=1}^{\infty} X_n$ is realcompact with respect to X.*

Proof. Consider the bicompactification $X' = \mathrm{cl}_{\beta X} Y$ of $Y = \bigcup_{n=1}^{\infty} X_n$ and let $p \in X' - Y$. For each n with $p \notin \mathrm{cl}_{\beta X} X_n$, we can use complete regularity to find a continuous $f_n\colon \beta X \to R$ such that $f_n(p) = 0$ but $f_n(x) \neq 0$ for all $x \in X_n$. But for each n with $p \in \mathrm{cl}_{\beta X} X_n$, we can use Proposition 1.2 (it is easy to see that each X_n is realcompact with respect to $\mathrm{cl}_{\beta X} X_n$) to find a continuous $g_n\colon \mathrm{cl}_{\beta X} X_n \to R$ with $g_n(p) = 0$ but $g_n(x) \neq 0$ for all $x \in X_n$; then extending g_n from the bicompact space $\mathrm{cl}_{\beta X} X_n$ to all of βX, we again find a continuous $f_n\colon \beta X \to R$ such that $f_n(p) = 0$ but $f_n(x) \neq 0$ for all $x \in X_n$. Let:

$$f = \sum_{n=1}^{\infty} 2^{-n}(|f_n| \wedge 1)\,.$$

Then $f(x) \neq 0$ for all $x \in Y$, but $f(p) = 0$. Restricting f to X' and using Proposition 1.2, we have the desired result.

1.4. Corollary [4]. *If the Tychonoff space X is the union of a countable family of z-embedded realcompact subspaces, then X is realcompact.*

The following propositions are proved such as the respective propositions of the classical theory of realcompact spaces [8].

1.5. Proposition. *Let $X \subseteq Y$ and A is closed in X. If X is realcompact with respect to Y, then A is also realcompact with respect to Y.*

1.6. Proposition. *An arbitrary intersection of subspaces of a given space X, each of them is realcompact with respect to X, is realcompact with respect to X.*

1.7. Proposition. *Let $\mathcal{F}_\alpha \in \mathcal{L}(X_\alpha)$ and X_α is realcompact with respect to $w(\mathcal{F}_\alpha)$. Then $\prod_\alpha X_\alpha$ is realcompact with respect to $\prod_\alpha w(\mathcal{F}_\alpha)$.*

An essential role in the present paper play the notions of the relative dimensions d and I [5], [6]. Let us recall their definitions:

1.8. Definition. Let $X \subseteq Y$. The *covering dimension of X with respect to Y*, denoted by $d(X, Y)$, is defined as follows: $d(X, Y) \leqslant n$ if every $\mathcal{CZ}(X, Y)$-cover of X (under $\mathcal{CZ}(X, Y)$-cover of a space X we mean a finite cover of X members of which are elements of collection $\mathcal{CZ}(X, Y)$) has a $\mathcal{CZ}(X, Y)$-refinement of order at most $\leqslant n+1$.

1.9. Definition. Let $X \subseteq Y$. The *large inductive dimension of X with respect to Y*, denoted by $I(X, Y)$, is defined inductively as follows: $I(X, Y) = -1$ if and only if $X = \emptyset$. For a non-negative integer n, $I(X, Y) \leqslant n$ means that for every pair Z_1, Z_2 of disjoint elements of collection $\mathcal{Z}(X, Y)$ there are $Z \in \mathcal{Z}(X, Y)$, $O_1, O_2 \in \mathcal{CZ}(X, Y)$ with $X - Z = O_1 \cup O_2$, $O_1 \cap O_2 = \emptyset$, $Z_i \subseteq O_i$ $(i = 1, 2)$ and $I(Z, Y) \leqslant n-1$.

$\mathrm{Ind}_0 X$ denotes the *large inductive dimension of a space X with respect to X*.

2. Covering realcompactness degree.

2.1. Definition. Let $X \subseteq Y$. Under $R(X, Y)$-border cover of a space X we mean a finite family $\{O_i\}$ of elements of collection $\mathcal{CZ}(X, Y)$ such that $X - \bigcup O_i$ is realcompact with respect to Y. The *covering realcompactness degree of a space X with respect to Y*, denoted by $R\text{-}d(X, Y)$, is defined as follows: $R\text{-}d(X, Y) \leqslant n$ if every $R(X, Y)$-border cover has a $R(X, Y)$-refinement of order at most $\leqslant n+1$.

$R\text{-}\dim X$ denotes the *covering realcompactness degree of a space X with respect to X*.

2.2. Lemma. *Let $\mathcal{F} \in \mathcal{L}(X)$. A family $\{O_1, \ldots, O_k\}$ of elements of collection $\mathcal{CF}$ is a $R(X, w(\mathcal{F}))$-border cover of X if and only if there are cozero-sets $H_1, \ldots, H_k$ of $w(\mathcal{F})$ with $H_i \cap X = O_i$ and $v(\mathcal{F}) - X \subseteq \bigcup H_i$.*

Proof. Let $\{O_i\}$ be a $R(X, w(\mathcal{F}))$-border cover of X. By Definition 2.1, there are cozero-sets H_i of $w(\mathcal{F})$ with $H_i \cap X = O_i$. Consider sets $H_i' = H_i \cap v(\mathcal{F})$. It is easy to see that $H_i' \in \mathcal{CZ}(v(\mathcal{F}), w(\mathcal{F}))$ and hence, by Proposition 0.4 from [5], $v(\mathcal{F}) - H_i' = \mathrm{cl}_{v(\mathcal{F})}(X - O_i)$. So, by Proposition 0.2 from [5],

$$v(\mathcal{F}) - \bigcup H_i' = \mathrm{cl}_{v(\mathcal{F})}(X - \bigcup O_i)\,.$$

By Definition 2.1, Propositions 0.3 and 0.4 from [5], $\mathrm{cl}_{v(\mathcal{F})}(X - \bigcup O_i) = X - \bigcup O_i$. Thus $v(\mathcal{F}) - \bigcup H_i' \subseteq X$ and hence $v(\mathcal{F}) - X \subseteq H_i'$. It is clear now that $v(\mathcal{F}) - X \subseteq \bigcup H_i$.

Conversely, suppose $\{H_i\}$ are cozero-sets of $w(\mathcal{F})$ with $H_i \cap X = O_i$ and

$v(\mathcal{F})-X\subseteq\bigcup H_i$. Consider sets $H_i'=H_i\cap v(\mathcal{F})$. It is easy to see that $v(\mathcal{F})-X\subseteq\bigcup H_i'$ and $H_i'\cap X=O_i$. Obviously, $X-\bigcup O_i=v(\mathcal{F})-\bigcup H_i'\subseteq X$. Hence $X-\bigcup O_i$ is an element of collection $\mathcal{Z}(v(\mathcal{F}), w(\mathcal{F}))$ and by Proposition 1.5, $X-\bigcup O_i$ is realcompact with respect to $w(\mathcal{F})$. Finally, by Theorem 2.2 from [12],

$$O_i\in\mathcal{C}\mathcal{F}=\mathcal{C}\mathcal{Z}(X, w(\mathcal{F}))\,.$$

Thus, $\{O_i\}$ is a $R(X, w(\mathcal{F}))$-border cover of X.

2.3. THEOREM. *If $\mathcal{F}\in\mathcal{L}(X)$, then $R\text{-}d(X, w(\mathcal{F}))=d(v(\mathcal{F})-X, w(\mathcal{F}))$.*

Proof. Suppose $d(v(\mathcal{F})-X, w(\mathcal{F}))\leqslant n$, and let $\{O_i\}$ be a $R(X, w(\mathcal{F}))$-border cover of X. By Lemma 2.2, there are cozero-sets H_i of $w(\mathcal{F})$ with $H_i\cap X=O_i$ and $v(\mathcal{F})-X\subseteq\bigcup H_i$. By Theorem 3 from [6], there are cozero-sets H_i' of $w(\mathcal{F})$ of order $\leqslant n+1$ with $H_i'\subseteq H_i$ and $v(\mathcal{F})-X\subseteq\bigcup H_i'$. By Lemma 2.2, $\{H_i'\cap X\}$ is a $R(X, w(\mathcal{F}))$-border cover of X of order $\leqslant n+1$, refining $\{O_i\}$. Hence $R\text{-}d(X, w(\mathcal{F}))\leqslant n$.

Conversely, suppose $R\text{-}d(X, w(\mathcal{F}))\leqslant n$, and let $\{O_i\}$ be a $\mathcal{C}\mathcal{Z}(v(\mathcal{F})-X, w(\mathcal{F}))$-cover of $v(\mathcal{F})-X$. There are cozero-sets O_i' of $w(\mathcal{F})$ with $O_i'\cap(v(\mathcal{F})-X)=O_i$. It is easy to see that $v(\mathcal{F})-X\subseteq\bigcup O_i'$. By Lemma 2.2, $\{O_i'\cap X\}$ is a $R(X, w(\mathcal{F}))$-border cover of X and hence it is refined by a $R(X, w(\mathcal{F}))$-border cover $\{G_i\}$ of X of order $\leqslant n+1$. By Lemma 2.2, there are cozero-sets $\{G_i'\}$ of $w(\mathcal{F})$ with $G_i'\cap X=G_i$ and $v(\mathcal{F})-X\subseteq\bigcup G_i'$. It follows from the density of X in $w(\mathcal{F})$ that order $\{G_i'\}\leqslant n+1$. Consider sets $G_i'\cap(v(\mathcal{F})-X)$. It is easy to see that $\{G_i'\cap(v(\mathcal{F})-X)\}$ is a $\mathcal{C}\mathcal{Z}(v(\mathcal{F})-X, w(\mathcal{F}))$-cover of $v(\mathcal{F})-X$ of order $\leqslant n+1$. In order to prove our theorem it will suffice to show that $G_i'\cap(v(\mathcal{F})-X)\subseteq O_i$. It follows from the preceding reasonings that $G_i\subseteq O_i'\cap X$ and hence $X-G_i\supseteq X-(O_i'\cap X)$. Then

$$\mathrm{cl}_{v(\mathcal{F})}(X-G_i)\supseteq\mathrm{cl}_{v(\mathcal{F})}(X-(O_i'\cap X))$$

and

$$v(\mathcal{F})-\mathrm{cl}_{v(\mathcal{F})}(X-G_i)\subseteq v(\mathcal{F})-\mathrm{cl}_{v(\mathcal{F})}(X-(O_i'\cap X))\,.$$

On the other hand,

$$G_i'\cap v(\mathcal{F})=v(\mathcal{F})-\mathrm{cl}_{v(\mathcal{F})}(X-G_i)$$

and

$$O_i'\cap v(\mathcal{F})=v(\mathcal{F})-\mathrm{cl}_{v(\mathcal{F})}(X-(O_i'\cap X))\,.$$

(Propositions 0.3 and 0.4 from [5]). Hence $G_i'\cap v(\mathcal{F})\subseteq O_i'\cap v(\mathcal{F})$ and so,

$$G_i'\cap(v(\mathcal{F})-X)\subseteq O_i'\cap(v(\mathcal{F})-X)=O_i\,.$$

The proof is complete.

2.4. COROLLARY. $R\text{-}\dim X=d(vX-X, \beta X)$.

2.5. COROLLARY. *Let $\mathcal{F}\in\mathcal{L}(X)$ and $v(\mathcal{F})-X$ is z-embedded in $w(\mathcal{F})$, then* $R\text{-}d(X, w(\mathcal{F}))=\dim(v(\mathcal{F})-X)$.

2.6. COROLLARY. *If bX be an arbitrary bicompactification of a pseudocompact space X satisfying the bicompact axiom of countability* [10], *then* $R\text{-}d(X, bX)=\dim(bX-X)$.

2.7. COROLLARY. *Let $X\subseteq Y$. Then $R\text{-}d(X, Y)\leqslant d(X, Y)$.*

Proof. Let $\mathcal{F}$ denote a collection $\mathcal{Z}(X, Y)$. It is easy to see that $\mathcal{F}\in\mathcal{L}(X)$. By Theorem 2.2 from [12], $\mathcal{Z}(X, w(\mathcal{F}))=\mathcal{F}$ and hence $R\text{-}d(X, Y)=R\text{-}d(X, w(\mathcal{F}))$ and $d(X, Y)=d(X, w(\mathcal{F}))$. So, in order to prove our corollary it will suffice to show that $R\text{-}d(X, w(\mathcal{F}))\leqslant d(X, w(\mathcal{F}))$. By Theorem 2.3, $R\text{-}d(X, w(\mathcal{F}))=d(v(\mathcal{F})-X, w(\mathcal{F}))$; by Theorem 2 and Proposition 1 from [6], $d(X, w(\mathcal{F}))=\dim w(\mathcal{F})$ and $d(v(\mathcal{F})-X, w(\mathcal{F}))\leqslant\dim w(\mathcal{F})$. Thus $R\text{-}d(X, w(\mathcal{F}))\leqslant d(X, w(\mathcal{F}))$.

2.8. THEOREM. *Let $X\subseteq Y$ and $X=\bigcup_{i=1}^{\infty}B_i$ with $B_i\in\mathcal{Z}(X, Y)$ and $R\text{-}d(B_i, Y)\leqslant n$. Then $R\text{-}d(X, Y)\leqslant n$.*

Proof. Let $\mathcal{F}$ denote a collection $\mathcal{Z}(X, Y)$. It is easy to see that $\mathcal{F}\in\mathcal{L}(X)$. By Theorem 2.2 from [12], $Z(X, w(\mathcal{F}))=\mathcal{F}$ and hence $R\text{-}d(X, Y)=R\text{-}d(X, w(\mathcal{F}))$. It is not difficult to see that the following equalities also holds: $R\text{-}d(B_i, Y)=R\text{-}d(B_i, w(\mathcal{F}))$. So, in order to prove our theorem it will suffice to show that $R\text{-}d(X, w(\mathcal{F}))\leqslant n$.

It is clear that $X\subseteq\bigcup\mathrm{cl}_{v(\mathcal{F})}(B_i)\subseteq v(\mathcal{F})$. By Proposition 1.5, $\mathrm{cl}_{v(\mathcal{F})}(B_i)$ is realcompact with respect to $w(\mathcal{F})$ and hence, by Proposition 1.3, $\bigcup\mathrm{cl}_{v(\mathcal{F})}(B_i)$ also is realcompact with respect to $w(\mathcal{F})$. By Proposition 0.1 from [5], $v(\mathcal{F})$ is the smallest space between X and $w(\mathcal{F})$, which is realcompact with respect to $w(\mathcal{F})$. Hence $\bigcup\mathrm{cl}_{v(\mathcal{F})}(B_i)=v(\mathcal{F})$. Then we have

$$v(\mathcal{F})-X\subseteq\bigcup\{\mathrm{cl}_{v(\mathcal{F})}(B_i)-B_i\}\,.$$

It is easy to see (each B_i, as an element of a collection $\mathcal{F}$, is a closed subset of X), that the converse inclusion also holds. Thus

$$v(\mathcal{F})-X=\bigcup\{\mathrm{cl}_{v(\mathcal{F})}(B_i)-B_i\}\,.$$

By Theorem 2.3, $d(\mathrm{cl}_{v(\mathcal{F})}(B_i)-B_i, w(\mathcal{F}))=R\text{-}d(B_i, w(\mathcal{F}))\leqslant n$. By Proposition 0.3 from [5], $\mathrm{cl}_{v(\mathcal{F})}(B_i)$ is an element of a collection $Z(v(\mathcal{F}), w(\mathcal{F}))$ and hence $\mathrm{cl}_{v(\mathcal{F})}(B_i)-B_i$ is an element of $Z(v(\mathcal{F})-X, w(\mathcal{F}))$. By the countable sum theorem for relative covering dimension d (Theorem 1 from [6]), $d(v(\mathcal{F})-X, w(\mathcal{F}))\leqslant n$ and hence, by Theorem 2.3, $R\text{-}d(X, w(\mathcal{F}))\leqslant n$.

It is easy to find realcompact spaces whose subspaces are not necessarily realcompact. Hence a general monotone property for covering realcompactness degree is not possible. But we have the following

2.9. PROPOSITION. *If $X\subseteq Y$ and $B\in\mathcal{Z}(X, Y)$, then $R\text{-}d(B, Y)\leqslant R\text{-}d(X, Y)$.*

Proof. Suppose $R\text{-}d(X, Y)\leqslant n$, and let $\{O_i\}$ be a $R(B, Y)$-border cover of B. By Definition 2.1, there are sets O_i' with $O_i'\in\mathcal{C}\mathcal{Z}(X, Y)$ and $O_i'\cap B=O_i$. It is easy to see that $\{O_i'\}\cup\{X-B\}$ is a $R(X, Y)$-border cover of X, and hence it is refined by a $R(X, Y)$-border cover $\{H_i\}$ of order $\leqslant n+1$. Then, by Proposition 1.5,

$\{H_i \cap B\}$ is a $R(B, Y)$-border cover of B of order $\leqslant n+1$, refining $\{O_i\}$. Hence $R\text{-}d(B, Y) \leqslant n$.

2.10. COROLLARY. *Let $X \subseteq Y$ and $R\text{-}d(X, Y) \leqslant n$. If B be a subset of X which is a countable union of elements of collection $\mathscr{Z}(X, Y)$, then $R\text{-}d(B, Y) \leqslant n$.*

Proof. This follows from Theorem 2.8 and Proposition 2.9.

2.11. COROLLARY. *If G be a cozero-set of a space X, then $R\text{-}\dim G \leqslant R\text{-}\dim X$.*

2.12. Remark. Let $A \subseteq B \subseteq C$ and B is realcompact with respect to C. If B satisfies the first axiom of countability, then A is realcompact with respect to C. Really, it is easy to see, that each point b of B is an element of a collection $\mathscr{Z}(B, C)$, and hence, by Proposition 1.3, $B-\{b\}$ is realcompact with respect to C. By Proposition 1.6, $A = \bigcap_{b \in B-A} (B-\{b\})$ is also realcompact with respect to C.

2.13. THEOREM. *Let $M \subseteq N \subseteq X$ and N satisfies the first axiom of countability, then $R\text{-}d(M, X) \leqslant R\text{-}d(N, X)$.*

Proof. Suppose $R\text{-}d(N, X) \leqslant n$, and let $\{O_i\}$ be a $R(M, X)$-border cover of M. There are elements H_i of collection $\mathscr{C}\mathscr{Z}(N, X)$ with $H_i \cap M = O_i$. A set $H = \bigcup H_i$, as a finite union of elements of collection $\mathscr{C}\mathscr{Z}(N, X)$, also is an element of $\mathscr{C}\mathscr{Z}(N, X)$ and hence, by Corollary 2.10, $R\text{-}d(H, X) \leqslant n$. It is easy to see that a cover $\{H_i\}$ of H is a $R(H, X)$-border cover of H and hence it is refined by a $R(H, X)$-border cover $\{G_i\}$ of H of order $\leqslant n+1$. A set $H-\bigcup G_i$, as a subspace of N, satisfies the first axiom of countability and hence, by Definition 2.1 and Remark 2.12, $M \cap (H-\bigcup G_i)$ is realcompact with respect to X. Consider a collection $\{G_i \cap M\}$. It is easy to see that $\{G_i \cap M\}$ consisting of elements of collection $\mathscr{C}\mathscr{Z}(M, X)$, refining $\{O_i\}$ and has order $\leqslant n+1$. Now we show that $\{G_i \cap M\}$ is a $R(M, X)$-border cover of M. Consider a set $M-\bigcup(G_i \cap M)$. It follows from the above reasonings that

$$M-\bigcup(G_i \cap M) = \{M-\bigcup O_i\} \cup \{M \cap (H-\bigcup G_i)\}.$$

Evidently, both members in the right part of this equality are realcompact with respect to X and hence, by Proposition 1.3, their union $M-\bigcup(G_i \cap M)$ is also realcompact with respect to X. Thus $\{G_i \cap M\}$ is a $R(M, X)$-border cover of M. Hence $R\text{-}d(M, X) \leqslant n$.

2.14. COROLLARY. *If A be an arbitrary subspace of perfectly normal space X, then $R\text{-}\dim A \leqslant R\text{-}\dim X$.*

2.15. THEOREM. *If $M, N \subseteq X$, then $R\text{-}d(M \cup N, X) \leqslant R\text{-}d(M, X) + R\text{-}d(N, X) + 1$.*

Proof. Suppose $R\text{-}d(M, X) = n$, $R\text{-}d(N, X) = m$, and let $\{O_i\}$ be a $R(M \cup N, X)$-border cover of $M \cup N$. Consider a collection $\{O_i \cap M\}$. It is easy to see that $O_i \cap M \in \mathscr{C}\mathscr{Z}(M, X)$ and hence $M' = \bigcup(O_i \cap M)$ is also an element of collection $\mathscr{C}\mathscr{Z}(M, X)$. By Corollary 2.10, $R\text{-}d(M', X) \leqslant n$. Hence a cover $\{O_i \cap M\}$ of M' is refined by a $R(M', X)$-border cover $\{G_i\}$ of order $\leqslant n+1$. By Definition 2.1, $M'-\bigcup G_i = Z_1$ is realcompact with respect to X. By Lemma 1 from [6], there are sets $H_1, \ldots, H_k$ of order $\leqslant n+1$ with $H_i \in \mathscr{C}\mathscr{Z}(M \cup N, X)$, $H_i \cap M = G_i$ and $H_i \subseteq O_i$.

We can use the preceding process (but with respect to N) to find sets $T_1, \ldots, T_k$ of order $\leqslant m+1$ with $T_i \in \mathscr{C}\mathscr{Z}(M \cup N, X)$, $T_i \subseteq O_i$ and a set Z_2, which is realcompact with respect to X.

Consider a family $\omega = \{H_i\} \cup \{T_i\}$. In order to prove that ω is a $R(M \cup N, X)$-border cover it will suffice to show that $(M \cup N)-\bigcup(H_i \cup T_i)$ is realcompact with respect to X. It is not difficult to see that

$$(M \cup N)-\bigcup(H_i \cup T_i) \subseteq \{M \cup N)-\bigcup O_i\} \cup Z_1 \cup Z_2 .$$

Each member in the right part of this inclusion is realcompact with respect to X, and hence, by Proposition 1.3, their union is also realcompact with respect to X. $(M \cup N)-\bigcup(H_i \cup T_i)$ is closed in this union and hence, by Proposition 1.5, ω is a $R(M \cup N, X)$-border cover of $M \cup N$. It is easy to see that ω refines $\{O_i\}$ and order $\{\omega\} \leqslant n+m+2$. Thus $R\text{-}d(M \cup N, X) \leqslant n+m+1$.

2.16. COROLLARY. *Let M and N be subsets of X. If $M \cup N$ satisfies the first axiom of countability and N is realcompact with respect to X, then $R\text{-}d(M \cup N, X) = R\text{-}d(M, X)$.*

Proof. By Theorem 2.15, $R\text{-}d(M \cup N, X) \leqslant R\text{-}d(M, X) + R\text{-}d(N, X) + 1 = R\text{-}d(M, X)$. By Theorem 2.13, the converse inequality also holds.

2.17. COROLLARY. *Let X be a perfectly normal space with $X = M \cup N$ and N is realcompact, then $R\text{-}\dim X = R\text{-}\dim M$.*

In general, when the first axiom of countability does not hold, we have

2.18. PROPOSITION. *Let $B \subseteq X \subseteq Y$ and B be a countable union of elements of collection $\mathscr{Z}(X, Y)$. If $X-B$ is realcompact with respect to Y, then $R\text{-}d(X, Y) = R\text{-}d(B, Y)$.*

Proof. By Theorem 2.15, $R\text{-}d(X, Y) \leqslant R\text{-}d(B, Y) + R\text{-}d(X-B, Y) + 1 = R\text{-}d(B, Y)$. By Corollary 2.10, the converse inequality also holds.

2.19 THEOREM. *Let $\mathscr{F} \in \mathscr{L}(X)$. If $d(B, w(\mathscr{F})) \leqslant n$ for every element B of collection $\mathscr{F}$ which is realcompact with respect to $w(\mathscr{F})$, then $\dim w(\mathscr{F}) \leqslant R\text{-}d(X, w(\mathscr{F})) + n + 1$.*

Proof. Suppose $\{O_i\}$ be a $\mathscr{C}\mathscr{F}$-cover of X. By Definition 2.1, $\{O_i\}$ is a $R(X, w(\mathscr{F}))$-border cover of X and hence it is refined by a $R(X, w(\mathscr{F}))$-border cover $\{H_i\}$ of order $\leqslant R\text{-}d(X, w(\mathscr{F})) + 1$. Consider set $B = X - \bigcup H_i$. By Definition 2.1, $B \in \mathscr{F}$ and is realcompact with respect to $w(\mathscr{F})$. So, $d(B, w(\mathscr{F})) \leqslant n$. By Theorem 3 from [6], there are elements $\{G_i\}$ of collection $\mathscr{C}\mathscr{F}$ with $B \subseteq \bigcup G_i$, $G_i \subseteq O_i$ and order $\{G_i\} \leqslant n+1$. Evidently $\omega = \{H_i\} \cup \{G_i\}$ is a $\mathscr{C}\mathscr{F}$-cover of X of order $\leqslant R\text{-}d(X, w(\mathscr{F})) + n + 2$ refining $\{O_i\}$. Thus $d(X, w(\mathscr{F})) \leqslant R\text{-}d(X, w(\mathscr{F})) + n + 1$. It is easy to see now that $\dim w(\mathscr{F}) \leqslant R\text{-}d(X, w(\mathscr{F})) + n + 1$ (Proposition 1 from [6]).

2.20. COROLLARY. *Let bX be an arbitrary bicompactification of a pseudocompact space X. If $\dim B \leqslant n$ for every bicompact element of collection $\mathscr{Z}(X, bX)$, then*

$\dim bX \leqslant R\text{-}d(X, bX)+n+1$. *Furthermore, if X satisfies the bicompact axiom of countability, then* $\dim bX \leqslant \dim(bX-X)+n+1$.

2.21. Corollary. *Let bX be an arbitrary bicompactification of a pseudocompact space X satisfying the bicompact axiom of countability. If $\dim B \leqslant 0$ for every bicompact element of collection $\mathcal{Z}(X, bX)$, then* $\dim bX \leqslant \dim(bX-X)+1$.

3. Large inductive realcompactness degree. By replacing the empty set in the definition of relative large inductive dimension (Definition 1.9) we get

3.1. Definition. Let $X \subseteq Y$. The *large inductive realcompactness degree of a space X with respect to Y*, denoted by $R\text{-}I(X, Y)$, is defined inductively as follows: $R\text{-}I(X, Y) = -1$ if and only if X is realcompact with respect to Y. For a non-negative integer n, $R\text{-}I(X, Y) \leqslant n$ means that for each pair Z_1, Z_2 of disjoint elements of collection $\mathcal{Z}(X, Y)$ there are $Z \in \mathcal{Z}(X, Y)$, $O_1, O_2 \in \mathcal{CZ}(X, Y)$ with $X-Z = O_1 \cup O_2$, $O_1 \cap O_2 = \emptyset$, $Z_i \subseteq O_i$ $(i = 1, 2)$ and $R\text{-}I(Z, Y) \leqslant n-1$. $R\text{-}\mathrm{Ind}_0 X$ denotes the large inductive realcompactness degree of X with respect to X.

The small inductive realcompactness degree $R\text{-}i(X, Y)$ of X with respect to Y is defined by analogy with definition of relative small inductive dimension $i(X, Y)$ [5].

3.2. Theorem. *If $\mathcal{F} \in \mathcal{L}(X)$, then* $R\text{-}I(X, w(\mathcal{F})) = I(v(\mathcal{F})-X, w(\mathcal{F}))$.

Proof of this theorem is given in [5] (Theorem 1.9).

3.3. Corollary. $R\text{-}\mathrm{Ind}_0 X = I(vX-X, \beta X)$.

3.4. Corollary. *Let $\mathcal{F} \in \mathcal{L}(X)$ and $v(\mathcal{F})-X$ is z-embedded in $w(\mathcal{F})$, then* $R\text{-}I(X, w(\mathcal{F})) = \mathrm{Ind}_0(v(\mathcal{F})-X)$.

3.5. Corollary. *If bX be an arbitrary bicompactification of a pseudocompact space X satisfying the bicompact axiom of countability, then* $R\text{-}I(X, bX) = \mathrm{Ind}_0(bX-X)$.

3.6. Corollary. *Let $X \subseteq Y$. Then* $R\text{-}I(X, Y) \leqslant I(X, Y)$.

Proof. Let $\mathcal{F}$ denote a collection $\mathcal{Z}(X, Y)$. It is easy to see that $\mathcal{F} \in \mathcal{L}(X)$. By Theorem 2.2 from [12], $\mathcal{F} = \mathcal{Z}(X, w(\mathcal{F}))$ and hence $R\text{-}I(X, Y) = R\text{-}I(X, w(\mathcal{F}))$ and $I(X, Y) = I(X, w(\mathcal{F}))$. So, in order to prove our corollary it will suffice to show that $R\text{-}I(X, w(\mathcal{F})) \leqslant I(X, w(\mathcal{F}))$. But this follows from Theorem 3.2, Theorems 1.1 and 1.9 from [5].

3.7. Corollary. *Let $X \subseteq Y$. Then* $R\text{-}d(X, Y) \leqslant R\text{-}I(X, Y)$.

3.8. Theorem. *Let $X \subseteq Y$ and $X = \bigcup_{i=1}^{\infty} B_i$ with $B_i \in \mathcal{Z}(X, Y)$ and $R\text{-}I(B_i, Y) \leqslant n$. Then $R\text{-}I(X, Y) \leqslant n$.*

Proof of this theorem is such as a proof of Theorem 2.8.

3.9. Proposition. *If $X \subseteq Y$ and $B \in \mathcal{Z}(X, Y)$, then* $R\text{-}I(B, Y) \leqslant R\text{-}I(X, Y)$.

Proof. Suppose $R\text{-}I(X, Y) = k$. For $k = -1$ the result follows from Proposition 1.5. We assume its validity for $k \leqslant n-1$ and suppose $k \leqslant n$.

Let Z_1, Z_2 be disjoint elements of collection $\mathcal{Z}(B, Y)$. It is easy to see that $Z_1, Z_2 \in \mathcal{Z}(X, Y)$, and hence there are $Z \in \mathcal{Z}(X, Y)$, $O_1, O_2 \in \mathcal{CZ}(X, Y)$ with $X-Z = O_1 \cup O_2$, $O_1 \cap O_2 = \emptyset$, $Z_i \subseteq O_i$ $(i = 1, 2)$ and $R\text{-}I(Z, Y) \leqslant n-1$. Consider sets $Z \cap B \in \mathcal{Z}(B, Y)$, $O_1 \cap B$, $O_2 \cap B \in \mathcal{CZ}(B, Y)$. Clearly $B-(Z \cap B) = (O_1 \cap B) \cup (O_2 \cap B)$, $(O_1 \cap B) \cap (O_2 \cap B) = \emptyset$, $Z_i \subseteq O_i \cap B$ $(i = 1, 2)$ and by the induction hypothesis $R\text{-}I(Z \cap B, Y) \leqslant R\text{-}I(Z, Y) \leqslant n-1$. Hence $R\text{-}I(B, Y) \leqslant n$.

3.10. Corollary. *Let $X \subseteq Y$ and $R\text{-}I(X, Y) \leqslant n$. If B be a subset of X which is a countable union of elements of collection $\mathcal{Z}(X, Y)$, then* $R\text{-}I(B, Y) \leqslant n$.

Proof. This follows from Theorem 2.8 and Proposition 2.9.

3.11. Corollary. *If G be a cozero-set of a space X, then* $R\text{-}\mathrm{Ind}_0 G \leqslant R\text{-}\mathrm{Ind}_0 X$.

As in the case of covering realcompactness degree, a general monotone property for large inductive realcompactness degree is not possible. But we have the following

3.12. Theorem. *Let $M \subseteq N \subseteq X$ and N satisfies the first axiom of countability, then* $R\text{-}I(M, X) \leqslant R\text{-}I(N, X)$.

Proof. Suppose $R\text{-}I(N, X) = k$. For $k = -1$ the result follows from the Remark 2.12. We assume its validity for $k \leqslant n-1$ and suppose $k \leqslant n$.

Let Z_1, Z_2 be disjoint elements of collection $\mathcal{Z}(M, X)$. There are $F_1, F_2 \in \mathcal{Z}(N, X)$ with $F_i \cap M = Z_i$ $(i = 1, 2)$. Evidently, $G = N-(F_1 \cap F_2)$ is an element of $\mathcal{CZ}(N, X)$, and hence, by Corollary 3.10, $R\text{-}I(G, X) \leqslant n$. This means that there are $F \in \mathcal{Z}(G, X)$, $G_1, G_2 \in \mathcal{CZ}(G, X)$ with $G-F = G_1 \cup G_2$, $G_1 \cap G_2 = \emptyset$, $F_i \cap G \subseteq G_i$ $(i = 1, 2)$ and $R\text{-}I(F, X) \leqslant n-1$. Consider sets $Z = F \cap M$ and $O_i = G_i \cap M$. Evidently, $Z \in \mathcal{Z}(M, X)$, $O_1, O_2 \in \mathcal{CZ}(M, X)$, $M-Z = O_1 \cup O_2$, $O_1 \cap O_2 = \emptyset$, $Z_i \subseteq O_i$ and by the inductive hypothesis (clearly, $F \subseteq N$), $R\text{-}I(Z, X) \leqslant R\text{-}I(F, X) \leqslant n-1$. Hence $R\text{-}I(M, X) \leqslant n$.

3.13. Corollary. *If A be an arbitrary subspace of perfectly normal space X, then* $R\text{-}\mathrm{Ind}_0 A \leqslant R\text{-}\mathrm{Ind}_0 X$.

3.14. Theorem. *If $M, N \subseteq X$, then* $R\text{-}I(M \cup N, X) \leqslant R\text{-}I(M, X)+R\text{-}I(N, X)+1$.

Proof. Let $R\text{-}I(M, X) = k_1$, $R\text{-}I(N, X) = k_2$. For $k_1 = k_2 = -1$ the result follows from Proposition 1.3. Let $k_1 \leqslant n$, $k_2 \leqslant m$ and assume the theorem for the cases $k_1 \leqslant n$, $k_2 \leqslant m-1$ and $k_1 \leqslant n-1$, $k_2 \leqslant m$.

Let Z_1, Z_2 be disjoint elements of collection $\mathcal{Z}(Y, X)$ where $Y = M \cup N$. Choose $O_1, O_2 \in \mathcal{CZ}(Y, X)$ and $F_1, F_2 \in Z(Y, X)$ with $Z_i \subseteq O_i \subseteq F_i$ $(i = 1, 2)$ and $F_1 \cap F_2 = \emptyset$. Since $R\text{-}I(M, X) \leqslant n$, there are $G_1, G_2 \in \mathcal{CZ}(M, X)$ and $D \in \mathcal{Z}(M, X)$ with $M-D = G_1 \cup G_2$, $G_1 \cap G_2 = \emptyset$, $F_i \cap M \subseteq G_i$ and $R\text{-}I(D, X) \leqslant n-1$. By Lemma 1 from [6], there are $V_1, V_2 \in \mathcal{CZ}(Y, X)$ with $V_i \cap M = G_i$ and $V_1 \cap V_2 = \emptyset$. Then $U_1 = (V_1-F_2) \cup O_1$ and $U_2 = (V_2-F_1) \cup O_2$ are disjoint elements of collection $\mathcal{CZ}(Y, X)$ with $Z_i \subseteq U_i$ $(i = 1, 2)$ and $M-(U_1 \cup U_2) = D$, and hence $R\text{-}I(M-(U_1 \cup U_2), X) = R\text{-}I(D, X) \leqslant n-1$. By Proposition 3.9 (evidently, $N-(U_1 \cup U_2) \in \mathcal{Z}(N, X)$), $R\text{-}I(N-(U_1 \cup U_2), X) \leqslant R\text{-}I(N, X) \leqslant m$. By the induction hypothesis

$$R\text{-}I(Y-(U_1 \cup U_2), X) \leqslant n+m.$$

Thus $R\text{-}I(Y, X) \leqslant n+m+1$.

As in the case of covering realcompactness degree we have the following corollaries.

3.15. COROLLARY. *Let M and N be subsets of X. If $M \cup N$ satisfies the first axiom of countability and N is realcompact with respect to X, then $R\text{-}I(M \cup N, X) = R\text{-}I(M, X)$.*

3.16. COROLLARY. *Let X be a perfectly normal space with $X = M \cup N$ and N is realcompact, then $R\text{-Ind}_0 X = R\text{-Ind}_0 M$.*

3.17. COROLLARY. *Let $B \subseteq X \subseteq Y$ and B be a countable union of elements of collection $\mathscr{Z}(X, Y)$. If $X - B$ is realcompact with respect to Y, then $R\text{-}I(X, Y) = R\text{-}I(B, Y)$.*

3.18. THEOREM. *Let $\mathscr{F} \in \mathscr{L}(X)$. If $I(B, w(\mathscr{F})) \leqslant n$ for every element B of collection $\mathscr{F}$ which is realcompact with respect to $w(\mathscr{F})$, then $I(v(\mathscr{F}), w(\mathscr{F})) \leqslant R\text{-}I(X, w(\mathscr{F}))+n+1$.*

Proof. Suppose $R\text{-}I(X, w(\mathscr{F})) = k$. For $k = -1$ the result follows from the Definition 3.1. We assume its validity for $k \leqslant n'-1$ and suppose $k \leqslant n'$.

Let Z_1, Z_2 be disjoint elements of collection $\mathscr{F}$. There are $Z \in \mathscr{F}$, $O_1, O_2 \in \mathscr{C}\mathscr{F}$ with $X - Z = O_1 \cup O_2$, $O_1 \cap O_2 = \varnothing$, $Z_i \subseteq O_i$ $(i = 1, 2)$ and $R\text{-}I(Z, w(\mathscr{F})) \leqslant n'-1$. By the induction hypothesis and by Theorem 1.7 from [5], $I(Z, w(\mathscr{F})) \leqslant (n'-1)+ +n+1$ and hence $I(X, w(\mathscr{F})) \leqslant n'+n+1$. By Theorem 1.7 from [5],

$$I(v(\mathscr{F}), w(\mathscr{F})) \leqslant R\text{-}I(X, w(\mathscr{F}))+n+1 .$$

3.19. COROLLARY. *Let bX be an arbitrary bicompactification of a pseudocompact space X. If $\text{Ind}_0 B \leqslant n$ for every bicompact element of collection $\mathscr{Z}(X, bX)$, then $\text{Ind}_0 bX \leqslant R\text{-}I(X, bX)+n+1$. Furthermore, if X satisfies the bicompact axiom of countability, then $\text{Ind}_0 bX \leqslant \text{Ind}_0(bX - X)+n+1$.*

3.20. COROLLARY. *Let bX be an arbitrary bicompactification of a pseudocompact space X satisfying the bicompact axiom of countability. If $\dim B \leqslant 0$ for every bicompact element of collection $\mathscr{Z}(X, bX)$, then $\text{Ind}_0 bX \leqslant \text{Ind}_0(bX - X)+1$.*

4. Product theorems. The following theorem is to be contrasted with an absolute case ([8], Exercise 9I and 9.15), where counterexamples to the corresponding statements for the Hewitt extensions to be found; these counterexamples involve products of just two factors.

Suppose $\mathscr{F}_\alpha \in \mathscr{L}(X_\alpha)$, $\alpha \in A$. Consider a collection $\mathscr{Z}(\prod_\alpha X_\alpha, \prod_\alpha w(\mathscr{F}_\alpha))$ and denote it by $\prod_\alpha \mathscr{F}_\alpha$. Clearly $\prod_\alpha \mathscr{F}_\alpha \in \mathscr{L}(\prod_\alpha X_\alpha)$.

4.1. THEOREM. $\prod v(\mathscr{F}_\alpha) = v(\prod \mathscr{F}_\alpha)$.

Proof. By Proposition 1.7, $\prod v(\mathscr{F}_\alpha)$ is realcompact with respect to $\prod w(\mathscr{F}_\alpha)$. By Theorem 1.2 then, we need only show that every nonempty element of collection $\mathscr{Z}(\prod v(\mathscr{F}_\alpha), \prod w(\mathscr{F}_\alpha))$ meets $\prod X_\alpha$. Let B be such an element; it is not difficult to show that B is a countable intersection of finite unions of sets of the form $\text{pr}_\alpha^{-1} Z_\alpha$ with Z_α an element of collection $\mathscr{Z}(v(\mathscr{F}_\alpha), w(\mathscr{F}_\alpha))$ (pr_α is the projection of $\prod w(\mathscr{F}_\alpha)$ on $w(\mathscr{F}_\alpha)$). Fixing $b \in B$, we replace each of these finite unions by one of the sets whose union is taken, choosing this one to contain b. In this way, we find $B' \subseteq B$ such that B' is nonempty and is a countable intersection of sets of the form $\text{pr}_\alpha^{-1} Z_\alpha$. Thus we may write:

$$B' = \bigcap_{n=1}^{\infty} \text{pr}_{\alpha_n}^{-1} Z_{\alpha_n}$$

where $Z_{\alpha_n} \in \mathscr{Z}(v(\mathscr{F}_{\alpha_n}), w(\mathscr{F}_{\alpha_n}))$ and without loss of generality, suppose that $\alpha_1, \alpha_2, \ldots$ are distinct. Since each nonempty element of collection $\mathscr{Z}(v(\mathscr{F}_\alpha), w(\mathscr{F}_\alpha))$ meets X_α (Proposition 0.4 from [5]), we may choose, for each n, an $x_n \in Z_{\alpha_n} \cap X_{\alpha_n}$. Choosing any $x \in \prod X_\alpha$ such that $\text{pr}_{\alpha_n} x = x_n$ for all n, we have $x \in B' \cap \prod X_\alpha \subseteq B \cap \prod X_\alpha$. Hence $B \cap \prod X_\alpha \neq \varnothing$, which completes the proof.

4.2. COROLLARY. *Let $\mathscr{F}_i \in \mathscr{L}(X_i)$ $(i = 1, 2)$, then*

$$v(X_1 \times X_2, \mathscr{F}_1 \times \mathscr{F}_2) - (X_1 \times X_2) = \{(v(\mathscr{F}_1) - X_1) \times v(\mathscr{F}_2)\} \cup \{v(\mathscr{F}_1) \times (v(\mathscr{F}_2) - X_2)\} .$$

4.3. EXAMPLE. Now we show that a natural formula

$$R\text{-}d(X_1 \times X_2, w(\mathscr{F}_1) \times w(\mathscr{F}_2)) \leqslant R\text{-}d(X_1, w(\mathscr{F}_1)) + R\text{-}d(X_2, w(\mathscr{F}_2))$$

does not hold in general.

Let Y be any space with $\dim Y = n$ $(n \geqslant 1)$. Clearly, $\dim \beta Y = n$. Let $X = \beta Y - \{p\}$ with $p \in \beta Y - Y$. It is easy to see that $\dim X = n$ and $R\text{-}\dim X = 0$. By Corollary 4.2,

$$K = v(X^2, \mathscr{Z}^2(X)) - X^2 = (\{p\} \times \beta X) \cup (\beta X \times \{p\}) .$$

Evidently, $d(K, (\beta X)^2) = \dim K \geqslant n$ and hence

$$R\text{-}d(X^2, (\beta X)^2) = d(K, (\beta X)^2) > 2(R\text{-}\dim X) = 0 .$$

4.4. PROPOSITION. *Let $\mathscr{F}_i \in \mathscr{L}(X_i)$ $(i = 1, 2)$. If X_i is not realcompact with respect to $w(\mathscr{F}_i)$, then*

$$R\text{-}I(X_1 \times X_2, w(\mathscr{F}_1) \times w(\mathscr{F}_2)) \leqslant I(X_1, w(\mathscr{F}_1)) + I(X_2, w(\mathscr{F}_2)) .$$

Proof. By Corollary 3.6,

$$R\text{-}I(X_1 \times X_2, w(\mathscr{F}_1) \times w(\mathscr{F}_2)) \leqslant I(X_1 \times X_2, w(\mathscr{F}_1) \times w(\mathscr{F}_2)) .$$

By Proposition 1 from [7],

$$I(X_1 \times X_2, w(\mathscr{F}_1) \times w(\mathscr{F}_2)) \leqslant I(X_1, w(\mathscr{F}_1)) + I(X_2, w(\mathscr{F}_2)) .$$

This completes the proof.

4.5. PROPOSITION. *Let $\mathscr{F}_i \in \mathscr{L}(X_i)$ $(i = 1, 2)$. If X_1 is not realcompact with respect to $w(\mathscr{F}_1)$ and X_2 is realcompact with respect to $w(\mathscr{F}_2)$, then*

$$R\text{-}I(X_1 \times X_2, w(\mathscr{F}_1) \times w(\mathscr{F}_2)) \leqslant R\text{-}I(X_1, w(\mathscr{F}_1)) + I(X_2, w(\mathscr{F}_2)) .$$

Proof. By Corollary 4.2, we may write

$$K = v(\mathscr{F}_1 \times \mathscr{F}_2) - (X_1 \times X_2) = (v(\mathscr{F}_1) - X_1) \times v(\mathscr{F}_2) .$$

By Proposition 1 from [7],

$$I(K, w(\mathscr{F}_1)\times w(\mathscr{F}_2))\leqslant I(v(\mathscr{F}_1)-X_1, w(\mathscr{F}_1))+I(v(\mathscr{F}_2), w(\mathscr{F}_2)).$$

It is easy to see that (Theorem 3.2 and Theorem 1.7 from [5]) the right part of this inequality is equal to $R\text{-}I(X_1, w(\mathscr{F}_1))+I(X_2, w(\mathscr{F}_2))$. Finally, it is easy to see that Theorem 3.2 completes the proof.

4.6. Corollary. *If X_1 is not realcompact and X_2 is realcompact, then*

$$R\text{-}I(X_1\times X_2, \beta X_1\times\beta X_2)\leqslant R\text{-}\mathrm{Ind}_0 X_1+\mathrm{Ind}_0 X_2 .$$

Furthermore, if $X_1\times X_2$ is z-embedded in $\beta X_1\times\beta X_2$, then

$$R\text{-}\mathrm{Ind}_0(X_1\times X_2)\leqslant R\text{-}\mathrm{Ind}_0 X_1+\mathrm{Ind}_0 X_2 .$$

It should be observed that the corresponding statements (Propositions 4.4, 4.5 and Corollary 4.6) hold also for covering realcompactness degree.

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DEPARTMENT OF MECHANICS AND MATHEMATICS
TBILISI STATE UNIVERSITY
Tbilisi, Georgian SSR

Accepté par la Rédaction le 24. 9. 1978

The cohomological dimension of the ordered set of real numbers equals three

by

Stanisław Balcerzyk (Toruń)

Abstract. The purpose of the paper is to show that the cohomological dimension of the ordered set of real numbers equals three. An appropriate resolution is constructed.

We preserve the terminology and the notation of [1].

Let C be a small K-category where K denotes a commutative ring, then $C^e = C^*\otimes_K C$ is an enveloping category of C and $\mathrm{Hom}_C(= C$ for abbreviation) is a K-functor $C^e \to K\text{-Mod}$. The cohomological dimension $\dim_K C$ is defined as homological (projective) dimension of C in the category of K-functors $K\text{-Mod}^{C^e}$.

Any partially ordered set π may be viewed as a small category with a set of objects π and a unique map $x\to y$ for any $x\leqslant y$ in π. $\dim_K K\pi$ is denoted by $\dim_K \pi$, where $K\pi$ is a K-category generated by π. Let R denote the ordered set of real numbers.

The purpose of the present paper is to show that $\dim_K R = 3$ for any commutative ring K. We construct a particular projective resolution of R. In [1] Mitchell proved that $2\leqslant \dim_K R\leqslant 3$ assuming continuum hypothesis and expected this dimension to be 3; he proved even more, that $\dim_K R\leqslant n+2$ if $|R| = \aleph_n$.

I like to thank Dr. Andrzej Prószyński for correcting a detail of the proof.

1. We denote by $R(x, y)$ for $x, y\in R$ a K-free generator of $\mathrm{Hom}_{KR}(x, y)$ (i.e. a unique map $x\to y$ of R) if $x\leqslant y$ and zero in the opposite case. $\otimes$ means $\otimes_K$. We remind that $R(., a)\otimes R(b, .)$ denotes a K-functor $R^e\to K\text{-Mod}$ which is represented by the object (a, b) of R^e. It associates with an object (x, y) of R^e the free K-module on $R(x, a)\otimes R(b, y)$ if $x\leqslant a$, $b\leqslant y$ and zero in the opposite case. Functors $R(., a)\otimes R(b, .)$ are projective in the category $(K\text{-Mod})^{R^e}$ of K-functors.

We denote by Q the ordered set of 2-rational numbers, i.e., numbers of the form $m/2^n$ for some $n = 0, 1, \ldots$ and some integer m. We define a projective resolution $0\to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} Q\to 0$ of the functor Q in the category $(K\text{-Mod})^{Q^e}$ as follows

$$P_0 = \bigoplus_{a\in Q} Q(., a)\otimes Q(a, .),$$

$$P_1 = P_2 = \bigoplus_{n=0}^{\infty} \bigoplus_{m=-\infty}^{\infty} Q(., m/2^n)\otimes Q((m+1)/2^n, .)$$