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ON THE DIMENSION OF INJECTIVE BANACH SPACES

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ABSTRACT. The purpose of this note is to give an affirmative answer, assuming the generalized continuum hypothesis, to a problem of H. Rosenthal on the cardinality of the dimension on injective Banach spaces.

The problem in question is contained in [**4**, Problem 7.a]; in this connection we prove the following result.

THEOREM 1. *Assume the G.C.H. If X is an infinite dimensional injective Banach space with* $\dim X = \alpha$, *then* $\alpha^\omega = \alpha$.

We start with some preliminaries.

We denote cardinals by α, β; ω denotes the cardinality of natural numbers. We denote by α^ω the cardinality of the family of countable subsets of α. For a cardinal α, we denote by $\mathrm{cf}(\alpha)$ the least cardinal β such that α is the cardinal sum of β many cardinals, each smaller than α. A cardinal α is *regular* if $\alpha = \mathrm{cf}(\alpha)$, and *singular* if $\mathrm{cf}(\alpha) < \alpha$. The least cardinal strictly greater than β is denoted by β^+. The cardinality of a set A is denoted by $|A|$. The generalised continuum hypothesis (G.C.H.) is the statement that $\alpha^+ = 2^\alpha$ for all infinite cardinals α.

A real Banach space X is injective if for every Banach space Y and every bounded linear isomorphism $T\colon X \to Y$, there is a bounded linear projection $P\colon Y \to T(X)$. If Γ is a set, we denote by $l^1(\Gamma)$ the Banach space of real-valued functions on Γ which are absolutely summable. If X is a Banach space we denote with $\dim X$ the least cardinal α such that there is a family $F = \{x_\xi\colon \xi < \alpha\}$ of elements of X with the property that X is the closed linear span of F.

LEMMA 2. *Let X be an injective Banach space with* $\dim X = \alpha$. *Then $l^1(\alpha)$ is isomorphic to a subspace of X^*.*

PROOF. Since X is a complemented subspace of $C(S)$ for some compact space S, X^* is a complemented subspace of $L^1(\lambda)$ for some measure λ. So the conclusion is a direct consequence of Theorem 2.5 of [**3**].

PROOF OF THEOREM 1. Let us assume that the conclusion is false. Then there is an injective Banach space X with $\dim X = \alpha$ and $\alpha^\omega > \alpha$. Under the G.C.H., $\alpha^\omega > \alpha$ means that $\mathrm{cf}(\alpha) = \omega$ and since $l^\infty(\mathbf{N})$ is isomorphic to a subspace of X [**5**] it follows that $\alpha > \mathrm{cf}(\alpha)$.

We choose a sequence $\{\alpha_\eta\colon \eta < \omega\}$ of regular cardinals such that $\alpha_1 = \omega^+$, $\alpha_{\eta+1} > 2^{\alpha_\eta}$ and $\sum_{\eta<\omega} \alpha_\eta = \alpha$.

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From Lemma 2 there is a family $\{e_\xi: \xi < \alpha\}$ of elements of the unit ball of X^* equivalent to the canonical basis for $l^1(\alpha)$.

Let, also, $\{x_\xi: \xi < \alpha\}$ be a norm dense subset of X. Using finite induction we choose a family $\{A_\eta: \eta < \omega\}$ of subsets of α such that:

(i) $A_\eta \subset \{\xi: \alpha_\eta < \xi < \alpha_{\eta+1}\}$,

(ii) $|A_\eta| > 2$, and

(iii) for $\eta < \omega$ and $\xi_1, \xi_2 \in A_\eta$

$$e_{\xi_1}(x_\xi) = e_{\xi_2}(x_\xi) \quad \text{for all } \xi < \alpha_\eta.$$

For every $\eta < \omega$ we choose $\xi_1^\eta \neq \xi_2^\eta$ elements of A_η, and we set $e_\eta = e_{\xi_1^\eta} - e_{\xi_2^\eta}$. Then the sequence $\{e_\eta: \eta < \omega\}$ converges weak* to $0 \in X^*$, and since X is injective, $\{e_\eta: \eta < \omega\}$ is in fact weakly convergent [2]. On the other hand, $\{e_\eta: \eta < \omega\}$ is equivalent to the usual basis for $l^1(\mathbf{N})$, a contradiction.

REMARK 1. As the referee has remarked, the proof shows immediately the following more general statement:

If X is an $\mathcal{L}_\infty$ Grothendieck space, then under the G.C.H. we have $(\dim X)^\omega = \dim X$. (Recall that a Banach space X is a Grothendieck space if every sequence in X^* which is weak* convergent necessarily converges weakly.)

REMARK 2. We do not know what happens without any set-theoretical assumption. In this direction we proved in [1] the following.

THEOREM A. *If X is an injective Banach space in which each weakly compact subset is separable and* $\dim X = \alpha$ *then* $\alpha^\omega = \alpha$.

THEOREM B. *Let α be a cardinal and X be an injective Banach space such that $l^1(\alpha)$ is isomorphic to a subspace of X. Then X contains isomorphically a copy of $l^1(\alpha^\omega)$.*

REFERENCES

1. S. Argyros, *Weak compactness in $L^1(\lambda)$ and injective Banach spaces* (to appear).

2. A. Grothendieck, *Sur les applications lineaires faiblement compactes d' espaces du type $C(K)$*, Canad. J. Math. **5** (1953), 129–173.

3. R. Haydon, *On dual L^1-spaces and injective bidual Banach spaces*, Israel J. Math. **31** (1979), 142–152.

4. H. P. Rosenthal, *On injective Banach spaces $L^\infty(\mu)$ for finite measures μ*, Acta Math. **124** (1970), 205–247.

5. ______, *On relatively disjoint families of measures with some applications to Banach spaces theory*, Studia Math. **37** (1970), 13–36.

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