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ON THE DIMENSION OF THE l_p^n-SUBSPACES OF BANACH SPACES, FOR $1 \leqslant p < 2$

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ABSTRACT. We give an estimate relating the stable type p constant of a Banach space X with the dimension of the l_p^n-subspaces of X. Precisely, let C be this constant and assume $1 < p < 2$. We show that, for each $\varepsilon > 0$, X must contain a subspace $(1+\varepsilon)$-isomorphic to l_p^k, for every k less than $\delta(\varepsilon)C^{p'}$ where $\delta(\varepsilon) > 0$ is a number depending only on p and ε.

Introduction. It is known (cf. [**11, 7**]) that if a Banach space X is not of stable type p, for $1 \leqslant p < 2$, then X must contain almost isometric copies of l_p^n for every integer n. The aim of this paper is to give a quantitative estimate relating the stable type p constant of a finite-dimensional space X with the dimension of the l_p^n-subspaces of X.

Precisely, let $ST_p(X)$ denote the stable type p constant of X. Assume for simplicity that $1 < p < 2$. We show in this paper that, for each $\varepsilon > 0$, there is a number $\delta(\varepsilon) > 0$ depending only on ε and p such that the following holds: Any Banach space X contains a subspace $(1+\varepsilon)$-isomorphic to l_p^k for every k such that

$$(1) \qquad k \leqslant \delta(\varepsilon)\big(ST_p(X)\big)^{p'} \quad \text{where } 1/p + 1/p' = 1.$$

In the particular case $X = l_1^n$, it is easy to see that $ST_p(l_1^n) \sim n^{1/p'}$, so that our result implies that $l_p^{\delta n}$ is $(1+\varepsilon)$-isomorphic to a subspace of l_1^n for some $\delta = \delta(\varepsilon, p) > 0$. This last result was discovered recently by Johnson and Schechtman [**6**], and it strongly motivated the present paper.

Our proof is different from that of [**6**], although it rests on the same basic ingredients (i.e. p-stable random variables and the exponential inequality stated in this paper as Lemma 1.5).

It is worthwhile to note that our result also implies the theorem of Krivine [**7**], but only for $1 < p < 2$; (indeed, if X is isomorphic to l_p, then X is not of stable type p, so that $ST_p(X) = \infty$, and we can take any k in (1)). Moreover, our paper yields a new proof, rather direct, of the main results of [**11**], but only for the section devoted to the "type" of Banach spaces.

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1. Preliminary results. Throughout this paper, we will write simply i.i.d. for "independent and identically distributed". We recall that a real-valued symmetric random variable θ is called p-stable if its Fourier transform is as follows: for some $\sigma \geq 0$, $\mathbf{E} \exp it\,\theta = \exp - \sigma |t|^p$ for all real t. When $\sigma = 1$, we will say that θ is standard. A symmetric Banach space valued random variable S is called p-stable if $\xi(S)$ is p-stable for any $\mathbf{R}$-linear continuous linear form ξ on the Banach space. Throughout this paper, we consider only real Banach spaces, but the complex case can be treated similarly.

DEFINITION 1.1. Let $1 \leq p < 2$. A Banach space X is said to be of stable type p if, for each $r < p$, there is a constant C such that, for any finite sequence $x_1,\ldots,x_n$ in X, we have

$$\left(\mathbf{E}\left\|\sum \theta_i x_i\right\|^r\right)^{1/r} \leq C\left(\sum \|x_i\|^p\right)^{1/p} \tag{1.1}$$

where $\theta_1,\theta_2,\ldots,\theta_n,\ldots$ is an i.i.d. sequence of standard p-stable random variables as above. If $p > 1$, we will denote by $ST_p(X)$ the smallest constant C such that (1.1) holds with $r = 1$. If $p = 1$, we will denote by $ST_1(X)$ the smallest constant C such that (1.1) holds with (say) $r = \frac{1}{2}$. For more details on this notion, cf. **[11]**. We recall only that if the above property (in Definition 1.1) holds for *some* $r < p$, then it also holds for *all* $r < p$. We will use repeatedly the fact that if S_1, $S_2,\ldots,S_k$ are i.i.d., p-stable, Banach space valued random variables, then any linear combination of them, $\sum_1^k \alpha_i S_i$, has the same distribution as the variable $(\sum_1^k |\alpha_i|^p)^{1/p} S_1$. In particular, we have, for each $r < p$,

$$\left(\mathbf{E}\left\|\sum_1^k \alpha_i S_i\right\|^r\right)^{1/r} = \left(\sum |\alpha_i|^p\right)^{1/p}(\mathbf{E}\|S_1\|^r)^{1/r}.$$

(For $p < 2$, $\mathbf{E}\|S_1\|^r$ is finite only if $r < p$.) It will be convenient to record also the following simple observation.

PROPOSITION 1.2. *Let $r = 1$ if $p > 1$ and $r = \frac{1}{2}$ if $p = 1$. The constant $ST_p(X)$ is equal to the smallest constant C such that, for any sequence $x_1,\ldots,x_n$ in X, we have*

$$\left(\mathbf{E}\left\|\sum_1^n \theta_i x_i\right\|^r\right)^{1/r} \leq C n^{1/p} \sup_{i \leq n} \|x_i\|. \tag{1.2}$$

PROOF (SKETCH). It is clear that (1.1) implies (1.2), so that it is enough to prove the converse. Let us assume that (1.2) holds for arbitrary sequences (x_i). We claim that we then have $\forall(\alpha_i) \in \mathbf{R}^n$,

$$\left(\mathbf{E}\left\|\sum_1^n \alpha_i\theta_i x_i\right\|^r\right)^{1/r} \leq C \sup\|x_i\|\left(\sum |\alpha_i|^p\right)^{1/p}. \tag{1.3}$$

Clearly, (1.3) implies (1.1), so that it is enough to prove (1.3). Now if we apply (1.2) to a sequence $(y_i) = (x_1, x_1,\ldots,x_1, x_2,\ldots,x_2, x_3,\ldots)$ where x_i is repeated k_i times, we find, with $N = \sum_{i=1}^n k_i$,

$$\left(\mathbf{E}\left\|\sum_{i=1}^N y_i\theta_i\right\|^r\right)^{1/r} = \left(\mathbf{E}\left\|\sum_{i=1}^n k_i^{1/p}\theta_i x_i\right\|^r\right)^{1/r} \tag{1.4}$$

and (1.2) implies

$$\left(\mathbf{E}\left\|\sum_1^N y_i\theta_i\right\|^r\right)^{1/r} \leqslant CN^{1/p}\sup\|x_i\|.$$

Therefore, we have by (1.4),

$$\left(\mathbf{E}\left\|\sum_1^n \left(k_iN^{-1}\right)^{1/p}\theta_i x_i\right\|^r\right)^{1/r} \leqslant C\sup\|x_i\|,$$

and this last result clearly implies (1.3) by a density argument.To state the next result, we will need more notation. Consider $x_1,\dots,x_n$ in a Banach space X. We will denote by $(Y_j)_{j\geqslant 1}$ an i.i.d. sequence of random variables uniformly distributed on the set $\{\pm x_1,\dots,\pm x_n\}$. In other words, the distribution of each variable Y_j is equal to the probability $\frac{1}{2n}\sum_{i=1}^n\delta_{x_i}+\delta_{-x_i}$. Let $(A_j)_{j\geqslant 1}$ be an i.i.d. sequence of exponential random variables (i.e. $\mathbf{P}(A_j>\lambda)=e^{-\lambda}$ for any $\lambda\geqslant 0$). We will always assume that $(A_j)_{j\geqslant 1}$ is independent of the sequence $(Y_j)_{j\geqslant 1}$. Finally, we set $\Gamma_j=\sum_{k=1}^{k=j}A_k$. It is well known (cf. [3, p. 10]) that

$$\mathbf{P}\left(\Gamma_j<\lambda\right)=\int_0^\lambda \frac{t^{j-1}}{(j-1)!}e^{-t}\,dt,$$

for all $\lambda>0$. We can now state the representation which we will use.

PROPOSITION 1.3. *There is a number* $C_p>0$ *depending only on* p *such that* $(1/n^{1/p})\sum_{i=1}^n\theta_i x_i$ *has the same distribution as* $C_p\sum_{j=1}^\infty(\Gamma_j)^{-1/p}Y_j$.

This result follows from [9] (for more details, see [10]). More generally, it is known (cf. [9]) that if $(Y_j)_{j\geqslant 1}$ is an i.i.d. sequence of symmetric X-valued random variables, then the variable $S=\sum_{j=1}^\infty\Gamma_j^{-1/p}Y_j$ is p-stable, and we have

$$\forall\xi\in X^*\ \mathbf{E}\exp i\langle\xi,S\rangle=\exp\left(-\mathbf{E}\,|\langle\xi,Y_1\rangle|^p/\left(C_p\right)^p\right).$$

For more information we refer the reader to [10].

For the proof of the main result, we will use the fact that, on the average, S behaves very much like the series $\sum_{j\geqslant 1}j^{-1/p}Y_j$, which is obtained from S by replacing Γ_j by j. The next lemma, which is entirely elementary, will allow us to do this substitution:

LEMMA 1.4. *For any* p *such that* $1<p<2$, *we have*

$$\Phi=\sum_{j\geqslant 1}\mathbf{E}\,|\Gamma_j^{-1/p}-j^{-1/p}|<\infty.$$

Moreover,

$$\sum_{j\geqslant 2}\mathbf{E}\,|\Gamma_j^{-1}-j^{-1}|<\infty. \tag{1.5}$$

PROOF. Recall that

$$\forall x>0\quad \mathbf{P}\{\Gamma_j<x\}=\int_0^x\frac{u^{j-1}}{(j-1)!}e^{-u}\,du;$$

therefore

$$\Phi = \int \sum_{j\geq 1} |u^{-1/p} - j^{-1/p}| \frac{u^{j-1}}{(j-1)!} e^{-u}\, du.$$

Elementary computations using Stirling's formula show that this integral converges. I am grateful to B. Maurey for showing me Lemma 1.4, which is an improvement of a previous version.

We will also need the following lemma, which can be proved by an argument similar (but simpler) to the one used in [**6**].

LEMMA 1.5. *Let* $1 < p < 2$ *and let* p' *be the conjugate of* p. *Let* $(Z_j)_{j\geq 1}$ *be a sequence of independent Banach space valued random variables which are uniformly bounded.*

Let $\lambda_j = \operatorname{ess\,sup}\|Z_j(\cdot)\|$; *we denote by* $(\lambda_j^*)_{j\geq 1}$ *the nonincreasing rearrangement of* $(\lambda_j)_{j\geq 1}$.

If $\|\{\lambda_j\}\|_{p\infty} = \sup_{j\geq 1} j^{1/p}\lambda_j^*$ *is finite, and if* $Z = \sum_{j=1}^{\infty} Z_j$ *converges a.s., then we have, for all* $c > 0$,

$$\mathbf{P}\{|\,\|Z\| - \mathbf{E}\|Z\|\,| > c\} \leq K \exp - \eta\left(\frac{c}{\|\{\lambda_j\}\|_{p\infty}}\right)^{p'}, \tag{1.6}$$

where K *and* $\eta > 0$ *are constants depending only on* p. (*Note that, by the result of Hoffmann-Jørgensen* [**5**], $\mathbf{E}\|Z\|$ *is necessarily finite.*)

PROOF (SKETCH). By some elementary arguments (see [**6**] for details) it is possible to prove that if $(d_j)_{j\geq 1}$ is a *scalar* martingale difference sequence such that $|d_j| \leq 2\lambda_j$ a.s. for all j, and if $\sup j^{1/p}\lambda_j^* < \infty$, then we have

$$\forall c > 0 \quad \mathbf{P}\left(\left|\sum_1^{\infty} d_j\right| > c\right) \leq K \exp - \eta\left(\frac{c}{\|\{\lambda_j\}\|_{p\infty}}\right)^{p'}. \tag{1.7}$$

This result immediately implies (1.6): indeed if we denote by $\mathcal{F}_j$ the σ-algebra generated by $\{Z_1, \ldots, Z_j\}$ then we have

$$|\mathbf{E}^{\mathcal{F}_j}\|Z\| - \mathbf{E}^{\mathcal{F}_{j-1}}\|Z\|| \leq 2 \operatorname{ess\,sup}\|Z_j\| \leq 2\lambda_j,$$

so that we may apply (1.7) to the sequence

$$d_j = \mathbf{E}^{\mathcal{F}_j}\|Z\| - \mathbf{E}^{\mathcal{F}_{j-1}}\|Z\|.$$

Since $\|Z\| - \mathbf{E}\|Z\| = \sum_{j=1}^{\infty} d_j$, this yields (1.6). In the case $p = 1$, the preceding result becomes

LEMMA 1.6. *Let* Z_j, λ_j *be as in Lemma* 1.5. *Assume that* $\|\{\lambda_j\}\|_{1\infty} = \sup j\lambda_j^* < \infty$. *Then, if* $Z = \sum Z_j$ *converges a.s., we have*

$$\forall c > 0 \quad \mathbf{P}\{|\,\|Z\| - \mathbf{E}\|Z\|\,| > c\} \leq K \exp - \left\{\exp \eta \frac{c}{\|\{\lambda_j\}\|_{1\infty}}\right\}$$

where K *and* $\eta > 0$ *are absolute constants.*

PROOF. The argument is the same as for Lemma 1.5 except that we use instead the following estimate:

$$\forall c > 0 \quad \mathbf{P}\left\{\left|\sum_1^\infty d_j\right| > c\right\} \leqslant K \exp - \left\{\exp \eta \frac{c}{\|\{\lambda_j\}\|_{1\infty}}\right\},$$

which can be proved by an argument similar to the one included in **[6]**.

REMARK 1.7. The preceding inequalities estimating the "rate of deviation" of $\|Z\|$ from its mean were first used in the vector valued case by Yurinski **[13]**. Further applications appear in **[2]**. Note that by orthogonality, we have

$$\mathbf{E}\,|\,\|Z\| - \mathbf{E}\|Z\|\,|^2 = \mathbf{E}\left|\sum d_j\right|^2 \leqslant 4\sum_1^\infty \lambda_j^2 \quad (\text{cf. } [\mathbf{2}]). \tag{1.8}$$

REMARK 1.8. Let $(\lambda_j)_{j\geqslant 1}$ be a sequence of scalars. We denote by λ_j^* the nonincreasing rearrangement of $(|\lambda_j|)_{j\geqslant 1}$. The space of all sequences $(\lambda_j)_{j\geqslant 1}$, such that $\sup_{j\geqslant 1} j^{1/p}\lambda_j^* < \infty$, is usually referred to as weak l_p and is denoted by $l_{p\infty}$. It is easy to check that

$$\|(\lambda_j)\|_{p\infty} = \sup_{j\geqslant 1} j^{1/p}\lambda_j^* = \left(\sup_{t>0} t^p \operatorname{card}\{j\,|\,|\lambda_j| > t\}\right)^{1/p}. \tag{1.9}$$

Now, let $E_1, E_2, \ldots$ be a sequence of subsets of $\mathbf{N}$ forming a partition of $\mathbf{N}$. If we set $\alpha_i = \|(\lambda_j)_{j\in E_i}\|_{p\infty}$, then from (1.9) it is easy to deduce

$$\|(\lambda_j)_{j\in\mathbf{N}}\|_{p\infty} \leqslant \left(\sum \alpha_i^p\right)^{1/p}. \tag{1.10}$$

2. The main result. The main result of this paper is

THEOREM 2.1. *Assume that* $1 \leqslant p < 2$, *and* $1/p + 1/p' = 1$. *For each* $\varepsilon > 0$, *there is a number* $\delta_p(\varepsilon) > 0$ *with the following property: Any Banach space X contains a subspace* $(1+\varepsilon)$*-isomorphic to* l_p^k *as long as*

$$k < \delta_p(\varepsilon) ST_p(X)^{p'} \quad \textit{if } 1 < p < 2, \tag{2.1}$$

$$\operatorname{Log} k < \delta_1(\varepsilon) ST_1(X) \quad \textit{if } p = 1. \tag{2.2}$$

PROOF OF THEOREM 2.1. We first consider the case $1 < p < 2$. By Proposition 1.2, we can find a finite sequence $x_1, \ldots, x_n$ in X such that

$$\operatorname{Sup}\|x_i\| \leqslant 1 \quad \text{and} \quad n^{-1/p}\mathbf{E}\left\|\sum_1^n \theta_i x_i\right\| \geqslant \frac{1}{2} ST_p(X).$$

By Proposition 1.3, we have (with (Y_j) as defined in Proposition 1.3)

$$\mathbf{E}\left\|\sum_{j\geqslant 1} \Gamma_j^{-1/p} Y_j\right\| \geqslant (2C_p)^{-1} ST_p(X). \tag{2.3}$$

Now let $(Y_{ji})_{j\geqslant 1}$ and $(\Gamma_{ji})_{j\geqslant 1}$ be i.i.d. copies of the sequences $(Y_j)_{j\geqslant 1}$ and $(\Gamma_j)_{j\geqslant 1}$ for $i = 1, 2, \ldots$ in such a way that if we set

$$S_i = \sum_{j\geqslant 1} (\Gamma_{ji})^{-1/p} Y_{ji}$$

then $(S_1, S_2, \dots, S_k)$ is a sequence of i.i.d. copies of

$$S = \sum_{j \geq 1} \Gamma_j^{-1/p} Y_j.$$

Since (by Proposition 1.3) S is p-stable, we have

$$\mathbf{E}\left\| \sum_1^k \alpha_i S_i \right\| = \left(\sum_1^k |\alpha_i|^p \right)^{1/p} \mathbf{E}\|S\|. \tag{2.4}$$

We will now compare $\sum_{i=1}^k \alpha_i S_i$ with $\sum_{i=1}^k \alpha_i \tilde{S}_i$ where we have set $\tilde{S}_i = \sum_{j \geq 1} j^{-1/p} Y_{ji}$. We may write clearly

$$\left| \mathbf{E}\left\| \sum_1^k \alpha_i S_i \right\| - \mathbf{E}\left\| \sum_1^k \alpha_i \tilde{S}_i \right\| \right| \leq \mathbf{E}\left\| \sum_1^k \alpha_i (S_i - \tilde{S}_i) \right\| \leq \sum_1^k |\alpha_i| \mathbf{E}\|S_i - \tilde{S}_i\|.$$

Hence by Lemma 1.4:

$$\leq \sum_1^k |\alpha_i| \Phi \leq \left(\sum_1^k |\alpha_i|^p \right)^{1/p} k^{1/p'} \Phi,$$

and by (2.1)

$$\leq \left(\delta_p(\varepsilon)^{1/p'} S T_p(X) \Phi \right) \left(\sum_1^k |\alpha_i|^p \right)^{1/p}.$$

It follows that if $\delta_p(\varepsilon)$ is chosen small enough, precisely if

$$k \leq \left(\delta \Phi^{-1} S T_p(X) \right)^{p'}, \tag{2.5}$$

then we have

$$\left| \mathbf{E}\left\| \sum_1^k \alpha_i S_i \right\| - \mathbf{E}\left\| \sum_1^k \alpha_i \tilde{S}_i \right\| \right| \leq \delta S T_p(X) \left(\sum_1^k |\alpha_i|^p \right)^{1/p}. \tag{2.6}$$

We now analyse the behaviour of $\|\sum_1^k \alpha_i \tilde{S}_i\|$ using Lemma 1.5. We first observe that

$$\|\alpha_i Y_{ji} j^{-1/p}\| \leq |\alpha_i| j^{-1/p}.$$

and using Remark 1.8,

$$\|\{|\alpha_i| j^{-1/p}\}_{i,j}\|_{p,\infty} \leq \left(\sum |\alpha_i|^p \right)^{1/p}.$$

Therefore, applying Lemma 1.5 we obtain

(2.7)

$$\forall c > 0 \quad \mathbf{P}\left\{ \left| \left\| \sum_1^k \alpha_i \tilde{S}_i \right\| - \mathbf{E}\left\| \sum_1^k \alpha_i \tilde{S}_i \right\| \right| > c \right\} \leq K \exp - \eta \left[c \Big/ \left(\sum_1^k |\alpha_i|^p \right)^{1/p} \right]^{p'}.$$

Now, let us fix (α_i) such that $\sum_1^k |\alpha_i|^p = 1$. Combining (2.6) and (2.7), we obtain

$$\mathbf{P}\left\{ \left| \left\| \sum_1^k \alpha_i \tilde{S}_i \right\| - \mathbf{E}\left\| \sum_1^k \alpha_i S_i \right\| \right| > c + \delta S T_p(X) \right\} \leq K \exp - \eta\, c^{p'}.$$

Therefore taking $c = \delta\mathbf{E}\|S\|$, we find by (2.3) and (2.4)

$$\text{(2.8)} \qquad \mathbf{P}\left\{\left|\left\|\sum_1^k \alpha_i \tilde{S}_i\right\| - \mathbf{E}\|S\|\right| > \delta K_1 \mathbf{E}\|S\|\right\} \leqslant K \exp - \eta(\delta\mathbf{E}\|S\|)^{p'}$$

with $K_1 = 1 + 2C_p$.

Of course, we may assume that $\delta K_1 < 1$, so that the last result tells us that, with "large" probability, $\|\sum_{i=1}^k \alpha_i \tilde{S}_i\|$ remains "close" to $\mathbf{E}\|S\|$. We can now conclude. By a well-known argument (cf., e.g., [**4**, Lemma 2.4]), there is a δ-net in the unit sphere of l_p^k of cardinality at most $(1 + 2/\delta)^k$, which is less than $\exp(2k/\delta)$. Therefore we deduce from (2.8) that, with probability greater than

$$1 - K\exp(2k/\delta)\exp - \eta(\delta\mathbf{E}\|S\|)^{p'},$$

we have, for each (α_i) in this δ-net,

$$\text{(2.9)} \qquad (1 - K_1\delta)\mathbf{E}\|S\| \leqslant \left\|\sum_1^k \alpha_i \tilde{S}_i(\omega)\right\| \leqslant \mathbf{E}\|S\|(1 + K_1\delta)$$

with K_1 as before.

Hence, if $2k/\delta \ll \eta(\delta\mathbf{E}\|S\|)^{p'}$—which is true, by (2.3), if

$$\text{(2.10)} \qquad k \leqslant \chi(\delta)\big(ST_p(X)\big)^{p'}$$

for some suitable $\chi(\delta)$—then the event considered in (2.9) has positive probability. Consequently, we can find an ω in our probability space such that $\tilde{S}_1(\omega),\ldots,\tilde{S}_k(\omega)$ verify (2.9) for all (α_1) in the δ-net. By another well-known argument (cf., e.g., [**4**, Lemma 2.5]), we may replace the δ-net by the whole sphere of l_p^k without spoiling too much the estimate (2.9). Precisely, there is a constant $\Delta(\delta)$, with $\Delta(\delta) \to 0$ when $\delta \to 0$, such that (if (2.5) and (2.10) hold) we can deduce from (2.9)

$$\mathbf{E}\|S\|(1 - K_1\delta)(1 - \Delta(\delta)) \leqslant \left\|\sum_1^k \alpha_i \tilde{S}_i(\omega)\right\| \leqslant \mathbf{E}\|S\|(1 + K_1\delta)(1 + \Delta(\delta))$$

for all (α_i) in the sphere of l_p^k. This means that the span of $\{\tilde{S}_1(\omega),\ldots,\tilde{S}_k(\omega)\}$ is $\Phi(\delta)$-isomorphic to l_p^k, with $\Phi(\delta) \to 1$ when $\delta \to 0$, and this concludes the proof of Theorem 2.1 in the case $1 < p < 2$.

Proof of Theorem 2.1, in the case $p = 1$. The basic idea is the same. By Proposition 1.2 we can find $x_1,\ldots,x_n$ in the unit ball of X such that

$$\frac{1}{n}\left(\mathbf{E}\left\|\sum_1^n \theta_i x_i\right\|^r\right)^{1/r} \geqslant \frac{1}{2} ST_1(X)$$

(recall that $r = \frac{1}{2}$). Let m be an integer which will be specified later. With the same notation as before, we deduce from Lemma 1.6 that if

$$\Phi_i = \sum_{j \geqslant m} j^{-1} Y_{ji}$$

we have, for all $c > 0$,

$$\mathbf{P}\left\{\left|\left\|\sum_1^k \alpha_i \Phi_i\right\| - \mathbf{E}\left\|\sum_1^k \alpha_i \Phi_i\right\|\right| > c\right\} \leqslant K \exp\left\{-\exp\frac{\eta c}{\sum|\alpha_i|}\right\} \tag{2.11}$$

for each (α_i) in $\mathbf{R}^k$. In the sequel, we will always assume that $\operatorname{Log} k \leqslant \chi(\delta) ST_1(X)$ for some number $\chi(\delta) > 0$ which will be specified later. We will establish below the following

CLAIM. The numbers $\chi(\delta) > 0$ and $m = m(\delta)$ can be chosen (depending only on δ) so that we have

$$A(1-\varphi(\delta))\sum|\alpha_i| \leqslant \mathbf{E}\left\|\sum_1^k \alpha_i \Phi_i\right\| \leqslant A(1+\varphi(\delta))\sum|\alpha_i| \tag{2.12}$$

for all (α_i) in $\mathbf{R}^k$, where A is a number such that

$$A \geqslant (1/2C_1) ST_1(X), \tag{2.13}$$

and where $\varphi(\delta) \to 0$ if $\delta \to 0$.

From this claim, it is easy to complete the proof of Theorem 2.1 by showing that, for some ω, the vectors $\Phi_1(\omega), \ldots, \Phi_k(\omega)$ span a subspace $(1+\varphi'(\delta))$-isomorphic to l_1^k, with $\varphi'(\delta) \to 0$ if $\delta \to 0$. We find for k the values indicated in Theorem 2.1; the proof of this part is the same as in the case $1 < p < 2$.

To complete this proof, we now prove the above claim. We define $\psi_i = \sum_{j \geqslant m} (\Gamma_{ji})^{-1} Y_{ji}$. By (1.5), if m is chosen large enough, say $m \geqslant m(\delta) > 1$, then we have

$$\left|\mathbf{E}\left\|\sum_1^k \alpha_i \Phi_i\right\| - \mathbf{E}\left\|\sum_1^k \alpha_i \psi_i\right\|\right| \leqslant \delta \sum|\alpha_i|.$$

Therefore, it remains only to show that we can obtain (2.12) and (2.13) with (ψ_i) in the place of (Φ_i). Let $\psi = \|\sum_1^k \alpha_i \psi_i\|$. Applying (1.8) (first for "fixed" Γ_{ji} and then integrating over Γ_{ji}) we obtain

$$\mathbf{E}(\psi - \mathbf{E}\psi)^2 \leqslant \sum_{i=1}^k \sum_{j \geqslant m} 4|\alpha_i|^2 \mathbf{E}\Gamma_{ji}^{-2} \leqslant \gamma_m^2 \left(\sum|\alpha_i|\right)^2,$$

where $\gamma_m = \{4\sum_{j \geqslant m} \mathbf{E}\Gamma_j^{-2}\}^{1/2}$ tends to zero when m tends to infinity. If $\sum_1^k |\alpha_i| = 1$, we have a fortiori $\|\psi - \mathbf{E}\psi\|_r \leqslant \gamma_m$ so that

$$\mathbf{E}\psi^r - \gamma_m^r \leqslant (\mathbf{E}\psi)^r \leqslant \mathbf{E}\psi^r + \gamma_m^r. \tag{2.14}$$

But, on the other hand, we know that $\chi_i = \sum_{j \geqslant 1} \Gamma_{ji}^{-1} Y_{ji}$ is 1-stable so that

$$\left(\mathbf{E}\left\|\sum_1^k \alpha_i \chi_i\right\|^r\right)^{1/r} = A \tag{2.15}$$

with $A = (\mathbf{E}\|\chi_1\|^r)^{1/r}$. Note that A verifies (2.13). Let $\mu_i = \sum_{j<m} \Gamma_{ji}^{-1} Y_{ji}$. We have

$$\left|\mathbf{E}\psi^r - \mathbf{E}\left\|\sum \alpha_i \chi_i\right\|^r\right|^{1/r} \leqslant \left(\mathbf{E}\left\|\sum \alpha_i \mu_i\right\|^r\right)^{1/r} \leqslant \left\|\sum_i |\alpha_i| \sum_{j<m} \frac{1}{\Gamma_{ji}}\right\|_r. \tag{2.16}$$

Let us assume that $m = m(\delta)$ is chosen so that $\gamma_m \leqslant \delta$. Then, by known estimates on $\mathbf{P}(\Gamma_j > x)$, we easily find a constant $B(\delta)$ such that

$$\mathbf{P}\Big(\sum_{j \leqslant m} \Gamma_j^{-1} > c\Big) \leqslant \frac{B(\delta)}{c}$$

for all $c > 0$. It is then easy to check that (for $k > 1$)

$$\left\| \sum_{i=1}^{k} |\alpha_i| \sum_{j \leqslant m} \frac{1}{\Gamma_{ji}} \right\|_r \leqslant \sum_{1}^{k} |\alpha_i| (\operatorname{Log} k) \cdot B'(\delta) \tag{2.17}$$

for some constant $B'(\delta)$. Combining (2.16), (2.17) and (2.15), we obtain finally that if $\operatorname{Log} k \leqslant \chi(\delta) ST_1(X)$ for some suitable $\chi(\delta)$, and if $m = m(\delta)$, we have

$$|\mathbf{E}\psi^r - A^r| \leqslant \varphi''(\delta)$$

with $\varphi''(\delta) \to 0$ when $\delta \to 0$. Taking (2.14) into account, this gives finally (as announced) (2.12) with the functions ψ_i instead of Φ_i and this concludes the proof.

REMARK. Actually, Proposition 1.2 is not really needed to prove Theorem 2.1. Indeed, let $x_1, \ldots, x_n$ be such that $\Sigma \|x_i\|^p = 1$ and $\mathbf{E}\|\Sigma_1^n \theta_i x_i\| \geqslant \frac{1}{2} ST_p(X)$. Let $y_i = x_i \|x_i\|^{-1}$. Let Y be a symmetric X-valued random variable with distribution equal to $\Sigma_{i=1}^n \|x_i\|^p (\delta_{y_i} + \delta_{-y_i})/2$, and let $Y_1, Y_2, \ldots$ be i.i.d. copies of Y. By the remarks following Proposition 1.3, we know that $\Sigma_{i=1}^n \theta_i x_i$ has the same distribution as $C_p \Sigma_{j \geqslant 1} \Gamma_j^{-1/p} Y_j$. Using this fact and the observation that $\|Y_j\| \leqslant 1$, we can prove Theorem 2.1 without referring to Proposition 1.2.

3. Applications. The most interesting case in Theorem 2.1 is probably the following, which was recently discovered by Johnson and Schechtman [**6**].

COROLLARY 3.1. *Let* $1 < p < 2$. *For each* $\varepsilon > 0$, *there is a number* $\eta_p(\varepsilon) > 0$ *such that* l_1^n *contains, for each* n, *a subspace* $(1 + \varepsilon)$-*isomorphic to* l_p^k *with* $k \geqslant \eta_p(\varepsilon) n$.

PROOF. Since $\mathbf{E}\Sigma_{i=1}^n |\theta_i| = n\mathbf{E}|\theta_1|$, it is obvious that $ST_p(l_1^n) \sim n^{1/p'}$ when $n \to \infty$. Therefore Corollary 3.1 follows from Theorem 2.1.

We can also derive from Theorem 2.1 a new proof of Krivine's theorem [**7**] (cf. also [**12, 8**]), but only for $1 < p < 2$: any space X which is isomorphic to l_p contains, for each n and each $\varepsilon > 0$, a subspace $(1 + \varepsilon)$-isomorphic to l_p^n. (In that case, we will say that X contains l_p^n's uniformly.) Indeed, this follows from Theorem 2.1, since $ST_p(X) = \infty$. More generally, we easily derive from Theorem 2.1 a new proof of the results of [**11**] on the type of Banach spaces.

COROLLARY 3.2. (a) *Let* $1 \leqslant p < 2$. *A Banach space* X *is of stable type* p *iff it does not contain* l_p^n*'s uniformly.*

(b) *If* X *is of stable type* p $(1 \leqslant p < 2)$, *then there is a* $q > p$ *such that* X *is of stable type* q.

(c) *Define* $p(X) = \operatorname{Sup}\{p \mid X$ *is of stable type* $p\}$; *then, for each integer* k *and each* $\varepsilon > 0$, X *contains a subspace* $(1 + \varepsilon)$-*isomorphic to* l_p^k.

PROOF. (a) The nontrivial part is to show that $ST_p(X) = \infty$ implies that X contains l_p^n's uniformly. This follows from Theorem 2.1.

(b) Suppose that there does not exist $q > p$ such that X is of stable type q. Then by (a) the space X must contain, for each n and each $\varepsilon > 0$, a subspace $(1+\varepsilon)$-isomorphic to l_q^n; a fortiori, this subspace is λ-isomorphic to l_p^n with $\lambda \leqslant (1+\varepsilon)n^{1/p-1/q}$. Letting $q \downarrow p$, we find that for each n and each $\delta > 0$, X contains a subspace $(1+\delta)$-isomorphic to l_p^n, so that X cannot be of stable type p. Finally, (c) follows from (a) if $p(X) < 2$. In the case $p(X) = 2$. (c) follows Dvoretzky's theorem (cf. [**4**]). This concludes the proof.

REMARK. Since our proof is more direct and constructive than that of [**11**], we can obtain some more precise estimates. For instance, in (b) of Corollary 3.2, we can obtain an estimate of $q > p$ in terms of $ST_p(X)$.

REMARK. It is worthwhile to recall that the notion of stable type p is closely related to the more usual notion of Rademacher type p (often referred to simply as "type p"). Indeed, it is known and rather easy to prove (cf. [**11**]) that for $1 \leqslant p < q \leqslant 2$, a Banach space X is of stable type p if it is of Rademacher type q; and, in the converse direction, X is of Rademacher type q if it is of stable type q. Therefore, in the definition of the index $p(X)$, we may use indifferently either one of these two notions of type. We can also obtain a quantitative version of Krivine's theorem as follows.

COROLLARY 3.3. *Let* $1 \leqslant p < 2$. *For each* $\varepsilon > 0$ *there is a constant* $\Delta_p(\varepsilon) > 0$ *such that any n-dimensional space which is C-isomorphic to* l_p^n *contains a subspace* $(1+\varepsilon)$-*isomorphic to* l_p^k *with*

$$k \geqslant \Delta_p(\varepsilon) C^{-p'} (\operatorname{Log} n)^{p'/p} \quad \text{if } p > 1$$

and

$$\operatorname{Log} k \geqslant \Delta_1(\varepsilon) C^{-1} \operatorname{Log} n \quad \text{if } p = 1.$$

For better estimates see [**1**].

PROOF. This follows from Theorem 2.1 and the easy observation that $ST_p(l_p^n) \sim (\operatorname{Log} n)^{1/p}$ when $n \to \infty$. (For more details see, e.g., [**11**, p. 80]).

REMARK. When $p = 1$ this result is already known and can be obtained by a rather standard "blocking" argument (cf., e.g., [**11**, Proposition 0.1]). When $p = 2$ the quantitative version of Dvoretzky's theorem proved in [**14** and **4**] yields an integer k proportional to n. These observations and the results of [**1**] show that the preceding estimate is not the "right" one, especially when p is close to 2, but we do not see how to improve it.

REMARK. Using similar ideas, one can obtain some estimates valid for any p but with some much more restrictive assumptions on the Banach space. For instance, let X be a Banach space with a 1-unconditional and 1-symmetric basis $(e_j)_{j \in \mathbf{N}}$. Let $1 \leqslant p < \infty$. Assume that $\sum_{j=1}^{\infty} j^{-1/p} e_j$ converges and let $M = \|\sum_{j \geqslant 1} j^{-1/p} e_j\|$. Then X contains, for each $\varepsilon > 0$, a subspace $(1+\varepsilon)$-isomorphic to l_p^k for every k such that $k \leqslant \lambda_p(\varepsilon) M^{p'}$ where $\lambda_p(\varepsilon) > 0$ is a constant depending only on p and ε. We only sketch briefly the argument: let $f\colon \mathbf{N} \times \mathbf{N} \to \mathbf{N}$ be a bijection; we consider

$$Z_i = \sum_{j \geqslant 1} (\Gamma_{ji})^{-1/p} e_{f(i,j)}.$$

It is possible to check that, for each (α_i) in $\mathbf{R}^k$, the distribution of $\|\sum_{i=1}^k \alpha_i Z_i\|$ is the same as that of $(\sum |\alpha_i|^p)^{1/p} \|Z_1\|$. Indeed, assume $\sum |\alpha_i|^p = 1$. Let us denote by $1/\gamma_j^*$ the decreasing rearrangement of the collection $\{|\alpha_i|^p / \Gamma_{ji} \mid i \leqslant k,\ j \in \mathbf{N}\}$. The distribution of $(1/\gamma_j^*)_{j \geqslant 1}$ is the same as that of $(1/\Gamma_j)_{j \geqslant 1}$ (cf., e.g., [**9**]). Hence, $\|\sum (\gamma_j^*)^{-1/p} e_j\|$, which is equal to $\|\sum \alpha_i Z_i\|$ (since (e_i) is a symmetric basis), has the same distribution as $\|Z_1\|$. Therefore, the above statement follows easily using Lemma 1.4 and the variables obtained from Z_i by replacing $\Gamma_{ji}^{-1/p}$ by $j^{-1/p}$.

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