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ON THE DIMENSION OF THE SOLUTION SET TO THE HOMOGENEOUS LINEAR FUNCTIONAL DIFFERENTIAL EQUATION OF THE FIRST ORDER

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Abstract. Consider the homogeneous equation

$$u'(t) = \ell(u)(t) \qquad \text{for a.e. } t \in [a,b]$$

where $\ell\colon C([a,b];\mathbb{R}) \to L([a,b];\mathbb{R})$ is a linear bounded operator. The efficient conditions guaranteeing that the solution set to the equation considered is one-dimensional, generated by a positive monotone function, are established. The results obtained are applied to get new efficient conditions sufficient for the solvability of a class of boundary value problems for first order linear functional differential equations.

Keywords: functional differential equation, boundary value problem, differential inequality, solution set

MSC 2010: 34K06, 34K10

1. Introduction

In many applications, it is of great importance to know whether a linear boundary value problem has a unique solution. Thus a lot of papers recently published are devoted to the study of the unique solvability of the general boundary value problem

$$u'(t) = \ell(u)(t) + q(t) \quad \text{for a.e. } t \in [a,b], \tag{1.1}$$

$$h(u) = c \tag{1.2}$$

where $\ell\colon C([a,b];\mathbb{R}) \to L([a,b];\mathbb{R})$ and $h\colon C([a,b];\mathbb{R}) \to \mathbb{R}$ are linear bounded operators, $q \in L([a,b];\mathbb{R})$, and $c \in \mathbb{R}$ (see, e.g., [1]–[14] and the references therein).

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A well-known result, the so-called Fredholm Theorem, describes the relation between the unique solvability of the inhomogeneous and the corresponding homogeneous linear boundary value problems. To be more precise, the following theorem is well-known from the general theory of boundary value problems for functional differential equations:

Theorem 1.1. *The problem* (1.1), (1.2) *is uniquely solvable if and only if the corresponding homogeneous problem*

$$u'(t) = \ell(u)(t), \tag{1.1$_0$}$$

$$h(u) = 0 \tag{1.2$_0$}$$

has only the trivial solution.

Remark 1.1. If the problem (1.1$_0$), (1.2$_0$) has only the trivial solution then the solution to the problem (1.1), (1.2) can be expressed in the form

$$u(t) = cu_0(t) + \int_a^b G(t,s)q(s)\,\mathrm{d}s \quad \text{for } t \in [a,b], \tag{1.3}$$

where $G\colon [a,b] \times [a,b] \to \mathbb{R}$ is the so-called Green's function of the problem (1.1$_0$), (1.2$_0$) and u_0 is a nontrivial solution to the homogeneous equation (1.1$_0$) satisfying $h(u_0) = 1$.

It is also known that the dimension of the solution set to the homogeneous equation (1.1$_0$) plays an important role in the theory. Although we are dealing with a first order linear ordinary differential equation, the dimension of the solution set U to the equation (1.1$_0$) can be any natural number. More precisely, it is known that $\dim U \geqslant 1$ (see Section 4 in [6]), and if $\dim U \geqslant 2$ then for every linear bounded operator $h\colon C([a,b];\mathbb{R}) \to \mathbb{R}$, the problem (1.1$_0$), (1.2$_0$) has a nontrivial solution (see [6, Remark 4.7]). Therefore, it is of great importance to find conditions guaranteeing the relation $\dim U = 1$, and, in general, to study the structure of the solution set U.

1.1. Basic notation and definitions.

The following notation is used throughout the paper:

$\mathbb{N}$ is the set of all natural numbers.

$\mathbb{R}$ is the set of all real numbers; $\mathbb{R}_+ = [0, +\infty[$.

$C([a,b];\mathbb{R})$ is the Banach space of continuous functions $u\colon [a,b] \to \mathbb{R}$ with the norm

$$\|u\|_C = \max\{|u(t)|\colon t \in [a,b]\}.$$

$L([a,b];\mathbb{R})$ is the Banach space of Lebesgue-integrable functions $p\colon [a,b]\to\mathbb{R}$ with the norm

$$\|u\|_L=\int_a^b |p(t)|\,\mathrm{d}t.$$

$AC([a,b];\mathbb{R})$ is the set of absolutely continuous functions $u\colon [a,b]\to\mathbb{R}$.
$C([a,b];\mathbb{R}_+)=\{u\in C([a,b];\mathbb{R})\colon\ u(t)\geqslant 0 \quad \text{for } t\in[a,b]\}$.
$L([a,b];\mathbb{R}_+)=\{p\in L([a,b];\mathbb{R})\colon\ p(t)\geqslant 0 \quad \text{for a.e. } t\in[a,b]\}$.
$\mathcal{L}_{ab}$ is the set of all linear bounded operators $\ell\colon C([a,b];\mathbb{R})\to L([a,b];\mathbb{R})$.
$\mathcal{P}_{ab}$ is the set of all positive operators $\ell\in\mathcal{L}_{ab}$, i.e., such operators that transform the set $C([a,b];\mathbb{R}_+)$ into the set $L([a,b];\mathbb{R}_+)$.
For every $x\in\mathbb{R}$, $[x]_-=\max\{-x,0\}$.
An operator $\ell\in\mathcal{L}_{ab}$ is called an a-Volterra operator (or a b-Volterra operator, respectively) if for arbitrary $c\in\,]a,b]$ (or $c\in[a,b[$) and $v\in C([a,b];\mathbb{R})$ such that

$$v(t)=0 \quad \text{for } t\in[a,c] \qquad (\text{or } v(t)=0 \quad \text{for } \ t\in[c,b]),$$

the equality

$$\ell(v)(t)=0 \quad \text{for a.e. } t\in[a,c] \qquad (\text{or } \ell(v)(t)=0 \quad \text{for a.e. } t\in[c,b])$$

is fulfilled.

By a solution to the equation (1.1) or (1.1_0), we understand a function $u\in AC([a,b];\mathbb{R})$ satisfying (1.1) or (1.1_0), respectively, almost everywhere on $[a,b]$. By a solution to the problem (1.1), (1.2) or (1.1_0), (1.2_0), we understand a solution u to (1.1) or (1.1_0), satisfying (1.2) or (1.2_0), respectively.

Notation 1.1. Throughout the paper, by U we denote the set of all solutions u to the equation (1.1_0). Obviously, U is a linear vector space.

To formulate the main results it is convenient to introduce the following definitions:

Definition 1.1. An operator $\ell\in\mathcal{L}_{ab}$ is said to belong to the set $\mathcal{S}_{ab}(a)$ if every function $u\in AC([a,b];\mathbb{R})$ satisfying

$$u'(t)\geqslant \ell(u)(t) \quad \text{for a.e. } t\in[a,b], \tag{1.4}$$

$$u(a)\geqslant 0 \tag{1.5}$$

satisfies the inequality

$$u(t)\geqslant 0 \quad \text{for } \ t\in[a,b]. \tag{1.6}$$

Remark 1.2. The inclusion $\ell\in\mathcal{S}_{ab}(a)$ implies that the problem (1.1_0), (1.2_0) with $h(v)=v(a)$ has only the trivial solution. Therefore, according to Remark 1.1, every solution to (1.1), (1.2) (with $h(v)\overset{\text{def}}{=}v(a)$) has the representation (1.3). Thus the inclusion $\ell\in\mathcal{S}_{ab}(a)$ is equivalent to the non-negativity of u_0 and G.

Definition 1.2. An operator $\ell \in \mathcal{L}_{ab}$ is said to belong to the set $\mathcal{S}_{ab}(b)$ if every function $u \in AC([a,b];\mathbb{R})$ satisfying

$$u'(t) \leqslant \ell(u)(t) \quad \text{for a.e. } t \in [a,b], \tag{1.7}$$

$$u(b) \geqslant 0 \tag{1.8}$$

satisfies the inequality (1.6).

Remark 1.3. The inclusion $\ell \in \mathcal{S}_{ab}(b)$ is equivalent to the non-negativity of u_0 and non-positivity of G in (1.3) with $h(v) \stackrel{\text{def}}{=} v(b)$.

Definition 1.3. An operator $\ell \in \mathcal{L}_{ab}$ is said to belong to the set $\mathcal{S}'_{ab}(a)$ if every function $u \in AC([a,b];\mathbb{R})$ satisfying (1.4) and (1.5) satisfies the inequalities (1.6) and

$$u'(t) \geqslant 0 \quad \text{for a.e. } t \in [a,b]. \tag{1.9}$$

Remark 1.4. It follows from the integral representation (1.3) (with $h(v) \stackrel{\text{def}}{=} v(a)$) that the inclusion $\ell \in \mathcal{S}'_{ab}(a)$ is connected, in a certain sense, with the non-negativity of u'_0, G, and G'_t (see Remark 1.2 and [1, §3.4, Theorem 4.2]).

Definition 1.4. An operator $\ell \in \mathcal{L}_{ab}$ is said to belong to the set $\mathcal{S}'_{ab}(b)$ if every function $u \in AC([a,b];\mathbb{R})$ satisfying (1.7) and (1.8) satisfies the inequalities (1.6) and

$$u'(t) \leqslant 0 \quad \text{for a.e. } t \in [a,b]. \tag{1.10}$$

Remark 1.5. It follows from the integral representation (1.3) (with $h(v) \stackrel{\text{def}}{=} v(b)$) that the inclusion $\ell \in \mathcal{S}'_{ab}(b)$ is connected, in a certain sense, with the sign-properties of u'_0, G, and G'_t (see Remark 1.2 and [1, §3.4, Theorem 4.2]).

Definition 1.5. An operator $\ell \in \mathcal{L}_{ab}$ belongs to the set $\mathcal{P}^+_{ab}$ (or $\mathcal{P}^-_{ab}$, respectively) if it transforms non-negative non-decreasing (or non-increasing) absolutely continuous functions into non-negative functions.

Similarly, we say that an operator $\ell \in \mathcal{L}_{ab}$ belongs to the set $\mathcal{N}^+_{ab}$ (or $\mathcal{N}^-_{ab}$, respectively) if it transforms non-negative non-decreasing (or non-increasing) absolutely continuous functions into non-positive functions.

Remark 1.6. In contrast with the set $\mathcal{P}_{ab}$ (of positive operators), having $\ell \in \mathcal{P}^+_{ab}$ (or $\ell \in \mathcal{P}^-_{ab}$, respectively), it is not supposed that the operator ℓ transforms every non-negative function into a non-negative function. For example, let

$$\ell(u)(t) \stackrel{\text{def}}{=} p(t)u(\tau(t)) - g(t)u(\mu(t)) \quad \text{for } t \in [a,b]$$

be an operator with $p, g \in L([a,b];\mathbb{R}_+)$, $\tau, \mu\colon\ [a,b] \to [a,b]$ measurable functions. If $p(t) \geqslant g(t)$ for a.e. $t \in [a,b]$ and $g(t)(\tau(t) - \mu(t)) \geqslant 0$ for a.e. $t \in [a,b]$, then ℓ belongs to $\mathcal{P}_{ab}^+$ but does not belong to $\mathcal{P}_{ab}$.

More precisely, $\mathcal{P}_{ab} \subset \mathcal{P}_{ab}^+ \cap \mathcal{P}_{ab}^-$. However, $\mathcal{P}_{ab} \neq \mathcal{P}_{ab}^+ \cap \mathcal{P}_{ab}^-$ as the following example illustrates: Let

$$\ell(u)(t) \overset{\text{def}}{=} \frac{2}{b-a}\int_a^b u(s)\,\mathrm{d}s - u\Big(\frac{a+b}{2}\Big) \quad \text{for } t \in [a,b].$$

Then, supposing u to be a non-negative, non-decreasing function, we obtain $\ell(u)(t) \geqslant 0$ for $t \in [a,b]$. We obtain the same inequality, i.e. the inequality $\ell(u)(t) \geqslant 0$ for $t \in [a,b]$, whenever u is a non-negative and non-increasing function. Therefore, $\ell \in \mathcal{P}_{ab}^+ \cap \mathcal{P}_{ab}^-$.

On the other hand, let $\varepsilon \in\,]0, (b-a)/2[$ and put

$$u_0(t) = \begin{cases} 0 & \text{for } [a, a+\varepsilon[\, \cup\,]b-\varepsilon, b], \\ \dfrac{2(t-a-\varepsilon)}{b-a-2\varepsilon} & \text{for } \Big[a+\varepsilon, \dfrac{a+b}{2}\Big[, \\ \dfrac{2(b-\varepsilon-t)}{b-a-2\varepsilon} & \text{for } \Big[\dfrac{a+b}{2}, b-\varepsilon\Big]. \end{cases}$$

Obviously, $u_0(t) \geqslant 0$ for $t \in [a,b]$, but

$$\ell(u_0)(t) = -\frac{2\varepsilon}{b-a} < 0 \quad \text{for } t \in [a,b].$$

Consequently, $\ell \notin \mathcal{P}_{ab}$.

Remark 1.7. Define the operator $\varphi\colon\ C([a,b];\mathbb{R}) \to C([a,b];\mathbb{R})$ as follows:

$$\varphi(v)(t) = v(a+b-t) \quad \text{for } t \in [a,b], \qquad v \in C([a,b];\mathbb{R}).$$

Put

$$\tilde{\ell}(v)(t) = -\ell(\varphi(v))(a+b-t) \quad \text{for a.e. } t \in [a,b], \qquad v \in C([a,b];\mathbb{R}).$$

Then it can be easily verified that $\tilde{\ell} \in \mathcal{S}_{ab}(a)$ or $\tilde{\ell} \in \mathcal{S}'_{ab}(a)$ if and only if $\ell \in \mathcal{S}_{ab}(b)$ or $\ell \in \mathcal{S}'_{ab}(b)$, respectively. Furthermore, $\tilde{\ell} \in \mathcal{P}_{ab}^+$ (or $\tilde{\ell} \in \mathcal{P}_{ab}^-$), if and only if $\ell \in \mathcal{N}_{ab}^-$, (or $\ell \in \mathcal{N}_{ab}^+$, respectively).

Remark 1.8. Note that the inclusion $\ell \in \mathcal{S}_{ab}(a)$, and consequently also the inclusion $\ell \in \mathcal{S}'_{ab}(a)$, guarantees that the unique solution to (1.1_0) satisfying $u(a) = 0$ is a trivial solution. Similarly, the inclusion $\ell \in \mathcal{S}_{ab}(b)$, or $\ell \in \mathcal{S}'_{ab}(b)$, implies that the unique solution to (1.1_0) satisfying $u(b) = 0$ is the trivial solution.

For monotone operators the following theorems can be found in [5].

Theorem 1.2 (Theorem 1.1 in [5]). *Let $\ell_0 \in \mathcal{P}_{ab}$. Then $\ell_0 \in \mathcal{S}_{ab}(a)$ if and only if there exists $\gamma \in AC([a,b];\mathbb{R})$ such that*

$$\begin{aligned} \gamma(t) &> 0 && \text{for } t \in [a,b], \\ \gamma'(t) &\geqslant \ell_0(\gamma)(t) && \text{for a.e. } t \in [a,b]. \end{aligned}$$

Theorem 1.3 (Theorem 1.5 in [5]). *Let $\ell_1 \in \mathcal{P}_{ab}$. Then $-\ell_1 \in \mathcal{S}_{ab}(b)$ if and only if there exists $\gamma \in AC([a,b];\mathbb{R})$ such that*

$$\begin{aligned} \gamma(t) &> 0 && \text{for } t \in [a,b], \\ \gamma'(t) &\leqslant -\ell_1(\gamma)(t) && \text{for a.e. } t \in [a,b]. \end{aligned}$$

Remark 1.9. Note that the function γ appearing in Theorem 1.2 and Theorem 1.3 is non-decreasing and non-increasing, respectively. Therefore, from Theorem 1.2 it follows that if $\ell_0 \in \mathcal{S}_{ab}(a)$ is a b-Volterra positive operator then $\tilde{\ell}_0 \in \mathcal{S}_{cb}(c)$ for every $c \in [a,b[$, where $\tilde{\ell}_0$ is the restriction of ℓ_0 to the space $C([c,b];\mathbb{R})$ defined by

$$\begin{aligned} \tilde{\ell}_0(y)(t) &= \ell_0(\vartheta(y))(t) \quad \text{for a.e. } t \in [c,b],\ y \in C([c,b];\mathbb{R}), \\ \vartheta(y)(t) &= \begin{cases} y(c) & \text{for } t \in [a,c[\,, \\ y(t) & \text{for } t \in [c,b]. \end{cases} \end{aligned}$$

Similarly, from Theorem 1.3 it follows that if $-\ell_1 \in \mathcal{S}_{ab}(b)$ is an a-Volterra non-increasing operator then $-\tilde{\ell}_1 \in \mathcal{S}_{ac}(c)$ for every $c \in \,]a,b]$, where $\tilde{\ell}_1$ is the restriction of ℓ_1 to the space $C([a,c];\mathbb{R})$ defined by

$$\begin{aligned} \tilde{\ell}_1(y)(t) &= \ell_1(\vartheta(y))(t) \quad \text{for a.e. } t \in [a,c],\ y \in C([a,c];\mathbb{R}), \\ \vartheta(y)(t) &= \begin{cases} y(t) & \text{for } t \in [a,c], \\ y(c) & \text{for } t \in \,]c,b]. \end{cases} \end{aligned}$$

2. Main results

In this section, the main results are formulated using the general terms that the operator ℓ, or its positive or negative part, belong to one of the sets $\mathcal{S}_{ab}(a)$, $\mathcal{S}_{ab}(b)$, $\mathcal{S}'_{ab}(a)$, and $\mathcal{S}'_{ab}(b)$. The effective criteria guaranteeing such an inclusion can be found in Section 4 of the paper. For more conditions guaranteeing the mentioned inclusions one can see [2], [5] or the monograph [7] in which also a detailed introduction to the problem is contained.

Proposition 2.1. *Let $\ell \in \mathcal{S}'_{ab}(a)$. Then $\dim U = 1$ and the set U is generated by a positive non-decreasing function.*

Theorem 2.1. *Let $\ell \in \mathcal{P}^+_{ab}$ admit the representation $\ell = \ell_0 - \ell_1$ with $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ and let $\ell_0 \in \mathcal{S}_{ab}(a)$. Then $\dim U = 1$ and the set U is generated by a positive non-decreasing function.*

Theorem 2.2. *Let $\ell \in \mathcal{P}^+_{ab}$ admit the representation $\ell = \ell_0 - \ell_1$ with $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ and let $-\ell_1 \in \mathcal{S}_{ab}(b)$. Let, moreover, there exist $\gamma \in AC([a,b];\mathbb{R})$ satisfying*

$$\gamma(t) > 0 \quad \text{for } t \in [a,b], \tag{2.1}$$

$$\gamma'(t) \geqslant \ell(\gamma)(t) \quad \text{for a.e. } t \in [a,b]. \tag{2.2}$$

Then $\dim U = 1$ and the set U is generated by a positive non-decreasing function.

Theorem 2.3. *Let $\ell \in \mathcal{N}^+_{ab}$ admit the representation $\ell = \ell_0 - \ell_1$ with $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ and let $-\ell_1 \in \mathcal{S}_{ab}(b)$ be an a-Volterra operator. Then $\dim U = 1$ and the set U is generated by a positive function u with the following property: the relation*

$$u(a) = \max\{u(t)\colon\ t \in [a,b]\} \tag{2.3}$$

holds and, in addition, if there exists $c \in \,]a,b]$ such that $u(c) = u(a)$ then

$$u(t) = u(c) \quad \text{for } t \in [a,c]. \tag{2.4}$$

Theorem 2.4. *Let $\ell \in \mathcal{P}^-_{ab}$ admit the representation $\ell = \ell_0 - \ell_1$ with $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ and let $-\ell_1 \in \mathcal{S}_{ab}(b)$ be an a-Volterra operator. Let, moreover, there exist $\gamma \in AC([a,b];\mathbb{R})$ satisfying (2.1) and (2.2). Then $\dim U = 1$ and the set U is generated by a positive function u with the following property: the relation*

$$u(a) = \min\{u(t)\colon\ t \in [a,b]\} \tag{2.5}$$

holds and, in addition, if there exists $c \in \,]a,b]$ such that $u(c) = u(a)$ then (2.4) is fulfilled.

According to Remark 1.7, the following assertions immediately follow from Proposition 2.1 and Theorems 2.1–2.4:

Proposition 2.2. *Let $\ell \in \mathcal{S}'_{ab}(b)$. Then $\dim U = 1$ and the set U is generated by a positive non-increasing function.*

Theorem 2.5. *Let $\ell \in \mathcal{N}_{ab}^-$ admit the representation $\ell = \ell_0 - \ell_1$ with $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ and let $-\ell_1 \in \mathcal{S}_{ab}(b)$. Then $\dim U = 1$ and the set U is generated by a positive non-increasing function.*

Theorem 2.6. *Let $\ell \in \mathcal{N}_{ab}^-$ admit the representation $\ell = \ell_0 - \ell_1$ with $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ and let $\ell_0 \in \mathcal{S}_{ab}(a)$. Let, moreover, there exist $\gamma \in AC([a,b];\mathbb{R})$ satisfying* (2.1) *and*

$$\gamma'(t) \leqslant \ell(\gamma)(t) \quad \text{for a.e. } t \in [a,b]. \tag{2.6}$$

Then $\dim U = 1$ and the set U is generated by a positive non-increasing function.

Theorem 2.7. *Let $\ell \in \mathcal{P}_{ab}^-$ admit the representation $\ell = \ell_0 - \ell_1$ with $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ and let $\ell_0 \in \mathcal{S}_{ab}(a)$ be a b-Volterra operator. Then $\dim U = 1$ and the set U is generated by a positive function u with the following property: the relation*

$$u(b) = \max\{u(t) \colon \ t \in [a,b]\} \tag{2.7}$$

holds and, in addition, if there exists $c \in [a,b[$ such that $u(c) = u(b)$ then

$$u(t) = u(c) \quad \text{for } t \in [c,b]. \tag{2.8}$$

Theorem 2.8. *Let $\ell \in \mathcal{N}_{ab}^+$ admit the representation $\ell = \ell_0 - \ell_1$ with $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ and let $\ell_0 \in \mathcal{S}_{ab}(a)$ be a b-Volterra operator. Let, moreover, there exist $\gamma \in AC([a,b];\mathbb{R})$ satisfying* (2.1) *and* (2.6)*. Then $\dim U = 1$ and the set U is generated by a positive function u with the following property: the relation*

$$u(b) = \min\{u(t) \colon \ t \in [a,b]\} \tag{2.9}$$

holds and, in addition, if there exists $c \in [a,b[$ such that $u(c) = u(b)$ then (2.8) *is fulfilled.*

3. Auxiliary propositions

Lemma 3.1. *Let $\ell \in \mathcal{P}_{ab}^+$ admit the representation $\ell = \ell_0 - \ell_1$ with $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ and let $-\ell_1 \in \mathcal{S}_{ab}(b)$. Let, moreover, $u \in AC([a,b];\mathbb{R})$ satisfy* (1.4) *and* (1.6)*. Then u satisfies also* (1.9).

P r o o f. Put

$$w(t) = \min\{u(s)\colon\ s \in [t,b]\} \quad \text{for } t \in [a,b] \tag{3.1}$$

and

$$A = \{t \in [a,b]\colon\ w(t) = u(t)\}. \tag{3.2}$$

Obviously,

$$w \in AC([a,b];\mathbb{R}), \tag{3.3}$$

$$w(t) \geqslant 0 \qquad \text{for } t \in [a,b], \tag{3.4}$$

$$w'(t) \geqslant 0 \qquad \text{for a.e. } t \in [a,b], \tag{3.5}$$

$$w(t) \leqslant u(t) \quad \text{for } t \in [a,b], \tag{3.6}$$

and

$$w'(t) = \begin{cases} u'(t) & \text{for a.e. } t \in A, \\ 0 & \text{for a.e. } t \in [a,b] \setminus A. \end{cases} \tag{3.7}$$

Put

$$q(t) = u'(t) - \ell(u)(t) \quad \text{for a.e. } t \in [a,b]. \tag{3.8}$$

Then in view of (1.4) we have

$$q(t) \geqslant 0 \quad \text{for a.e. } t \in [a,b]. \tag{3.9}$$

Moreover, according to (3.6) from (3.8) it follows that

$$u'(t) = \ell(u)(t) + q(t) \leqslant \ell_0(u)(t) - \ell_1(w)(t) + q(t) \quad \text{for a.e. } t \in [a,b]. \tag{3.10}$$

On the other hand, in view of the inclusion $\ell \in \mathcal{P}_{ab}^{+}$, on account of (3.3)–(3.6), and (3.9), we have

$$\ell_0(u)(t) - \ell_1(w)(t) + q(t) \geqslant \ell(w)(t) + q(t) \geqslant 0 \quad \text{for a.e. } t \in [a,b]. \tag{3.11}$$

Now from (3.7), (3.10), and (3.11) we get

$$w'(t) \leqslant \ell_0(u)(t) - \ell_1(w)(t) + q(t) \quad \text{for a.e. } t \in [a,b]. \tag{3.12}$$

Put

$$z(t) = w(t) - u(t) \quad \text{for } t \in [a, b]. \tag{3.13}$$

Then in view of (3.1), (3.8), (3.12), and (3.13) we have

$$z'(t) \leqslant -\ell_1(z)(t) \quad \text{for a.e. } t \in [a, b], \qquad z(b) = 0.$$

Now the inclusion $-\ell_1 \in \mathcal{S}_{ab}(b)$ implies $z(t) \geqslant 0$ for $t \in [a, b]$, whence in view of (3.13) we obtain

$$w(t) \geqslant u(t) \quad \text{for } t \in [a, b]. \tag{3.14}$$

However, (3.14) together with (3.6) yields $w \equiv u$, and consequently, on account of (3.5), the inequality (1.9) is true. □

Lemma 3.2. *Let $\ell \in \mathcal{N}_{ab}^+$ admit the representation $\ell = \ell_0 - \ell_1$ with $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ and let $-\ell_1 \in \mathcal{S}_{ab}(b)$ be an a-Volterra operator. Let, moreover, $u \in AC([a, b]; \mathbb{R})$ satisfy* (1.4) *and let there exist $c \in \,]a, b]$ such that*

$$u(c) = \min\{u(t)\colon\ t \in [a, b]\} \leqslant 0. \tag{3.15}$$

Then (2.4) *holds.*

P r o o f. Define operators $\vartheta\colon C([a, c]; \mathbb{R}) \to C([a, b]; \mathbb{R})$ and $\tilde{\ell}_i\colon C([a, c]; \mathbb{R}) \to L([a, c]; \mathbb{R})$ $(i = 0, 1)$ as follows:

$$\vartheta(y)(t) = \begin{cases} y(t) & \text{for } t \in [a, c[\,, \\ y(c) & \text{for } t \in [c, b], \end{cases} \qquad y \in C([a, c]; \mathbb{R}), \tag{3.16}$$

$$\tilde{\ell}_i(y)(t) = \ell_i(\vartheta(y))(t) \quad \text{for a.e. } t \in [a, c], \quad y \in C([a, c]; \mathbb{R}) \quad (i = 0, 1). \tag{3.17}$$

Then it can be easily verified, according to Definition 1.5, (3.16), and (3.17), that

$$\tilde{\ell}_0 - \tilde{\ell}_1 \in \mathcal{N}_{ac}^+. \tag{3.18}$$

Moreover, according to the assumption that ℓ_1 is an a-Volterra operator,

$$\tilde{\ell}_1(\overline{y})(t) = \ell_1(y)(t) \quad \text{for a.e. } t \in [a, c],\ y \in C([a, b]; \mathbb{R}), \tag{3.19}$$

where $\overline{y}$ is the restriction of y to the interval $[a, c]$.

Let $\bar{u}$ be the restriction of u to the interval $[a,c]$. Thus the function $\bar{u}$, in view of (1.4), (3.15)–(3.17), and (3.19), satisfies the inequalities

$$\vartheta(\bar{u})(t) \leqslant u(t) \qquad \text{for } t \in [a,b], \tag{3.20}$$

$$\bar{u}'(t) \geqslant \tilde{\ell}_0(\bar{u})(t) - \tilde{\ell}_1(\bar{u})(t) \quad \text{for a.e. } t \in [a,c]. \tag{3.21}$$

Furthermore, note that according to Remark 1.9, the inclusion $-\ell_1 \in \mathcal{S}_{ab}(b)$ yields

$$-\tilde{\ell}_1 \in \mathcal{S}_{ac}(c). \tag{3.22}$$

Now put

$$w(t) = \max\{[\bar{u}(s)]_-\colon\ s \in [a,t]\} \quad \text{for } t \in [a,c], \tag{3.23}$$

and

$$A = \{t \in [a,c]\colon\ w(t) = -\bar{u}(t)\}.$$

Obviously,

$$w \in AC([a,c];\mathbb{R}), \tag{3.24}$$

$$w(t) \geqslant 0 \qquad \text{for } t \in [a,c], \tag{3.25}$$

$$w'(t) \geqslant 0 \qquad \text{for a.e. } t \in [a,c], \tag{3.26}$$

$$w(t) \geqslant [\bar{u}(t)]_- \geqslant -\bar{u}(t) \quad \text{for } t \in [a,c], \tag{3.27}$$

and

$$w'(t) = \begin{cases} -\bar{u}'(t) & \text{for a.e. } t \in A, \\ 0 & \text{for a.e. } t \in [a,c] \setminus A. \end{cases} \tag{3.28}$$

Put

$$q(t) = \bar{u}'(t) - \tilde{\ell}_0(\bar{u})(t) + \tilde{\ell}_1(\bar{u})(t) \quad \text{for a.e. } t \in [a,c]. \tag{3.29}$$

Then in view of (3.21) we have

$$q(t) \geqslant 0 \quad \text{for a.e. } t \in [a,c]. \tag{3.30}$$

Moreover, according to (3.27) from (3.29) we obtain

$$-\bar{u}'(t) = -\tilde{\ell}_0(\bar{u})(t) + \tilde{\ell}_1(\bar{u})(t) - q(t) \geqslant -\tilde{\ell}_0(\bar{u})(t) - \tilde{\ell}_1(w)(t) - q(t) \quad \text{for a.e. } t \in [a,c]. \tag{3.31}$$

On the other hand, in view of (3.18), on account of (3.24)–(3.27), and (3.30), we find

$$(3.32)\quad -\tilde{\ell}_0(\bar{u})(t)-\tilde{\ell}_1(w)(t)-q(t) \leqslant \tilde{\ell}_0(w)(t)-\tilde{\ell}_1(w)(t)-q(t) \leqslant 0 \quad \text{for a.e. } t \in [a,c].$$

Now from (3.28), (3.31), and (3.32) we get

$$(3.33)\qquad w'(t) \geqslant -\tilde{\ell}_0(\bar{u})(t) - \tilde{\ell}_1(w)(t) - q(t) \quad \text{for a.e. } t \in [a,c].$$

Put

$$(3.34)\qquad z(t) = w(t) + \bar{u}(t) \quad \text{for } t \in [a,c].$$

Then in view of (3.15), (3.23), (3.29), (3.33), and (3.34) we have

$$z'(t) \geqslant -\tilde{\ell}_1(z)(t) \quad \text{for a.e. } t \in [a,c], \qquad z(c) = 0.$$

Now the inclusion (3.22) implies $z(t) \leqslant 0$ for $t \in [a,c]$, whence, according to (3.34), we get

$$(3.35)\qquad w(t) \leqslant -\bar{u}(t) \quad \text{for } t \in [a,c].$$

However, (3.35) together with (3.27) yields $w \equiv -\bar{u}$, and consequently, on account of (3.25) and (3.26), we obtain

$$(3.36)\qquad \bar{u}(t) \leqslant 0 \quad \text{for } t \in [a,c],$$

$$(3.37)\qquad \bar{u}'(t) \leqslant 0 \quad \text{for a.e. } t \in [a,c].$$

However, (3.18), (3.36), and (3.37) result in

$$(3.38)\qquad \tilde{\ell}_0(\bar{u})(t) - \tilde{\ell}_1(\bar{u})(t) \geqslant 0 \quad \text{for a.e. } t \in [a,c],$$

whence, in view of (3.21), we have $\bar{u}'(t) \geqslant 0$ for a.e. $t \in [a,c]$. Consequently, the last inequality and (3.37) imply that $\bar{u}'(t) = 0$ for a.e. $t \in [a,c]$, i.e., (2.4) holds. □

Lemma 3.3. *Let $\ell \in \mathcal{P}_{ab}^{-}$ admit the representation $\ell = \ell_0 - \ell_1$ with $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ and let $-\ell_1 \in \mathcal{S}_{ab}(b)$ be an a-Volterra operator. Let, moreover, $u \in AC([a,b];\mathbb{R})$ satisfy* (1.4) *and* (1.6)*. Then* (2.5) *holds and, in addition, if there exists $c \in\,]a,b]$ such that $u(c) = u(a)$, then* (2.4) *holds.*

P r o o f. To prove the lemma it is sufficient to show that whenever there exists $c \in\,]a,b]$ such that

$$u(c) = \min\{u(t)\colon\ t \in [a,b]\} \tag{3.39}$$

then u satisfies (2.4) and so (2.5) holds necessarily. Therefore, let $c \in\,]a,b]$ be arbitrary but fixed, such that (3.39) holds. Define operators $\vartheta\colon C([a,c];\mathbb{R}) \to C([a,b];\mathbb{R})$, $\tilde{\ell}_i\colon C([a,c];\mathbb{R}) \to L([a,c];\mathbb{R})$ $(i = 0,1)$ by (3.16) and (3.17). Then it can be easily verified, according to Definition 1.5, (3.16), and (3.17), that

$$\tilde{\ell}_0 - \tilde{\ell}_1 \in \mathcal{P}_{ac}^{-}. \tag{3.40}$$

Moreover, according to the assumption that ℓ_1 is an a-Volterra operator, (3.19) is fulfilled, where $\overline{y}$ is the restriction of y to the interval $[a,c]$.

Let $\bar{u}$ be the restriction of u to the interval $[a,c]$. Thus the function $\bar{u}$, in view of (1.4), (3.16), (3.17), (3.19), and (3.39), satisfies the inequalities (3.20) and (3.21). Furthermore, note that according to Remark 1.9, the inclusion $-\ell_1 \in \mathcal{S}_{ab}(b)$ yields (3.22).

Now put

$$w(t) = \min\{\bar{u}(s)\colon\ s \in [a,t]\} \quad \text{for } t \in [a,c], \tag{3.41}$$

and

$$A = \{t \in [a,c]\colon\ w(t) = \bar{u}(t)\}.$$

Obviously, (3.24), (3.25),

$$w'(t) \leqslant 0 \qquad \text{for a.e. } t \in [a,c], \tag{3.42}$$

$$w(t) \leqslant \bar{u}(t) \quad \text{for } t \in [a,c], \tag{3.43}$$

and

$$w'(t) = \begin{cases} \bar{u}'(t) & \text{for a.e. } t \in A, \\ 0 & \text{for a.e. } t \in [a,c] \setminus A \end{cases} \tag{3.44}$$

hold. Define q by (3.29). Then in view of (3.21) we have (3.30). Moreover, according to (3.43) from (3.29) we obtain

$$\bar{u}'(t) = \tilde{\ell}_0(\bar{u})(t) - \tilde{\ell}_1(\bar{u})(t) + q(t) \leqslant \tilde{\ell}_0(\bar{u})(t) - \tilde{\ell}_1(w)(t) + q(t) \quad \text{for a.e. } t \in [a,c]. \tag{3.45}$$

On the other hand, in view of (3.40), on account of (3.24), (3.25), (3.30), (3.42), and (3.43), we find

$$\tilde{\ell}_0(\bar{u})(t) - \tilde{\ell}_1(w)(t) + q(t) \geqslant \tilde{\ell}_0(w)(t) - \tilde{\ell}_1(w)(t) + q(t) \geqslant 0 \quad \text{for a.e. } t \in [a,c]. \tag{3.46}$$

Now, from (3.44)–(3.46), we get

$$w'(t) \leqslant \tilde{\ell}_0(\bar{u})(t) - \tilde{\ell}_1(w)(t) + q(t) \quad \text{for a.e. } t \in [a,c]. \tag{3.47}$$

Put

$$z(t) = w(t) - \bar{u}(t) \quad \text{for } t \in [a,c]. \tag{3.48}$$

Then in view of (3.29), (3.39), (3.41), (3.47), and (3.48) we have

$$z'(t) \leqslant -\tilde{\ell}_1(z)(t) \quad \text{for a.e. } t \in [a,c], \qquad z(c) = 0.$$

Now the inclusion (3.22) implies $z(t) \geqslant 0$ for $t \in [a,c]$, whence according to (3.48) we get

$$w(t) \geqslant \bar{u}(t) \quad \text{for } t \in [a,c]. \tag{3.49}$$

However, (3.49) together with (3.43) yield $w \equiv \bar{u}$, and consequently, on account of (3.42), we obtain (3.37). Furthermore, (1.6), (3.37), and (3.40) result in (3.38), whence, in view of (3.21), we have $\bar{u}'(t) \geqslant 0$ for a.e. $t \in [a,c]$. Consequently, the last inequality and (3.37) imply that $\bar{u}'(t) = 0$ for a.e. $t \in [a,c]$, i.e., (2.4) holds. □

Lemma 3.4. *Let every nontrivial solution u to (1.1_0) satisfy*

$$u(t) \neq 0 \quad \textit{for } t \in [a,b].$$

Then $\dim U = 1$.

P r o o f. Let u and v be arbitrary nontrivial solutions to (1.1_0). Put

$$w(t) = u(a)v(t) - v(a)u(t) \quad \text{for } t \in [a,b].$$

Then, obviously,

$$w'(t) = \ell(w)(t) \quad \text{for a.e. } t \in [a,b], \qquad w(a) = 0.$$

Therefore, according to the assumptions we have $w \equiv 0$, i.e.,

$$v(t) = \frac{v(a)}{u(a)} u(t) \quad \text{for } t \in [a,b].$$

□

4. On the sets $\mathcal{S}_{ab}(a)$, $\mathcal{S}_{ab}(b)$, $\mathcal{S}'_{ab}(a)$, and $\mathcal{S}'_{ab}(b)$

Theorem 4.1. *Let $\ell \in \mathcal{P}^+_{ab}$ admit the representation $\ell = \ell_0 - \ell_1$ with $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ and let $\ell_0 \in \mathcal{S}_{ab}(a)$. Then $\ell \in \mathcal{S}'_{ab}(a)$.*

Proof. Let $u \in AC([a,b];\mathbb{R})$ satisfy (1.4) and (1.5). We will show that (1.6) and (1.9) hold.

Put

$$w(t) = \max\{u(s)\colon\ s \in [a,t]\} \quad \text{for } t \in [a,b] \tag{4.1}$$

and define the set A and the function q by (3.2) and (3.8), respectively. Then according to (1.4), (1.5), and (4.1) we have (3.3)–(3.5), (3.7), (3.9), and (3.14). Moreover, according to (3.14) from (3.8) it follows that

$$u'(t) = \ell(u)(t) + q(t) \leqslant \ell_0(w)(t) - \ell_1(u)(t) + q(t) \quad \text{for a.e. } t \in [a,b]. \tag{4.2}$$

On the other hand, in view of the inclusion $\ell \in \mathcal{P}^+_{ab}$, on account of (3.3)–(3.5), (3.9), and (3.14), we have

$$\ell_0(w)(t) - \ell_1(u)(t) + q(t) \geqslant \ell(w)(t) + q(t) \geqslant 0 \quad \text{for a.e. } t \in [a,b]. \tag{4.3}$$

Now from (3.7), (4.2), and (4.3) we get

$$w'(t) \leqslant \ell_0(w)(t) - \ell_1(u)(t) + q(t) \quad \text{for a.e. } t \in [a,b]. \tag{4.4}$$

Define z by (3.13). Then in view of (3.8), (3.13), (4.1), and (4.4) we have

$$z'(t) \leqslant \ell_0(z)(t) \quad \text{for a.e. } t \in [a,b], \qquad z(a) = 0.$$

Now the inclusion $\ell_0 \in \mathcal{S}_{ab}(a)$ implies $z(t) \leqslant 0$ for $t \in [a,b]$, whence in view of (3.13) we obtain (3.6). However, (3.6) together with (3.14) yields $w \equiv u$, and consequently, on account of (3.4) and (3.5) the inequalities (1.6) and (1.9) hold. □

Theorem 4.2. *Let $\ell \in \mathcal{P}^+_{ab}$ admit the representation $\ell = \ell_0 - \ell_1$ with $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ and let $-\ell_1 \in \mathcal{S}_{ab}(b)$. Then $\ell \in \mathcal{S}'_{ab}(a)$ if and only if there exists $\gamma \in AC([a,b];\mathbb{R})$ satisfying* (2.1) *and* (2.2).

P r o o f. If $\ell \in \mathcal{S}'_{ab}(a)$, then the problem

$$u'(t) = \ell(u)(t), \qquad u(a) = 1 \tag{4.5}$$

has a unique solution γ. Therefore, γ satisfies (2.2). Moreover, according to Definition 1.3, the function γ satisfies $\gamma'(t) \geqslant 0$ for a.e. $t \in [a,b]$, which together with $\gamma(a) = 1$ implies (2.1).

Let there exist $\gamma \in AC([a,b];\mathbb{R})$ satisfying (2.1) and (2.2), and let $u \in AC([a,b];\mathbb{R})$ satisfy (1.4) and (1.5). We have to show that (1.6) and (1.9) hold. However, according to Lemma 3.1, it is sufficient to show that (1.6) holds.

Assume on the contrary that there exists $t_0 \in \,]a,b]$ such that

$$u(t_0) < 0. \tag{4.6}$$

Put

$$\lambda = \max\left\{\frac{-u(t)}{\gamma(t)} \colon \ t \in [a,b]\right\}. \tag{4.7}$$

Then, in view of (2.1) and (4.6), we have

$$\lambda > 0. \tag{4.8}$$

Furthermore,

$$\lambda\gamma(t) + u(t) \geqslant 0 \quad \text{for } \ t \in [a,b] \tag{4.9}$$

and there exists $t_1 \in [a,b]$ such that

$$\lambda\gamma(t_1) + u(t_1) = 0. \tag{4.10}$$

From (1.4), (2.2), and (4.8) we get

$$\lambda\gamma'(t) + u'(t) \geqslant \ell(\lambda\gamma + u)(t) \quad \text{for a.e. } t \in [a,b]. \tag{4.11}$$

Now according to Lemma 3.1, in view of (4.9) and (4.11), the inequality

$$\lambda\gamma'(t) + u'(t) \geqslant 0 \quad \text{for a.e. } t \in [a,b] \tag{4.12}$$

holds. However, (4.9), (4.10), and (4.12) yield

$$\lambda\gamma(a) + u(a) = 0, \tag{4.13}$$

whence, on account of (1.5), (2.1), and (4.8), we obtain

$$0 < \lambda\gamma(a) = -u(a) \leqslant 0, \tag{4.14}$$

a contradiction. Therefore, (1.6) holds. □

Theorem 4.3. *Let $\ell \in \mathcal{N}_{ab}^{+}$ admit the representation $\ell = \ell_0 - \ell_1$ with $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ and let $-\ell_1 \in \mathcal{S}_{ab}(b)$ be an a-Volterra operator. Then $\ell \in \mathcal{S}_{ab}(a)$.*

P r o o f. Let $u \in AC([a,b];\mathbb{R})$ satisfy (1.4) and (1.5). We will show that (1.6) holds. Assume on the contrary that there exists $t_0 \in \,]a,b]$ such that (4.6) is fulfilled. Then the assumptions of Lemma 3.2 are satisfied for a suitable $c \in \,]a,b]$. Therefore, (2.4) holds and, in particular, $u(a) = u(c)$. However, the latter equality together with (1.5), (3.15), and (4.6) result in

$$0 \leqslant u(a) = u(c) \leqslant u(t_0) < 0,$$

a contradiction. Therefore, (1.6) holds and so $\ell \in \mathcal{S}_{ab}(a)$. □

Theorem 4.4. *Let $\ell \in \mathcal{P}_{ab}^{-}$ admit the representation $\ell = \ell_0 - \ell_1$ with $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ and let $-\ell_1 \in \mathcal{S}_{ab}(b)$ be an a-Volterra operator. Then $\ell \in \mathcal{S}_{ab}(a)$ if and only if there exists $\gamma \in AC([a,b];\mathbb{R})$ satisfying* (2.1) *and* (2.2).

P r o o f. If $\ell \in \mathcal{S}_{ab}(a)$, then the problem (4.5) has a unique solution γ. Therefore, γ satisfies (2.2). Moreover, according to Definition 1.1, the inequality $\gamma(t) \geqslant 0$ for $t \in [a,b]$ is fulfilled. Now from Lemma 3.3 it follows that

$$\gamma(a) = \min\{\gamma(t)\colon\ t \in [a,b]\},$$

which together with $\gamma(a) = 1$ implies (2.1).

Let now there exist a function $\gamma \in AC([a,b];\mathbb{R})$ satisfying (2.1) and (2.2), and let $u \in AC([a,b];\mathbb{R})$ satisfy (1.4) and (1.5). We will show that (1.6) holds.

Assume on the contrary that there exists $t_0 \in \,]a,b]$ such that (4.6) holds. Define λ by (4.7). Then, in view of (2.1) and (4.6), we have (4.8). Furthermore, (4.9) is fulfilled and there exists $t_1 \in [a,b]$ such that (4.10) holds true. Moreover, from (1.4), (2.2), and (4.8) we get (4.11). Now according to Lemma 3.3, in view of (4.9)–(4.11), the equality (4.13) is satisfied, whence, on account of (1.5), (2.1), and (4.8), we obtain (4.14), a contradiction. Therefore, (1.6) holds. □

5. Proofs of the main results

Proof of Proposition 2.1. Let u be an arbitrary nontrivial solution to (1.1_0). Without loss of generality we can assume that (1.5) holds. Therefore, in view of the inclusion $\ell \in \mathcal{S}'_{ab}(a)$ we have (1.6) and (1.9).

If $u(a) = 0$ then $v \equiv -u$ satisfies

$$v'(t) = \ell(v)(t) \quad \text{for a.e. } t \in [a,b], \qquad v(a) = 0,$$

and, consequently, due to the inclusion $\ell \in \mathcal{S}'_{ab}(a)$ again, we have $v(t) \geqslant 0$ for $t \in [a,b]$, i.e., $u(t) \leqslant 0$ for $t \in [a,b]$. Thus the latter inequality together with (1.6) yields $u \equiv 0$, a contradiction.

Therefore, $u(a) > 0$ and from (1.9) it follows that u is a positive non-decreasing function. Now the conclusion of the theorem follows from Lemma 3.4. □

Theorems 2.1 and 2.2 immediately follow from Proposition 2.1, according to Theorems 4.1 and 4.2.

Proof of Theorem 2.3. According to Theorem 4.3 we have

$$\ell \in \mathcal{S}_{ab}(a). \tag{5.1}$$

Let u be an arbitrary nontrivial solution to (1.1_0). Without loss of generality we can assume that (1.5) holds.

If $u(a) = 0$ then in view of (1.1_0) and (5.1) we have $u \equiv 0$, a contradiction.

Therefore, $u(a) > 0$. We will show that u is a positive function. Assume on the contrary that u has a zero. Then according to Lemma 3.2 we have $u(a) \leqslant 0$, a contradiction. Therefore, u is a positive function and the relation $\dim U = 1$ follows from Lemma 3.4.

Now we will prove the second part of the theorem. It is sufficient to show that if there exists $c \in \,]a,b]$ such that

$$u(c) = \max\{u(t)\colon\ t \in [a,b]\} \tag{5.2}$$

then (2.4) holds. Thus let $c \in \,]a,b]$ be such that (5.2) is fulfilled. Obviously,

$$u(c) > 0. \tag{5.3}$$

Put

$$v(t) = -u(t) \quad \text{for } t \in [a,b]. \tag{5.4}$$

Then in view of (1.1_0), (5.2), and (5.3) we have

$$v'(t) = \ell(v)(t) \quad \text{for a.e. } t \in [a,b], \qquad v(c) = \min\{v(t)\colon\ t \in [a,b]\} < 0.$$

Therefore, the assumptions of Lemma 3.2 are fulfilled and so we have $v(t) = v(c)$ for $t \in [a,c]$, i.e., in view of (5.4), (2.4) holds. □

P r o o f of Theorem 2.4. According to Theorem 4.4 we have (5.1). Let u be an arbitrary nontrivial solution to (1.1_0). Without loss of generality we can assume that (1.5) holds. Obviously, (5.1) yields (1.6).

If $u(a) = 0$ then in view of (1.1_0) and (5.1) we find $u \equiv 0$, a contradiction.

Therefore, $u(a) > 0$. Now the conclusion of the theorem follows from Lemmas 3.3 and 3.4. □

6. Application

The following general theorem is a simple consequence of Theorem 1.1.

Theorem 6.1. *Let for every $u \in U$ the following implication hold: if $h(u) = 0$ then $u \equiv 0$. Then the problem* (1.1), (1.2) *is uniquely solvable.*

From Theorem 6.1 it immediately follows that the knowledge of the structure of the solution set U allows us to find effective criteria guaranteeing the unique solvability of the problem (1.1), (1.2). In particular, the following consequence is true:

Corollary 6.1. *Let $\dim U = 1$ and let the set U be generated by a positive function. Let, moreover, the operator h have the following property: if $h(u) = 0$ then u has a zero. Then the problem* (1.1), (1.2) *is uniquely solvable.*

According to the results obtained in Section 2, Theorem 6.1, and Corollary 6.1, one can easily derive statements dealing with the solvability of special cases of the problem (1.1), (1.2). As an illustration, we give the results dealing with the initial, anti-periodic, and periodic boundary value problems:

Theorem 6.2. *The assumptions of each of Propositions* 2.1 *and* 2.2 *or Theorems* 2.1–2.8 *guarantee the existence of a unique solution u to the equation* (1.1) *satisfying*

$$u(t_0) = c,$$

where $t_0 \in [a,b]$ is arbitrary but fixed and $c \in \mathbb{R}$.

Theorem 6.3. *The assumptions of each of Propositions* 2.1 *and* 2.2 *or Theorems* 2.1–2.8 *guarantee the existence of a unique solution* u *to the equation* (1.1) *satisfying*

$$u(b) + u(a) = c,$$

where $c \in \mathbb{R}$.

The previous theorems immediately follow from Corollary 6.1 and the results established in the Section 2. Applying Theorem 6.1 and the statements of Section 2 we obtain

Theorem 6.4. *Let* $\ell(1) \not\equiv 0$. *Then the assumptions of each of Propositions* 2.1 *and* 2.2 *or Theorems* 2.1–2.8 *guarantee the existence of a unique solution* u *to the equation* (1.1) *satisfying*

$$u(b) - u(a) = c,$$

where $c \in \mathbb{R}$.

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