LETTER TO THE EDITOR

On the dimensionality of spacetime

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Abstract. Some superstring theories have more than one effective low-energy limit corresponding to classical spacetimes with different dimensionalities. We argue that all but the $(3+1)$-dimensional one might correspond to 'dead worlds', devoid of observers, in which case all such ensemble theories would actually *predict* that we should find ourselves inhabiting a $(3+1)$-dimensional spacetime. With more or less than one time dimension, the partial differential equations of nature would lack the hyperbolicity property that enables observers to make predictions. In a space with more than three dimensions, there can be no traditional atoms and perhaps no stable structures. A space with less than three dimensions allows no gravitational force and may be too simple and barren to contain observers.

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Many superstring theories have several stable (or extremely long-lived) states that constitute different effective low-energy theories with different spacetime dimensionalities, corresponding to different compactifications of the many (e.g. 11 or 26) dimensions of the fundamental manifold. Since the tunnelling probabilities between these states are negligible, such a theory for all practical purposes predicts an ensemble of classical $(n+m)$-dimensional spacetimes, and the prediction for the dimensionality takes the form of a probability distribution over n and m [1]. There are also inflationary models predicting a universe consisting of parts of exponentially large size having different dimensionality [2]. In this paper, we argue that this failure to make the unique prediction $(n, m) = (3, 1)$ is *not* a weakness of such theories, but a strength. To compute the theoretically predicted probability distribution for the dimensionality of our spacetime‡, we clearly need to take into account the selection effect arising from the fact that some of these states are more likely to contain self-aware observers such as us than others. This is completely analogous to the familiar selection effect in cosmological galaxy surveys, where we must take into account that bright galaxies are more likely to be sampled than faint ones [3]. Below we will argue that if observers can only exist in a world exhibiting a certain minimum complexity, predictability and stability, then all such ensemble theories may actually predict that we should find ourselves inhabiting a $(3+1)$-dimensional spacetime with 100% certainty, as illustrated in figure 1, and that the Bayesian prior probabilities of quantum-mechanical origin are completely irrelevant. We will first review some old but poorly known results regarding the number of spatial dimensions (when $m = 1$), then present some new arguments regarding

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‡ Hereafter, we let n and m refer to the number of *non-compactified* space and time dimensions, or more generally to the effective spacetime dimensionality that is relevant to the low-energy physics we will be discussing later.

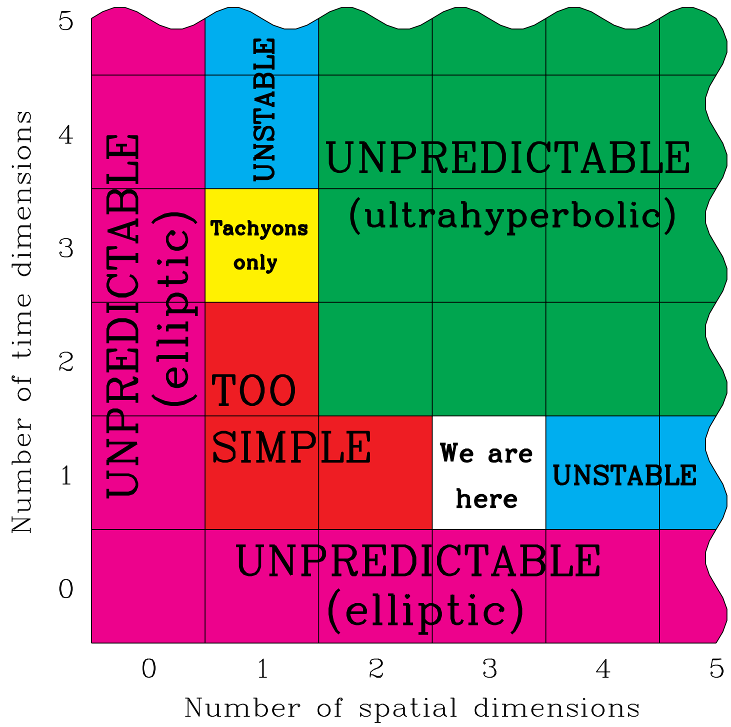

Figure 1. When the partial differential equations of nature are elliptic or ultrahyperbolic, physics has no predictive power for an observer. In the remaining (hyperbolic) cases, $n > 3$ may fail on the stability requirement (atoms are unstable) and $n < 3$ may fail on the complexity requirement (no gravitational attraction, topological problems).

(This figure can be viewed in colour in the electronic version of the article; see `http://www.iop.org/EJ/welcome`)

the number of time dimensions. In both cases, we are *not* attempting to rigorously show that merely $(n, m) = (3, 1)$ permits observers. Rather, we are simply arguing that it is far from obvious that any other (n, m) permits observers, since radical qualitative changes occur in all cases, so that the burden of proof of the contrary falls on the person wishing to criticize ensemble theories with fine-tuning arguments.

As was pointed out by Ehrenfest back in 1917 [4], neither classical atoms nor planetary orbits can be stable in a space with $n > 3$, and traditional quantum atoms cannot be stable either [5]. These properties are related to the fact that the fundamental Green's function of the Poisson equation $\nabla^2\phi = \rho$, which gives the electrostatic/gravitational potential of a point particle, is r^{2-n} for $n > 2$. Thus, the inverse-square law of electrostatics and gravity becomes an inverse-cube law if $n = 4$, etc. When $n > 3$, the two-body problem no longer has any stable orbits as solutions [6]. This is illustrated in figure 2, where a swarm of light test particles are incident from the left on a massive point particle (the black dot), all with the same momentum vector but with a range of impact parameters. There are two cases: those that start outside the shaded region escape to infinity, whereas those with smaller impact parameters spiral into a singular collision in a finite time. We can think of this as there being a finite cross section for annihilation. This is of course in stark contrast to the familiar case, $n = 3$, which gives either stable elliptic orbits or non-bound parabolic and hyperbolic orbits, and has no 'annihilation solutions' except for the measure

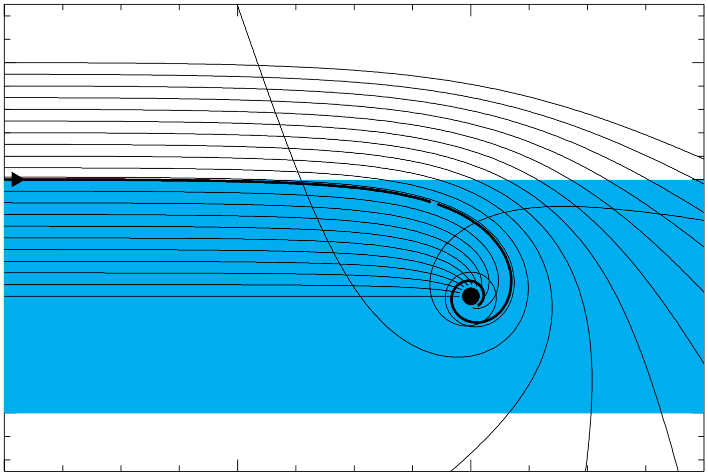

Figure 2. The-two body problem in four-dimensional space: the light particles that approach the heavy one at the centre either escape to infinity or get sucked into a cataclysmic collision. There are no stable orbits.

(This figure can be viewed in colour in the electronic version of the article; see `http://www.iop.org/EJ/welcome`)

zero case where the impact parameter is exactly zero. A similar disaster occurs in quantum mechanics, where a study of the Schrödinger equation shows that the hydrogen atom has no bound states for $n > 3$ [5]. Again, there is a finite annihilation cross section, which is reflected by the fact that the hydrogen atom has no ground state, but time-dependent states of arbitrarily negative energy. The situation in general relativity is analogous [5]. Modulo the important caveats in the discussion section, this means that such a world cannot contain any objects that are stable over time, and thus probably cannot contain stable observers.

What about $n < 3$? It has been argued [7] that organisms would face insurmountable topological problems if $n = 2$: for instance, two nerves cannot cross. Another problem, emphasized by Wheeler *et al* [8], is the well known fact (see e.g. [9]) that there is no gravitational force in general relativity with $n < 3$. We will not spend more time listing problems with $n < 3$, but simply conjecture that since $n = 2$ (let alone $n = 1$ and $n = 0$) offers vastly less complexity than $n = 3$, worlds with $n < 3$ are just too simple and barren to contain observers.

Here, we will present an argument for why a world with the same laws of physics as ours and with an $(n+m)$-dimensional spacetime can only contain observers if the number of time dimensions $m = 1$, regardless of the number of space dimensions, n. Before describing this argument, which involves hyperbolicity properties of partial differential equations (PDEs), let us make a few general comments about the dimensionality of time.

What would reality appear like to an observer in a manifold with more than one time-like dimension? Even when $m > 1$, there is no obvious reason why an observer could not, none the less, *perceive* time as being one-dimensional, thereby maintaining the pattern of having 'thoughts' in a one-dimensional succession that characterizes our own reality perception. If the observer is a localized object, it will travel along an essentially one-dimensional (time-like) world line through the $(n+m)$-dimensional spacetime manifold. The standard general relativity notion of its proper time is perfectly well defined, and we would expect this to be the time that it would measure if it had a clock and that it would subjectively experience.

Needless to say, many aspects of the world would none the less appear quite different. For instance, a re-derivation of relativistic mechanics for this more general case shows that energy now becomes an m-dimensional vector rather than a constant, whose direction determines in which of the many time directions the world line will continue, and in the non-relativistic limit, this direction is a constant of motion. In other words, if two non-relativistic observers that are moving in different time directions happen to meet at a point in spacetime, they will inevitably drift apart in separate time directions again, unable to stay together.

Another interesting difference, which can be shown by an elegant geometrical argument [10], is that particles become less stable when $m > 1$. For a particle to be able to decay when $m = 1$, it is not sufficient that a set of particles with the same quantum numbers exists. It is also necessary, as is well known, that the sum of their rest masses should be less than the rest mass of the original particle, regardless of how great its kinetic energy may be. When $m > 1$, this constraint vanishes [10]. For instance,

- a proton can decay into a neutron, a positron and a neutrino,
- an electron can decay into a neutron, an antiproton and a neutrino, and
- a photon of sufficiently high energy can decay into any particle and its antiparticle.

In addition to these two differences, one can concoct seemingly strange occurrences involving 'backward causation' when $m > 1$. None the less, although such unfamiliar behaviour may appear disturbing, it would seem unwarranted to assume that it would prevent any form of observer from existing. After all, we must avoid the fallacy of assuming that the design of our human bodies is the only one that allows self-awareness. Electrons, protons and photons would still be stable if their kinetic energies were low enough, so perhaps observers could still exist in rather cold regions of a world with $m > 1$†.

There is, however, an additional problem for observers when $m > 1$, which has not been previously emphasized even though the mathematical results on which it is based are well known. If an observer is to be able to make any use of its self-awareness and information-processing abilities, the laws of physics must be such that it can make at least some predictions. Specifically, within the framework of a field theory, it should, by measuring various nearby field values, be able to compute field values at some more distant spacetime points (ones lying along its future world line being particularly useful) with non-infinite error bars. If this type of well-posed causality were absent, then not only would there be no reason for observers to be self-aware, but it would appear highly unlikely that information processing systems (such as computers and brains) could exist at all.

Although this predictability requirement may sound modest, it is in fact only met by a small class of PDEs, essentially those which are hyperbolic. We will now discuss the classification and causal structure of PDEs in some detail. This mathematical material is well known, and can be found in more detail in [12]. Given an arbitrary second-order linear partial differential equation in $\mathbb{R}^d$,

$$\left[\sum_{i=1}^{d}\sum_{j=1}^{d}\mathsf{A}_{ij}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}+\sum_{i=1}^{d}\boldsymbol{b}_i\frac{\partial}{\partial x_i}+c\right]u=0, \tag{1}$$

where the matrix A (which we, without loss of generality, can take to be symmetric), the vector $\boldsymbol{b}$ and the scalar c are given differentiable functions of the d coordinates, it is customary to classify it depending on the signs of the eigenvalues of A. The PDE is said to be

- *elliptic* in some region of $\mathbb{R}^d$ if they are all positive or all negative there,

† It is, however, far from trivial to formulate a quantum field theory with a stable vacuum state when $m > 1$ [11].

- *hyperbolic* if one is positive and the rest are negative (or vice versa), and
- *ultrahyperbolic* in the remaining case, i.e. where at least two eigenvalues are positive and at least two are negative.

What does this have to do with the dimensionality of spacetime? For the various covariant field equations of nature that describe our world (the wave equation $u_{;\mu\mu} = 0$, the Klein–Gordon equation $u_{;\mu\mu} + m^2 u = 0$, etc†), the matrix $\mathbf{A}$ will clearly have the same eigenvalues as the metric tensor. For instance, they will be hyperbolic in a metric of the signature $(+ - - -)$, corresponding to $(n, m) = (3, 1)$, elliptic in a metric of the signature $(+ + + + +)$, and ultrahyperbolic in a metric of the signature $(+ + - -)$.

One of the merits of this standard classification of PDEs is that it determines their causal structure, i.e. how the boundary conditions must be specified to make the problem *well-posed*. Roughly speaking, the problem is said to be well-posed if the boundary conditions determine a unique solution, u, and if the dependence of this solution on the boundary data (which will always be linear) is *bounded*. The last requirement means that the solution u at a given point will only change by a finite amount if the boundary data is changed by a finite amount. Therefore, even if an ill-posed problem can be formally solved, this solution would in practice be useless to an observer, since it would need to measure the initial data with infinite accuracy to be able to place finite error bars on the solution (any measurement error would cause the error bars on the solution to be infinite).

Elliptic equations allow well-posed *boundary value problems*. On the other hand, giving 'initial' data for an elliptic PDE on a non-closed hypersurface, say a plane, is an ill-posed problem. This means that an observer in a world with no time dimensions ($m = 0$) would not be able do make any inferences at all about the situation in other parts of space based on what it observes locally.

Hyperbolic equations, on the other hand, allow well-posed *initial-value problems*. For example, specifying initial data (u and $\dot{u}$) for the Klein–Gordon equation on the shaded disk in figure 3 determines the solution in the volumes bounded by the two cones, including the (missing) tips. A localized observer can therefore make predictions about its future. If the matter under consideration is of such a low temperature that it is non-relativistic, then the fields will essentially contain only Fourier modes with wavenumbers $|\boldsymbol{k}| \ll m$, which means that for all practical purposes, the solution at a point is determined by the initial data in a 'causality cone' with an opening angle much narrower than 45°.

In contrast, if the initial data for a hyperbolic PDE is specified on a hypersurface that is not space-like, the problem becomes ill-posed. Figure 3, which is based on [12], provides an intuitive understanding of what goes wrong. A corollary of a remarkable theorem by Asgeirsson [13] is that if we specify u in the cylinder in figure 3, then this determines u throughout the region made up of the truncated double cones. Letting the radius of this cylinder approach zero, we obtain the disturbing conclusion that providing data in a, for all practical purposes, one-dimensional region determines the solution in a three-dimensional region. Such an apparent 'free lunch', where the solution seems to contain more information than input data, is a classical symptom of ill-posedness. The price that must be paid is specifying the input data with infinite accuracy, which is of course impossible given real-world measurement errors. Clearly, generic boundary data allows no solution at all, since it is not self-consistent. It is easy to see that the same applies when specifying 'initial' data on part of a non-space-like hypersurface, e.g. that given by $y = 0$. These properties are

† Our discussion will apply to matter fields with spin as well, e.g. fermions and photons, since spin does not alter the causal structure of the solutions. For instance, all four components of an electron–positron field obeying the Dirac equation satisfy the Klein–Gordon equation as well, and all four components of the electromagnetic vector potential in Lorentz gauge satisfy the wave equation.

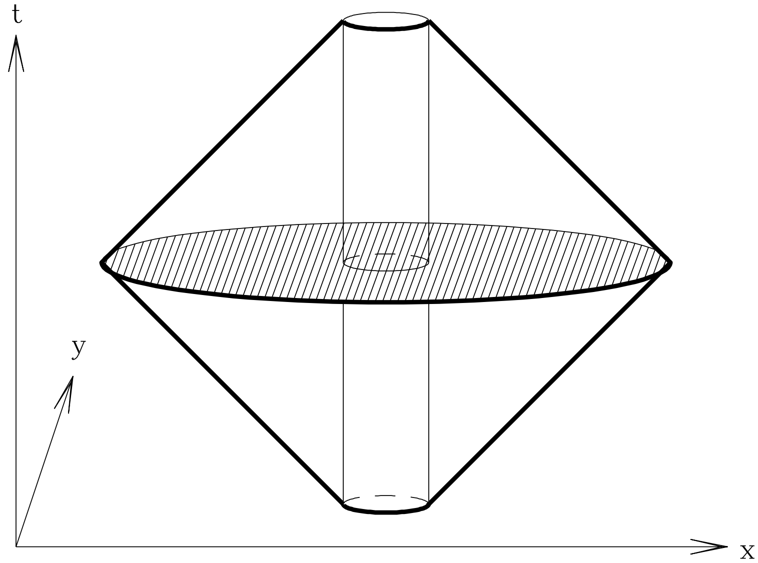

Figure 3. The causality structure for hyperbolic and ultrahyperbolic equations.

analogous in $n+1$ dimensions, and illustrate why an observer in an $(n+1)$-dimensional spacetime can only make predictions in time-like directions.

Asgeirsson's theorem also applies to the *ultrahyperbolic* case as well, showing that initial data on a hypersurface containing both space-like and time-like directions leads to an ill-posed problem. However, since a hypersurface by definition has a dimensionality which is one less than that of the spacetime manifold (data on a submanifold of lower dimensionality can never give a well-posed problem), *there are no space-like or time-like hypersurfaces* in the ultrahyperbolic case, and hence no well-posed problems†.

Since a mere minus sign distinguishes space from time, the remaining case $(n, m) = (1, 3)$ is mathematically equivalent to the case where $(n, m) = (3, 1)$ and all particles are tachyons [14] with imaginary rest mass. Also in this case, an observer would be unable to make any predictions since, as described in more detail in [15], well-posed problems require data to be specified in the non-local region *outside* the lightcones.

Above we discussed only linear PDEs, although the full system of coupled PDEs in nature is of course nonlinear. This in no way weakens our conclusions about only $m = 1$ giving well-posed initial-value problems. When PDEs give ill-posed problems even *locally*, in a small neighbourhood of a hypersurface (where we can generically approximate the nonlinear PDEs with linear ones), it is obvious that no nonlinear terms can make them well-posed in a larger neighbourhood.

Our conclusions are graphically illustrated in figure 1: given the other laws of physics, it is not implausible that only a $(3+1)$-dimensional spacetime can contain observers that are complex and stable enough to be able to understand and predict their world to any extent at all, for the following reasons.

- More or less than one time dimension: insufficient predictability.
- More than three space dimensions: insufficient stability.

† The only remaining possibility is the rather contrived case where data is specified on a null hypersurface. To measure such data, an observer would need to 'live on the light cone', i.e. travel with the speed of light, which means that it would subjectively not perceive any time at all (its proper time would stand still).

- Less than three space dimensions: insufficient complexity.

Thus, although application of the so-called weak anthropic principle [16] does in general *not* appear to give very strong predictions for physical constants [17], its dimensionality predictions may indeed turn out to give the narrowest probability distribution possible. Viewed in this light, the multiple dimensionality prediction of some superstring theories is a strength rather than a weakness, since it eliminates the otherwise embarrassing discrete fine-tuning problem of having to explain the 'lucky coincidence' that the compactification mechanism itself happened to single out only a $(3+1)$-dimensional spacetime.

Needless to say, we have not attempted to rigorously demonstrate that observers are impossible for other dimensionalities. For instance, within the context of specific models, one might consider exploring the possibility of stable structures in the case $(n, m) = (4, 1)$ based on short distance quantum corrections to the $1/r^2$ potential or on string-like (rather than point-like) particles. We have simply argued that it is far from obvious that any other combination other than $(n, m) = (3, 1)$ permits observers, since radical qualitative changes occur when n or m are altered. For this reason, a theory cannot be criticized for failing to predict a definite spacetime dimensionality until the stability and predictability issues raised here have been carefully analysed.

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