TOKYO J. MATH.
VOL. 25, NO. 2, 2002

On the Diophantine Equations $\binom{n}{2} = cx^l$ and $\binom{n}{3} = cx^l$

Nobuhiro TERAI

Ashikaga Institute of Technology

(Communicated by K. Komatsu)

Dedicated to Professor Norio Adachi on the occasion of his 60th birthday

Abstract. Upper bounds for the solution l of the equations of the title are derived by using results concerning the equation $ax^l - by^l = c$ with a, b, c non-zero integers. These solutions are also determined in some special cases.

1. Introduction.

The Diophantine equations on binomial coefficients have been studied by several authors. We consider the Diophantine equation

$$\binom{n}{k} = x^l \tag{1}$$

in integers $n,\ k,\ l,\ x$ with $k > 1, l > 1, x > 1$. There is no loss of generality in assuming $n \geq 2k$, since $\binom{n}{k} = \binom{n}{n-k}$. It is clear that (1) has infinitely many solutions if $k = l = 2$. For $k = 3$ and $l = 2$, equation (1) has only the solution $n = 50,\ x = 140$, as shown by Watson [W] and Ljunggren [Lj2] independently. In 1939, Erdös [P1] showed that no solutions exist if $k \geq 2^l$ or $l = 3$, and conjectured that if $l \geq 3$, then equation (1) has no solutions. Using an elementary way, Erdös [P2] established in 1951 that if $k > 3$, then equation (1) has no solutions.

For the remaining cases $k = 2$ and $k = 3$, Tijdeman [Ti] proved in 1976, by means of an effective inequality of Baker on linear forms in logarithms, that there are only finitely many solutions all of which can be effectively determined. Terai [Te] used a recent estimate of linear forms in two logarithms to show that $l < 4250$. It immediately follows from a result of Darmon-Merel [DM] that equation (1) has no solutions for $k = 2$. Their result is derived by the deep theory of elliptic curves, including Wiles' proof of most cases of Shimura-Taniyama conjecture. Györy [G2] settled the case $k = 3$ by combining a congruence result due to himself, a result of Darmon-Merel [DM] and a result of Bennett-de Weger [BW] obtained by linear forms in logarithms and the hypergeometric method.

Received September 22, 2000

In this paper, we consider the following more general equations than (1) for $k = 2, 3$:

$$\binom{n}{2} = cx^l , \tag{2}$$

$$\binom{n}{3} = cx^l \tag{3}$$

with c positive integer.

Our method is based on the deep results concerning the Diophantine equation

$$ax^l - by^l = c \tag{4}$$

with a, b, c non-zero integers.

Using an estimate of Mignotte [M], we derive an upper bound for the solution l of (2) in terms of c (Theorem 1). Further, combining the results of Ribet [Rt] and Darmon-Merel [DM], we also derive an absolute upper bound for the solution l of (3) (Theorem 7). The results of Győry [G1] and Bennett [Be] enable us to determine the solutions of (2) under some conditions when $c = p$, prime number of the form $p = 2a^l + 1$ (Theorem 2). It is worth noting that equation (2) for $l = 3, 4$ and equation (3) for $l = 2, 3, 4$ can be easily solved by using the program packages *KASH* (cf. [Ka]) and *SIMATH* (cf. [Si, Sections 4, 6]).

2. The results on the equation $ax^l - by^l = c$.

In the proof of our theorems, we need the deep results on equation (4).

The following proposition is shown by using a lower bound for linear forms in two logarithms.

PROPOSITION 1 (Mignotte [M]). *Let a, b be positive integers with $a \neq b$ and c be a non-zero integer. Put $A = \max\{a, b, 3\}$ and $\lambda = \log\left(1 + \dfrac{\log A}{|\log(a/b)|}\right)$. Let l be a positive integer ≥ 3. Suppose that equation* (4) *has integer solutions x, y with $y > |x| > 0$. Then*

$$l \leq \max\left\{3\log(1.5|c/b|),\ 7400\frac{\log A}{\lambda}\right\} .$$

The following powerful proposition depends on the multi-dimensional "hypergeometric method".

PROPOSITION 2 (Bennett [Be]). *Let a, b and l be positive integers with $l \geq 3$. Then the Diophantine equation*

$$|ax^l - by^l| = 1$$

has at most one solution in positive integers x, y.

Győry [G3] obtained the following lemma by combining Theorem 3 of Ribet [Rt] with Main theorem of Darmon-Merel [DM], which were proved by means of the theory of elliptic curves.

PROPOSITION 3 (Győry [G3]). *Let l, α be integers with $l \geq 3$ and $0 \leq \alpha < l$. Then the Diophantine equation*

$$x^l - 2^\alpha y^l = \pm 1$$

has no solutions in positive integers x, y except for $\alpha = 1$, where there is only the trivial solution $(x, y) = (1, 1)$.

3. The equation $\binom{n}{2} = cx^l$.

In what follows, n, x, l denote positive integers with $n \geq 2$, $x \geq 2$, $l \geq 3$.

We use Proposition 1 to obtain an upper bound for the solution l of equation (2).

THEOREM 1. *Let c be a positive integer with $c \geq 2$. If the equation* (2) *has solutions n, x, l, then*

$$l < 21352 \log c\,.$$

PROOF. Suppose that equation (2) has solutions n, x, l. Then $n(n-1) = 2cx^l$ and so equation (2) is reduced to solving the following Thue equation:

$$c_1 X^l - c_2 Y^l = 1$$

where c_1, c_2, X, Y are positive integers with $c_1 c_2 = 2c$ and $XY = x$. It follows from Proposition 1 that the equation above yields

$$l \leq 7400 \cdot \frac{\log 2c}{\log 2} < 21352 \log c\,. \qquad \square$$

In the following, under some conditions, we determine the solutions of equation (2). We follow a cogruence method due to Győry [G2] by means of Eisenstein's reciprocity laws.

LEMMA 1 (Győry [G1]). *Let l be a prime > 3. Let a, b, c be integers such that*

$$\frac{a^l + b^l}{a + b} = c^l\,, \quad (a, b) = 1\,, \quad (a^2 - b^2,\ l) = 1\,.$$

Then for each prime r with $r \neq l$ and $r \mid a + b$, we have

$$r^{l-1} \equiv 1 \pmod{l^2}\,.$$

Using Lemma 1, we show the following:

LEMMA 2. *Let l be a prime > 3. Let p be an odd prime with $p \neq l$ and $p \not\equiv 1 \pmod{l}$. If the Diophantine equation*

$$x^l - 2py^l = \pm 1$$

has solutions in positive integers x, y with $y \not\equiv 0 \pmod{l}$, then

$$p^{l-1} \equiv 1 \pmod{l^2}\,.$$

PROOF. Suppose that the equation $x^l - 2py^l = \pm 1$ has solutions in positive integers x, y with $y \not\equiv 0 \pmod{l}$. Since l is odd, we may assume that the right hand side of the equation is positive sign.

If $x + 1 \equiv 0 \pmod{l}$, then $p^{l-1} \equiv 1 \pmod{l^2}$. Indeed, since $x^l \equiv -1 \pmod{l^2}$, we have $-2 \equiv 2py^l \pmod{l^2}$ and hence $1 \equiv p^{l-1} \pmod{l^2}$.

Using Lemma 1, we also show that if $x + 1 \not\equiv 0 \pmod{l}$, then $p^{l-1} \equiv 1 \pmod{l^2}$. Since $p \not\equiv 1 \pmod{l}$ and $y \not\equiv 0 \pmod{l}$, it follows that

$$x - 1 = 2py_1^l\,, \quad \frac{x^l - 1}{x - 1} = y_2^l\,,$$

where y_1, y_2 are integers with $y = y_1 y_2$. If $x + 1 \not\equiv 0 \pmod{l}$, then Lemma 1 implies that $p^{l-1} \equiv 1 \pmod{l^2}$. $\square$

THEOREM 2. *Let l be a prime > 3. Let p be a prime such that $p = 2a^l + 1$ with $a \not\equiv 0 \pmod{l}$. Further, suppose that $p^{l-1} \not\equiv 1 \pmod{l^2}$. Then the Diophantine equation*

$$\binom{n}{2} = px^l$$

with $x \not\equiv 0 \pmod{l}$ has only the positive integer solution $(n, x) = (p, a)$.

PROOF. In view of the proof of Theorem 1, the equation above is reduced to solving the following Thue equations:

$$2X^l - pY^l = \pm 1\,, \tag{5}$$

$$X^l - 2pY^l = \pm 1\,, \tag{6}$$

where X, Y are positive integers with $x = XY$.

Since $p = 2a^l + 1$, it follows from Proposition 2 that the equation (5) has only the solution $(X, Y) = (a, 1)$ and so $x = XY = a$. Hence we obtain $n = p$.

We note that $p \not\equiv 1 \pmod{l}$, since $p = 2a^l + 1$ with $a \not\equiv 0 \pmod{l}$. By Lemma 2, we see that if (6) equation has positive integer solutions X, Y, then $p^{l-1} \equiv 1 \pmod{l^2}$. This completes the proof of Theorem 2. $\square$

We use Theorems 1, 2 to show the following:

COROLLARY 1. *Let l be an odd prime. The Diophantine equation*

$$\binom{n}{2} = 3x^l$$

with $x \not\equiv 0 \pmod{l}$ has only the positive integer solution $(n, x) = (3, 1)$.

PROOF. As shown in the forthcoming Theorem 3, when $l = 3$, the equation above has only the positive integer solution $(n, x) = (3, 1)$. We may assume that $l > 3$.

In view of the proof of Theorem 1, the equation above is reduced to solving the following Thue equations:

$$2X^l - 3Y^l = \pm 1\,, \tag{7}$$

$$X^l - 6Y^l = \pm 1\,, \tag{8}$$

where X, Y are positive integers with $x = XY$.

By Proposition 2, equation (7) has only the solution $X = Y = 1$ and hence $x = XY = 1$, $n = 3$.

By Theorem 1, equation (8) leads to

$$l < 21352 \log 3 = 23457.56958 \cdots .$$

By Lemma 2 with $l > 3$, we also have the congruence

$$3^{l-1} \equiv 1 \pmod{l^2}\,.$$

According to Ribenboim [Rm1, p. 170], it follows that $l = 11$ is the only prime l satisfying the congruence above in the range $3 < l < 23458$. But using *KASH*, we see that the equation

$$X^{11} - 6Y^{11} = \pm 1$$

has no positive integer solutions X, Y. □

Similarly, when $c = 5$, we obtain the following:

COROLLARY 2. *Let l be an odd prime. The Diophantine equation*

$$\binom{n}{2} = 5x^{2l}$$

with $x \not\equiv 0 \pmod{l}$ has no positive integer solutions (n, x).

PROOF. In view of the proof of Theorem 1, the equation above is reduced to solving the following Thue equations:

$$2X^{2l} - 5Y^{2l} = \pm 1\,, \tag{9}$$

$$X^{2l} - 10Y^{2l} = \pm 1\,, \tag{10}$$

where X, Y are positive integers with $x = XY$.

Equation (9) is impossible, because $-1 = \left(\frac{2}{5}\right) = \left(\frac{\pm 1}{5}\right) = 1$, where $\left(\frac{*}{*}\right)$ denotes the Jacobi symbol.

By Theorem 1, equation (10) leads to

$$l \leq \frac{1}{2} \cdot 21352 \log 5 = 17182.35915 \cdots .$$

By Lemma 2 with $l > 5$, we also have the congruence

$$5^{l-1} \equiv 1 \pmod{l^2}\,.$$

According to Ribenboim [Rm1, p. 170], there is no prime l satisfying the congruence above in the range $5 < l < 17183$. Using *KASH*, we see that equation (10) with $l = 3, 5$ has no positive integer solutions X, Y. □

4. The equation $\binom{n}{2} = cx^l$ $(l = 3, 4)$.

In this section, when $l = 3, 4$ and $p = 2a^l + 1$, we determine the solutions of equation (2) under a certain condition.

First consider equation (2) with $l = 3$. Then we need the following lemmas. For the proof of Lemma 3 below, see Ribenboim [Rm1, pp. 96, 106].

LEMMA 3. (i) *The Diophantine equation*

$$x^2 + x + 1 = y^3$$

has only the solutions in integers $(x, y) = (0, 1), (-1, 1), (18, 7), (-19, 7)$.

(ii) *The Diophantine equation*

$$x^2 + x + 1 = 3y^3$$

has only the solutions in integers $(x, y) = (1, 1), (-2, 1)$.

LEMMA 4. *Let* $p = 3$ *or* $p \equiv -1 \pmod 6$ *be a prime, and* m *a non-negative integer. The Diophantine equation*

$$x^3 - 2^m py^3 = \pm 1$$

has only the solutions $(p, m, x, y) = (5, 2, 19, 7), (17, 0, 18, 7)$.

PROOF. First consider the equation $x^3 - 2^m py^3 = \pm 1$ with $p \equiv -1 \pmod 6$. We may assume that the right hand side of the equation is equal to $+1$. Write the equation as $(x - 1)(x^2 + x + 1) = 2^m py^3$. Note that $x^2 + x + 1$ is odd and $p \equiv -1 \pmod 6$. If $y \not\equiv 0 \pmod 3$, then

$$x - 1 = 2^m pu^3 , \quad x^2 + x + 1 = v^3 ,$$

where u, v are integers with $y = uv$. If $y \equiv 0 \pmod 3$, then

$$x - 1 = 2^m p3^{3n-1}u^3 , \quad x^2 + x + 1 = 3v^3 ,$$

where u, v are integers with $y = 3^n uv$, $uv \not\equiv 0 \pmod 3$ and $n \geq 1$. Hence it follows from Lemma 3 that the equation $x^3 - 2^m py^3 = \pm 1$ has only the solutions $(p, m, x, y) = (5, 2, 19, 7), (17, 0, 18, 7)$.

Similarly the equation $x^3 - 2^m \cdot 3y^3 = \pm 1$ has no positive integer solutions. □

THEOREM 3. *Let* p *be a prime such that* $p = 2a^3 + 1$ *with* $a = 1$ *or* $a \equiv -1 \pmod 3$. *The Diophantine equation*

$$\binom{n}{2} = px^3 \tag{11}$$

has only the positive integer solution $(n, x) = (p, a)$.

PROOF. In view of the proof of Theorem 1, it suffices to consider the following equations:

$$2X^3 - pY^3 = \pm 1\,,$$

$$X^3 - 2pY^3 = \pm 1\,,$$

where X, Y are positive integers with $x = XY$.

Since $p = 2a^3 + 1$, it follows from Proposition 2 that the first equation above has only the solution $(X, Y) = (a, 1)$ and so $x = a$. Hence we obtain $n = p$.

We note that $p = 3$ or $p \equiv -1 \pmod 6$, since $p = 2a^3 + 1$ with $a = 1$ or $a \equiv -1 \pmod 3$. By Lemma 4 with $m = 1$, the second equation above has no positive integer solutions X, Y. □

THEOREM 4. *Let p be a prime with $3 \le p < 100$. Then equation* (11) *has only the following positive integer solutions n, x*:

$$p = 3 : (n, x) = (3, 1)\,; \quad p = 13 : (n, x) = (27, 3)\,;$$

$$p = 17 : (n, x) = (17, 2)\,; \quad p = 53 : (n, x) = (54, 3)\,.$$

PROOF. Since p is prime, equation (11) is reduced to solving the equations $2X^3 - pY^3 = \pm 1$ and $X^3 - 2pY^3 = \pm 1$. These Thue equations of third degree can be solved by using *KASH*. Then there are only the solutions listed above. □

Next consider equation (2) with $l = 4$. We need the following lemmas.

LEMMA 5 (Ljunggren [Lj1]). *The Diophantine equation*

$$x^2 + 1 = 2y^4$$

has only the positive integer solutions $(x, y) = (1, 1), (239, 13)$.

LEMMA 6. *Let $p \equiv 3 \pmod 4$ be a prime, and m a non-negative integer. The Diophantine equation*

$$x^4 - 2^m py^4 = \pm 1$$

has no positive integer solutions x, y.

PROOF. We may assume that y is odd. Since $p \equiv 3 \pmod 4$, the right hand side of the equation is equal to $+1$.

If $m > 0$, then from $p \equiv 3 \pmod 4$ the equation leads to

$$x^2 + 1 = 2u^4\,, \quad x^2 - 1 = 2^{m-1}pv^4\,,$$

where u, v are positive integers with $y = uv$. By Lemma 5, the first equation above has only the positive integer solutions $(x, u) = (1, 1), (239, 13)$, which do not satisfy the second equation.

If $m = 0$, then the equation leads to

$$x^2 + 1 = u^4\,, \quad x^2 - 1 = pv^4\,,$$

which has no positive integer solutions. □

THEOREM 5. *Let p be a prime such that $p = 2a^4+1$ with a odd ≥ 1. The Diophantine equation*

$$\binom{n}{2} = px^4 \tag{12}$$

has only the positive integer solution $(n, x) = (p, a)$.

PROOF. In view of the proof of Theorem 1, it suffices to consider the following equations:

$$2X^4 - pY^4 = \pm 1\,,$$

$$X^4 - 2pY^4 = \pm 1\,,$$

where X, Y are positive integers with $x = XY$.

Since $p = 2a^4 + 1$, it follows from Proposition 2 that the first equation above has only the solution $(X, Y) = (a, 1)$ and so $x = a$. Hence we obtain $n = p$.

We note that $p \equiv 3 \pmod 4$, since $p = 2a^4 + 1$ with a odd ≥ 1. By Lemma 6 with $m = 1$, the second equation above has no positive integer solutions X, Y. □

THEOREM 6. *Let p be a prime with $3 \leq p < 100$. Then equation* (12) *has only the following positive integer solutions n, x*:

$$p = 3 : (n, x) = (3, 1)\,; \quad p = 31 : (n, x) = (32, 2)\,; \quad p = 41 : (n, x) = (82, 3)\,;$$

PROOF. Since p is prime, equation (12) is reduced to solving the equations $2X^4 - pY^4 = \pm 1$ and $X^4 - 2pY^4 = \pm 1$. These Thue equations of fourth degree can be solved by using *KASH*. Then there are only the solutions listed above. □

5. The equation $\binom{n}{3} = cx^l$.

In what follows, n, x, l denote positive integers with $n \geq 3$, $x \geq 2$, $l \geq 3$.

In this section, we treat equation (3) when $c = p^m$ with p prime > 3 and $m \geq 1$. Using Propositions 1, 3, we show the following:

THEOREM 7. *Let p be a prime > 3 and m a positive integer. If the Diophantine equation*

$$\binom{n}{3} = p^m x^l$$

has solutions n, x, l, then

$$l \leq 19128\,.$$

PROOF. Suppose that the equation $\binom{n}{3} = p^m x^l$ has a solution n, x, l. Then

$$n(n-1)(n-2) = 2 \cdot 3 \cdot p^m x^l\,.$$

Now we distinguish two cases: (i) n is odd and (ii) n is even.

Case (i): n is odd. Then we have the following three subcases according as n, $n-1, n-2$ is divisible by p. In what follows, x_1 and x_2 denote a positive integer > 1.

(i,1) $(n-1)(n-2) = 2x_1^l$ or $6x_1^l$.

The first equation has no solutions by Proposition 3. The solution l of the second equation must satisfy $l \leq 19128$ by Proposition 1.

(i,2) $n(n-2) = x_1^l$ or $3x_1^l$.

It is clear that the first equation has no solutions since n is odd. From the second equation, we have

$$X^l - 3Y^l = \pm 2\,.$$

Thus it follows from Proposition 1 that $l \leq 11728$.

(i,3) $n(n-1) = 2x_1^l$ or $6x_1^l$.

This case is similar to the case (i,1).

Case (ii): n is even. Then we have the following three subcases according as n, $n-1, n-2$ is divisible by p.

(ii,1) $(n-1)(n-2) = 2^\alpha x_1^l$, $3x_1^l$ or $6x_1^l$ with $\alpha = 1, l$.

The first equation has no solutions by Proposition 3. From the second equation, we have

$$X^l - 3Y^l = \pm 1\,.$$

Thus it follows from Proposition 1 that $l \leq 11728$. The solution l of the third equation must satisfy $l \leq 19128$ by Proposition 1.

(ii,2) $n(n-2) = 2x_1^l$ or $6x_1^l$.

Note that n is even. From the first equation, we have $\frac{n}{2}\left(\frac{n}{2}-1\right) = 2^{l-1}x_2^l$, which has no solutions by Proposition 3. From the second equation, we have

$$X^l - 6Y^l = \pm 2$$

or

$$3X^l - 2Y^l = \pm 2\,.$$

By Proposition 1, the solution l of these equations must satisfy $l \leq 19128$, $l \leq 6201$, respectively.

(ii,3) $n(n-1) = 2^\alpha x_1^l$, $3x_1^l$ or $6x_1^l$ with $\alpha = 1, l$.

This case is similar to the case (ii,1). □

6. The equation $\binom{n}{3} = cx^l$ $(l = 2, 3, 4)$.

In this section, we treat some special cases of equation (3) for $l = 2, 3, 4$.

We first consider equation (3) when $l = 2$. More generally, consider the following:

$$n(n-1)(n-2) = dx^2 \tag{13}$$

for a fixed square-free positive integer d. Then equation (13) is easily reduced to the elliptic curve

$$E_d : \ Y^2 = X^3 - d^2 X \, ,$$

where X,Y are positive integers such that $Y = d^2 x, X = d(n-1)$. For each "small" d, all integral points on this elliptic curve E_d can be easily determined by using *SIMATH*. It should be noted that there is a close relationship between congruent numbers and the above E_d. Recall that a square-free positive integer d is called a *congruent number* if it is the area of some right triangle with rational sides. The following fact is well known (cf. Koblitz [Ko, p. 46]):

$$d \text{ is a congruent number } \Leftrightarrow \ E_d(\mathbf{Q}) \text{ has positive rank.}$$

It is conjectured that if $d \equiv 5, 6$, or $7 \pmod 8$, then d is a congruent number.

THEOREM 8. (i) *Let p_3, q_3, p_5, q_5 be a prime such that $p_3 \equiv q_3 \equiv 3 \pmod 8$ and $p_5 \equiv q_5 \equiv 5 \pmod 8$. If $d = p_3, p_3q_3, 2p_3q_3, 2p_5$, or $2p_5q_5$, then equation* (13) *has no positive integer solutions n, x.*

(ii) *Let d be a square-free positive integer with $1 \le d < 20$. Then equation* (13) *has only the following positive integer solutions n, x:*

$$d = 5 : (n, x) = (10, 12)\,; \quad d = 6 : (n, x) = (3, 1), (4, 2), (50, 140)\,;$$
$$d = 14 : (n, x) = (9, 6)\,; \quad d = 15 : (n, x) = (5, 2).$$

PROOF. (i) Any d of the above forms is not a congruent number. (See Serf [Se, p. 232]. See also Koblitz [Ko, p. 222] and Tunnell [Tu].) Note that d is not a congruent number if and only if all integral points on E_d are $(X, Y) = (0, 0), (d, 0), (-d, 0)$. Hence equation (13) has no positive integer solutions n, x.

(ii) Using *SIMATH*, we compute all integral points on elliptic curves E_d for the values of d above. Then we obtain all positive integer solutions n, x of equation (13) for each d. □

We next consider equation (3) when $l = 3$.

THEOREM 9. *Let p be a prime > 3 and m a positive integer. The Diophantine equation*

$$\binom{n}{3} = p^m x^3$$

has no positive integer solutions n, x.

PROOF. In view of the proof of Theorem 7, we see that the equation $\binom{n}{3} = p^m x^3$ is reduced to solving the Thue equations of third degree, which can be solved by Proposition 2, Lemma 4 and *KASH*. Note that *KASH* is used for only the equation

$$X^3 - 3Y^3 = \pm 2\,.$$

This Thue equation has only the solution $(X, Y) = (1, 1)$. Hence our assertion follows. □

We finally consider equation (3) when $l = 4$.

THEOREM 10. *Let p be a prime > 3 and m a positive integer. The Diophantine equation*

$$\binom{n}{3} = p^m x^4$$

has no positive integer solutions n, x.

PROOF. In view of the proof of Theorem 7, we see that the equation $\binom{n}{3} = p^m x^4$ is reduced to solving the Thue equations of fourth degree, which can be solved by Proposition 2, Lemma 6 and *KASH*. Note that *KASH* is used for only the equations

$$X^4 - 3Y^4 = \pm 2\,, \quad 8X^4 - 3Y^4 = \pm 1\,.$$

The first equation has only the solution $(X, Y) = (1, 1)$, and the second equation has no solutions. Hence our assertion follows. □

ACKNOWLEDGMENT. The author would like to thank Professor Györy for his valuable suggestions.

References

[Be] M. A. BENNETT, Rational approximation to algebraic numbers of small height: The diophantine equation $|ax^n - by^n| = 1$, J. Reine Angew. Math. **535** (2001), 1–49.

[BW] M. A. BENNETT and B. M. M. DE WEGER, On the diophantine equation $|ax^n - by^n| = 1$, Math. Comp. **67** (1998), 413–438.

[DM] H. DARMON and L. MEREL, Winding quotients and some variants of Fermat's Last Theorem, J. Reine Angew. Math. **490** (1997), 81–100.

[P1] P. ERDÖS, Note on the product of consecutive integers (II), J. London Math. Soc. **14** (1939), 245–249.

[P2] P. ERDÖS, On a Diophantine equation, J. London Math. Soc. **26** (1951), 176–178.

[G1] K. GYÖRY, Über die diophantische Gleichungen $x^p + y^p = cz^p$, Publ. Math. Debrecen **13** (1966), 301–305.

[G2] K. GYÖRY, On the diophantine equation $\binom{n}{k} = x^l$, Acta Arith. **80** (1997), 289–295.

[G3] K. GYÖRY, On the diophantine equation $n(n+1)\cdots(n+k-1) = bx^l$, Acta Arith. **83** (1998), 87–92.

[Ka] M. DABERKOW, C. FIEKER, J. KLÜNER, M. POHST, K. ROEGNER, M. SCHÖRNIG and K. WILDANGER, *KANT* V4, J. Symb. Comput. **24** (1997), 267–283, Computational algebra and number theory (London, 1993).

[Ko] N. KOBLITZ, *Introduction to Elliptic Curves and Modular Forms*, GTM 97, Springer (1984).

[Lj1] W. LJUNGGREN, Zur Theoie der Gleichung $x^2 + 1 = Dy^4$, Avh. Norske Vid. Akad. Oslo **5** (1942), 1–27.

[Lj2] W. LJUNGGREN, New solution of a problem proposed by E. Lucas, Norsk Mat. Tidsskr. **34** (1952), 65–72.

[M] M. MIGNOTTE, A note on the equation $ax^n - by^n = c$, Acta Arith. **75** (1996), 287–295.

[Rt] K.A. RIBET, On the equation $a^p + 2^\alpha b^p + c^p = 0$, Acta Arith. **79** (1997), 7–16.

[Rm1] P. RIBENBOIM, *The Little Book of Big Primes*, Springer (1991).

[Rm2] P. RIBENBOIM, *Catalan's Conjecture*, Academic Press (1944).

[Se] P. SERF, Congruent numbers and elliptic curves, *Computational Number Theory*, (ed. by A. Pethö et al.), Walter de Gruyter (1991), 227–238.

[Si] C. HOLLINGER and P. SERF, SIMATH, – a computer algebra system, *Computational Number Theory* (ed. by A. Pethö et al.), Walter de Gruyter (1991), 331–342.

[Ti] R. TIJDEMAN, Applications of the Gel'fond-Baker method to rational number theory, *Topics in Number Theory*, North-Holland (1976), 399–416.

[Te] N. TERAI, On a Diophantine equation of Erdös, Proc. Japan Acad. **70** Ser. A (1994), 213–217.

[Tu] J. TUNNELL, A classical Diophantine problem and mudular forms of weight $3/2$, Invent. Math. **72** (1983), 323–334.

[W] G.N. WATSON, The problem of the square pyramid, Messenger of Math. **48** (1918/19), 1–22.

Present Address:
DIVISION OF GENERAL EDUCATION, ASHIKAGA INSTITUTE OF TECHNOLOGY,
OMAE, ASHIKAGA, TOCHIGI, 326–8558 JAPAN.
e-mail: terai@ashitech.ac.jp