

Literatur

- [1] R. D. Carmichael, *Expansions of arithmetical functions in infinite series*, Proc. London Math. Soc. (2) 34 (1932), S. 1-26.
- [2] E. Cohen, *Fourier expansions of arithmetical functions*, Bull. Amer. Math. Soc. 67 (1961), S. 145-147.
- [3] — *A class of arithmetical functions*, Proc. Nat. Acad. Sci. USA 41 (1955), S. 939-944.
- [4] H. Delange, *Sur les fonctions arithmétiques multiplicatives*, Ann. Scient. de l'École Norm. Sup. 78 (1961), S. 273-304.
- [5] — *On a class of multiplicative arithmetical functions*, Scripta Math. 26 (1963), S. 121-141.
- [6] M. J. Delsarte, *Essai sur l'application de la théorie des fonctions presque périodiques à l'arithmétique*, Ann. Sci. de l'École Norm. Sup. (3) 62 (1945), S. 185-204.
- [7] G. Halász, *Über die Mittelwerte multiplikativer zahlentheoretischer Funktionen*, Acta Math. Acad. Sci. Hung. 19 (1968), S. 365-403.
- [8] G. H. Hardy, *Note on Ramanujan's trigonometrical function $c_Q(n)$ and certain series of arithmetical functions*, Proc. Cambr. Phil. Soc. 20 (1921), S. 263-271.
- [9] S. Ramanujan, *On certain trigonometrical sums and their applications in the theory of numbers*, Transact. Cambr. Phil. Soc. 22 (1918), S. 259-276.
- [10] A. Rényi, *A new proof of a theorem of Delange*, Publ. Math. Debrecen 12 (1965), S. 323-329.
- [11] W. Schwarz, *Einige Bemerkungen über periodische zahlentheoretische Funktionen*, Math. Nachr. 31 (1966), S. 125-136.
- [12] — und J. Spilker, *Eine Anwendung des Approximationssatzes von Weierstraß-Stone auf Ramanujan-Summen*, Nieuw Archief voor Wiskunde (3) 19 (1971), S. 198-209.
- [13] A. Wintner, *Eratosthenian averages*, Baltimore 1943.
- [14] E. Wirsing, *Das asymptotische Verhalten von Summen über multiplikative Funktionen*, Math. Ann. 143 (1961), S. 75-102.
- [15] — *Das asymptotische Verhalten von Summen über multiplikative Funktionen, II*, Acta Math. Acad. Sci. Hung. 18 (1967), S. 411-467.

MATHEMATISCHES INSTITUT
der JOHANN WOLFGANG GOETHE-UNIVERSITÄT

Eingegangen 6. 12. 1971

(243)

On the diophantine equations $\prod_0^k x_i - \sum_0^k x_i = n$ and $\sum_0^k \frac{1}{x_i} = \frac{a}{n}$ *

by

C. VIOLA (Pisa)

1. Introduction. We are concerned with the equations

$$(1) \quad \prod_0^k x_i - \sum_0^k x_i = n$$

and

$$(2) \quad \sum_0^k \frac{1}{x_i} = \frac{a}{n},$$

where a , k and n are given integers and the unknowns x_i take positive integral values. Equation (1) was first considered by Schinzel [9] in the case $n = 0$; he observed that for every k there exists a trivial solution, namely $(1, \dots, 1, 2, k+1)$. Misiurewicz (quoted in [9], Bemerkung) proved that in the case $n = 0$, apart from any permutation of the x_i 's, equation (1) has no solutions different from the trivial one when $k+1 = 2, 3, 4, 6, 24, 114, 174, 444$, while for any other $k < 1000$ there is at least one other solution⁽¹⁾. Later Schinzel conjectured (see [2], p. 238) that there is a $k > 1$ such that, for every sufficiently large n , (1) is soluble in integers $x_i > 1$. Note that equation (1) has for any n , k the trivial solution $(1, \dots, 1, 2, n+k+1)$.

Leonardo Pisano [5] proved in 1202 that for any $a > 0$, $n > 0$, there is a k such that (2) is soluble with $x_i \neq x_j$ for $i \neq j$; obviously k depends on a and n . Many authors (see e. g. [3], [4], [8], [10], [11]) have been concerned with related problems; a classical topic is the investigation of conditions for a positive rational number to be the sum of distinct

* Research supported by Consiglio Nazionale delle Ricerche.

⁽¹⁾ Note added in proof. With the aid of a computer, the result of Misiurewicz has been extended to all $k < 10^4$ (Amer. Math. Monthly 78(1971), pp. 1021-1022).

reciprocals of finitely many integers, belonging to a given sequence of natural numbers. Here we drop the restriction $x_i \neq x_j$; on the other hand we assume that k has a fixed value, independent of a and n . First of all we note that if $\frac{a}{n}$ is the sum of $k+1$ unit fractions:

$$\frac{a}{n} = \frac{1}{x_0} + \frac{1}{x_1} + \dots + \frac{1}{x_k},$$

then it is also the sum of $k+2, k+3, \dots$ unit fractions, since one may replace $\frac{1}{x_k}$ with $\frac{1}{2x_k} + \frac{1}{2x_k}$, etc. Hence (2) is trivially soluble when $a \leq k+1$.

In case $k=1$, one can easily prove that the equation $\frac{a}{n} = \frac{1}{x} + \frac{1}{y}$, $(a, n) = 1$, is soluble in positive integers x and y if and only if there exist d_1, d_2 such that $d_1|n, d_2|n$ and $d_1 + d_2 \equiv 0 \pmod{a}$. A proof is given in [11], Lemma 2. It follows that when $a > 2$ there are infinitely many n for which $\frac{a}{n} = \frac{1}{x} + \frac{1}{y}$ is insoluble. For instance, one may take n to be any prime $p \equiv 1 \pmod{a}$.

In case $k=2$, Schinzel conjectured ([10], p. 25) that for every $a > 0$, if $n > n_0(a)$, $\frac{a}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$ is soluble in positive integers x, y, z . Vaughan [12] has recently proved that the number of natural numbers $n \leq N$ for which $\frac{a}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$ is insoluble is $\ll N \exp\{-C(a)(\log N)^{2/3}\}$, $C(a) > 0$.

The aim of the present paper is to prove results of Vaughan's type for equations (1) and (2); it will then follow that for any a, k such that $a > k+1, k > 1$, the asymptotic density of the natural numbers n for which either (1) or (2) is insoluble is zero.

Since for $k > 1, x_i > 1$ the inequality $\prod_0^k x_i > \sum_0^k x_i$ holds, we may assume throughout that $n > 0, k > 1$ and, in order to avoid the trivial solution of equation (1), we shall impose the condition $x_i > 1$.

Our result is:

THEOREM 1. Let $E_k(N)$ denote the number of natural numbers $n \leq N$ for which

$$(1) \quad \prod_0^k x_i - \sum_0^k x_i = n$$

is insoluble in integers $x_i > 1$, and let $E_{a,k}(N)$ denote the number of natural numbers $n \leq N$ for which

$$(2) \quad \sum_0^k \frac{1}{x_i} = \frac{a}{n}$$

is insoluble in positive integers x_i . Then for $N \rightarrow \infty$

$$(3) \quad E_k(N) \ll N \exp\{-C(k)(\log N)^{1-\frac{1}{k}}\},$$

$$(4) \quad E_{a,k}(N) \ll N \exp\{-C(a, k)(\log N)^{1-\frac{1}{k}}\},$$

with $C(k) > 0, C(a, k) > 0$.

We follow the sifting process already used by Vaughan [12]. However, the method he uses to adapt equation (2) in case $k=2$ to the sieve (which gives the exponent $2/3$ rather than our $1/2$) does not extend to $k > 2$ in an obvious way. We outline here our method; consider for instance equation (1). Write (1) in the form

$$n + x_1 + \dots + x_k = x_0(x_1 \dots x_k - 1) \quad (x_i > 1),$$

or

$$n \equiv - \sum_1^k x_i \pmod{\prod_1^k x_i - 1} \quad (n \geq \prod_1^k x_i - 1);$$

if $m > 0$ is any integer and (x_1, \dots, x_k) is such that $\prod_1^k x_i - 1 = m, x_i > 1, x_i$ integers, then (1) is soluble for every $n \equiv - \sum_1^k x_i \pmod{m}, n \geq m$. A number of such n up to N can be sifted out, provided m runs through the sequence of all prime numbers up to \sqrt{N} . Any upper bound sieve estimate will therefore give us an upper bound for $E_k(N)$.

Some difficulties arise from the fact that for a given prime p two different k -factorizations $(x_1, \dots, x_k), (x'_1, \dots, x'_k)$ of $p+1$ may be such that $\sum_1^k x_i \equiv \sum_1^k x'_i \pmod{p}$. A lower bound $\omega_k(p)$ for the number of k -factorizations $(x_i), (x'_i), \dots$ of $p+1$, such that $\sum_1^k x_i \not\equiv \sum_1^k x'_i, \dots \pmod{p}$, is then obtained by means of Lemma 1, which allows us to apply the sieve for any k .

Theorem 1 is easily deduced from Theorem 2, which gives the average order of the above function $\omega_k(p)$.

The author wishes to thank E. Bombieri and R. C. Vaughan for comments and helpful suggestions.

2. First some preliminary definitions. The ordered k -tuple (x_1, \dots, x_k) is called an *admissible k -factorization* of ν whenever $\prod_1^k x_i = \nu$, $x_i > 1$, x_i integers. Let (x_i) and (x'_i) be two admissible k -factorizations of ν . We say that (x_i) is *r -equivalent* to (x'_i) and write $(x_i) \approx_r (x'_i)$ ($1 \leq r \leq k$), when

$$\sum_1^r x_i = \sum_1^r x'_i.$$

LEMMA 1. Let $(x_1, \dots, x_k), (x'_1, \dots, x'_k)$ be two admissible k -factorizations of ν ; let $1 \leq s \leq r \leq k$ and

$$(x_1, \dots, x_k) \approx_r (x'_1, \dots, x'_k).$$

If

$$x_i \geq \nu^{2/3^i}, \quad x'_i \geq \nu^{2/3^i} \quad (i = 1, \dots, s),$$

then

$$x_i = x'_i \quad (i = 1, \dots, s).$$

Proof. Clearly we may assume $r \geq 2$. Suppose first $s = 1$. We have

$$\xi = x_2 \dots x_k = \frac{\nu}{x_1} \leq \nu^{1/3},$$

$$\xi' = x'_2 \dots x'_k = \frac{\nu}{x'_1} \leq \nu^{1/3};$$

hence

$$\begin{aligned} |x_1 - x'_1| &= \left| \sum_2^r x_i - \sum_2^r x'_i \right| < \max \left\{ \sum_2^r x_i, \sum_2^r x'_i \right\} \leq \max \left\{ \prod_2^k x_i, \prod_2^k x'_i \right\} \\ &= \max \{ \xi, \xi' \} \leq \nu^{1/3}. \end{aligned}$$

If $x_1 \neq x'_1$, then $|\xi - \xi'| \geq 1$, whence

$$|x_1 - x'_1| = \nu \frac{|\xi - \xi'|}{\xi \xi'} \geq \nu^{1/3},$$

a contradiction.

In the general case the proof is by induction on s . We assume that $x_i \geq \nu^{2/3^i}$, $x'_i \geq \nu^{2/3^i}$ ($i = 1, \dots, s-1$) implies $x_i = x'_i$ ($i = 1, \dots, s-1$).

Let $x_i \geq \nu^{2/3^i}$, $x'_i \geq \nu^{2/3^i}$ ($i = 1, \dots, s$). Then

$$(x_1 \dots x_{s-1})^{2/3} x_s \geq \nu^{\frac{2}{3} \sum_1^{s-1} \frac{2}{3^i} + \frac{2}{3^s}} = \nu^{2/3},$$

$$(5) \quad x_s \geq \left(\frac{\nu}{x_1 \dots x_{s-1}} \right)^{2/3},$$

and

$$(6) \quad x'_s \geq \left(\frac{\nu}{x'_1 \dots x'_{s-1}} \right)^{2/3}.$$

It follows from $x_i = x'_i$ ($i = 1, \dots, s-1$) that $(x_s, \dots, x_r, \dots, x_k), (x'_s, \dots, x'_r, \dots, x'_k)$ are admissible $(k-s+1)$ -factorizations of

$$\frac{\nu}{x_1 \dots x_{s-1}} = \frac{\nu}{x'_1 \dots x'_{s-1}},$$

and

$$(x_s, \dots, x_r, \dots, x_k) \approx_{r-s+1} (x'_s, \dots, x'_r, \dots, x'_k);$$

from (5), (6) and the previous argument for the case $s = 1$ we deduce that $x_s = x'_s$, which completes the proof of the lemma.

Note that if (x_i) and (x'_i) are admissible k -factorizations of $p+1$, then it follows from

$$1 < \sum_1^k x_i \leq \prod_1^k x_i = p+1,$$

$$1 < \sum_1^k x'_i \leq \prod_1^k x'_i = p+1$$

that

$$(x_1, \dots, x_k) \approx_k (x'_1, \dots, x'_k)$$

if and only if

$$(7) \quad \sum_1^k x_i \equiv \sum_1^k x'_i \pmod{p}.$$

Also if $p \equiv -1 \pmod{a}$ and $(y_i), (y'_i)$ are admissible k -factorizations of $\frac{p+1}{a}$, we have

$$(y_1, \dots, y_k) \approx_{k-1} (y'_1, \dots, y'_k)$$

if and only if

$$(8) \quad \sum_1^{k-1} y_i \equiv \sum_1^{k-1} y'_i \pmod{p}.$$

We also require some well-known results which we state here as lemmas.

LEMMA 2 (Brun-Titchmarsh). If $q \leq x^a$, $0 < a < 1$, $(q, l) = 1$, then

$$\pi(x; q, l) \ll \frac{x}{\varphi(q) \log x}.$$



LEMMA 3 (Bombieri [1]). For any $A > 0$ there is $B > 0$ such that

$$\sum_{q \leq x^{1/2}(\log x)^{-B}} \max_{y \leq x} \max_{(a, l)=1} \left| \pi(y; q, l) - \frac{\text{li } y}{\varphi(q)} \right| \ll x(\log x)^{-A}.$$

LEMMA 4. Let $d_k(n) = \sum_{\substack{(x_1, \dots, x_k) \\ x_1 \dots x_k = n}} 1$. Then

$$\sum_{n \leq x} \frac{d_k^l(n)}{\varphi(n)} \ll (\log x)^{k^l}.$$

Proof.

$$\begin{aligned} \sum_{n \leq x} \frac{d_k^l(n)}{\varphi(n)} &= \sum_{n \leq x} \frac{d_k^l(n)}{n} \sum_{s|n} \frac{\mu^2(s)}{\varphi(s)} = \sum_{s \leq x} \frac{\mu^2(s)}{\varphi(s)} \sum_{\substack{n \leq x \\ s|n}} \frac{d_k^l(n)}{n} \\ &= \sum_{s \leq x} \frac{\mu^2(s)}{s \varphi(s)} \sum_{r \leq x/s} \frac{d_k^l(rs)}{r} \ll \sum_{s \leq x} \frac{\mu^2(s) d_k^l(s)}{s \varphi(s)} \sum_{r \leq x/s} \frac{d_k^l(r)}{r} \\ &\ll \left(\sum_{s=1}^{\infty} \frac{d_k^l(s)}{s \varphi(s)} \right) \left(\sum_{r \leq x} \frac{d_k^l(r)}{r} \right). \end{aligned}$$

The lemma follows from [6], Lemma 1.1.2 by partial summation.

LEMMA 5 (Montgomery [7]). If $\omega(p)$ ($0 \leq \omega(p) < p$) residue classes modulo p are removed from the first N natural numbers for each prime $p \leq \sqrt{N}$, then the number Z of natural numbers which remain satisfies

$$Z \leq \frac{4N}{\sum_{m \leq \sqrt{N}} \mu^2(m) \prod_{p|m} \frac{\omega(p)}{p - \omega(p)}}.$$

3. First consider equation (1). Let $\sqrt{N} \leq n \leq N$; suppose there are a prime $p \leq \sqrt{N}$ and an admissible k -factorization (x_i) of $p+1$ such that $n \equiv -\sum_1^k x_i \pmod{p}$. Then

$$x_0 = \frac{n + \sum_1^k x_i}{p} > 1,$$

whence

$$\prod_0^k x_i - \sum_0^k x_i = n, \quad x_i > 1.$$

Suppose now that, for every prime $p, p \leq \sqrt{N}$, there are at least $\omega_k(p)$ admissible k -factorizations of $p+1$, no two of which are k -equivalent

to one another. By (7), there are at least $\omega_k(p)$ residue classes modulo p such that, for any n belonging to one of them, $\sqrt{N} \leq n \leq N$, equation (1) is soluble. Hence, by Lemma 5,

$$(9) \quad E_k(N) \leq \frac{4N}{\sum_{m \leq \sqrt{N}} \mu^2(m) \prod_{p|m} \frac{\omega_k(p)}{p - \omega_k(p)}} + \sqrt{N}.$$

If $(x_1, \dots, x_k) \neq (x'_1, \dots, x'_k)$ are two k -factorizations of $p+1$ such that $x_i \geq (p+1)^{2/3^i}, x'_i \geq (p+1)^{2/3^i}$ ($i = 1, \dots, k$), then $(x_1, \dots, x_k), (x'_1, \dots, x'_k)$ are admissible and, by Lemma 1 with $s = r = k$,

$$(x_1, \dots, x_k) \not\approx_k (x'_1, \dots, x'_k).$$

We may therefore define

$$(10) \quad \omega_k(p) = f_k(p+1, p+1),$$

with

$$(11) \quad f_k(p, \xi) = \sum_{\substack{(x_1, \dots, x_k) \\ x_1 \dots x_k = p \\ x_i \geq \xi^{2/3^i}}} 1.$$

Consider now equation (2). Suppose there are a prime $p \equiv -1 \pmod{a}$ and an admissible k -factorization (y_1, \dots, y_k) of $\frac{p+1}{a}$ such that $n \equiv -\sum_1^{k-1} y_i \pmod{p}$; then

$$y_1 + \dots + y_{k-1} + n = y_0 p = y_0 (a y_1 \dots y_k - 1),$$

$$y_0 + y_1 + \dots + y_{k-1} + n = a y_0 y_1 \dots y_k,$$

$$\frac{a}{n} = \frac{y_0 + y_1 + \dots + y_{k-1} + n}{n y_0 y_1 \dots y_k} = \frac{1}{x_0} + \frac{1}{x_1} + \dots + \frac{1}{x_k},$$

with

$$x_0 = n y_1 \dots y_k, \quad x_1 = n y_0 y_2 \dots y_k, \quad \dots, \quad x_k = y_0 y_1 \dots y_k.$$

Suppose that for each $p \equiv -1 \pmod{a}$ there are at least $\omega_{a,k}(p)$ admissible k -factorizations of $\frac{p+1}{a}$, no two of which are $(k-1)$ -equivalent to one another. By (8), there are at least $\omega_{a,k}(p)$ residue classes modulo p such that, for any n belonging to one of them, equation (2) is soluble. It follows from Lemma 5

$$(12) \quad E_{a,k}(N) \leq \frac{4N}{\sum_{m \leq \sqrt{N}} \mu^2(m) \prod_{p|m} \frac{\omega_{a,k}(p)}{p - \omega_{a,k}(p)}},$$

with

$$(13) \quad \omega_{a,k}(p) = \begin{cases} f_k\left(\frac{p+1}{a}, \frac{p+1}{a}\right), & \text{if } p \equiv -1 \pmod{a}, p \geq a, \\ 0, & \text{otherwise,} \end{cases}$$

where f_k is defined by (11). For if $(y_1, \dots, y_k) \neq (y'_1, \dots, y'_k)$ are k -factorizations of $\frac{p+1}{a} > 1$ such that

$$y_i \geq \left(\frac{p+1}{a}\right)^{2/3^i}, \quad y'_i \geq \left(\frac{p+1}{a}\right)^{2/3^i} \quad (i = 1, \dots, k),$$

then

$$(y_1, \dots, y_k) \not\sim_{k-1} (y'_1, \dots, y'_k),$$

by Lemma 1 with $s = r = k - 1$.

By (10) and (13),

$$(14) \quad \omega_k(p) = \omega_{1,k}(p).$$

4. THEOREM 2.

$$(\log \xi)^{k-1} \ll_{a,k} \sum_{p \leq \xi} \frac{\omega_{a,k}(p)}{p} \ll (\log \xi)^{k-1}.$$

Proof. By partial summation, it suffices to prove that

$$(15) \quad \xi (\log \xi)^{k-2} \ll_{a,k} \sum_{p \leq \xi} \omega_{a,k}(p) \ll \xi (\log \xi)^{k-2}.$$

The upper bound. If $p \equiv -1 \pmod{a}$, $p \geq a$, then

$$\begin{aligned} \omega_{a,k}(p) &= \sum_{\substack{x_1 \dots x_k = \frac{p+1}{a} \\ x_i \geq \left(\frac{p+1}{a}\right)^{2/3^i}}} 1 \leq \sum_{\substack{x_1 \dots x_k = \frac{p+1}{a} \\ x_i \geq \left(\frac{p+1}{a}\right)^{2/3^i}}} 1 = \sum_{\substack{x_1 \left|\frac{p+1}{a}\right. \\ x_1 \geq \left(\frac{p+1}{a}\right)^{2/3}}} \sum_{\substack{x_2 \dots x_k = \frac{p+1}{ax_1}}} 1 \\ &= \sum_{\substack{r \left|\frac{p+1}{a}\right. \\ r \leq \left(\frac{p+1}{a}\right)^{1/3}}} \sum_{\substack{x_2 \dots x_k = r}} 1 = \sum_{\substack{r \left|\frac{p+1}{a}\right. \\ r \leq \left(\frac{p+1}{a}\right)^{1/3}}} d_{k-1}(r). \end{aligned}$$

Hence

$$\begin{aligned} \sum_{p \leq \xi} \omega_{a,k}(p) &\leq \sum_{\substack{p \leq \xi \\ p \equiv -1 \pmod{a}}} \sum_{\substack{r \left|\frac{p+1}{a}\right. \\ r \leq \left(\frac{p+1}{a}\right)^{1/3}}} d_{k-1}(r) \leq \sum_{\substack{r \leq \left(\frac{\xi+1}{a}\right)^{1/3}}} d_{k-1}(r) \sum_{\substack{p \leq \xi \\ p \equiv -1 \pmod{a}}} 1 \\ &\leq \sum_{r \leq \left(\frac{\xi+1}{a}\right)^{1/3}} d_{k-1}(r) \pi(\xi; r, -1). \end{aligned}$$

Therefore, by Lemmas 2 and 4,

$$\sum_{p \leq \xi} \omega_{a,k}(p) \ll \frac{\xi}{\log \xi} \sum_{r \leq (\xi+1)^{1/3}} \frac{d_{k-1}(r)}{\varphi(r)} \ll \xi (\log \xi)^{k-2}.$$

The lower bound.

$$\begin{aligned} \sum_{p \leq \xi} \omega_{a,k}(p) &\geq \sum_{\substack{\frac{\xi}{2} < p \leq \xi \\ p \equiv -1 \pmod{a}}} f_k\left(\frac{p+1}{a}, \xi\right) = \sum_{\substack{\frac{\xi}{2} < p \leq \xi \\ p \equiv -1 \pmod{a}}} \sum_{\substack{x_1 \dots x_k = \frac{p+1}{a} \\ x_i \geq \xi^{2/3^i}}} 1 \\ &= \sum_{\substack{\frac{\xi}{2} < p \leq \xi \\ p \equiv -1 \pmod{a}}} \sum_{\substack{x_1 \left|\frac{p+1}{a}\right. \\ x_1 \geq \xi^{2/3}}} \sum_{\substack{x_2 \dots x_k = \frac{p+1}{ax_1} \\ x_i \geq \xi^{2/3^i}}} 1 \geq \sum_{\substack{\frac{\xi}{2} < p \leq \xi \\ p \equiv -1 \pmod{a}}} \sum_{\substack{r \left|\frac{p+1}{a}\right. \\ r \leq \frac{\xi}{2a}}} \sum_{\substack{x_2 \dots x_k = r \\ x_i \geq \xi^{2/3^i}}} 1 \\ &= \sum_{\substack{r \leq \frac{\xi^{1/3}}{2a}}} \sum_{\substack{x_1 \dots x_{k-1} = r \\ x_i \geq \xi^{2/3^i+1}}} \sum_{\substack{\frac{\xi}{2} < p \leq \xi \\ p \equiv -1 \pmod{ar}}} 1 \\ &= \sum_{r \leq \frac{\xi^{1/3}}{2a}} f_{k-1}(r, \xi^{1/3}) \left\{ \pi(\xi; ar, -1) - \pi\left(\frac{\xi}{2}; ar, -1\right) \right\}, \end{aligned}$$

whence

$$(16) \quad \sum_{p \leq \xi} \omega_{a,k}(p) \geq \left(\text{li } \xi - \text{li } \frac{\xi}{2} \right) \sum_{\substack{r \leq \frac{\xi^{1/3}}{2a}}} \frac{f_{k-1}(r, \xi^{1/3})}{\varphi(ar)} + \sum_{\substack{r \leq \frac{\xi^{1/3}}{2a}}} f_{k-1}(r, \xi^{1/3}) \left\{ \left[\pi(\xi; ar, -1) - \frac{\text{li } \xi}{\varphi(ar)} \right] - \left[\pi\left(\frac{\xi}{2}; ar, -1\right) - \frac{\text{li } \frac{\xi}{2}}{\varphi(ar)} \right] \right\}.$$

First consider the term

$$\begin{aligned} R_{a,k}(\xi) &= \sum_{\substack{r \leq \frac{\xi^{1/3}}{2a}}} f_{k-1}(r, \xi^{1/3}) \times \\ &\times \left\{ \left[\pi(\xi; ar, -1) - \frac{\text{li } \xi}{\varphi(ar)} \right] - \left[\pi\left(\frac{\xi}{2}; ar, -1\right) - \frac{\text{li } \frac{\xi}{2}}{\varphi(ar)} \right] \right\}. \end{aligned}$$

The Cauchy-Schwarz inequality yields

$$(17) \quad |R_{a,k}(\xi)| \leq \left(\sum_{r \leq \xi^{1/3}} \frac{f_{k-1}^2(r, \xi^{1/3})}{\varphi(ar)} \right)^{1/2} \left(\sum_{r \leq \frac{\xi^{1/3}}{a}} \varphi(ar) \left\{ \left[\pi(\xi; ar, -1) - \frac{\text{li } \xi}{\varphi(ar)} \right] - \left[\pi\left(\frac{\xi}{2}; ar, -1\right) - \frac{\text{li } \frac{\xi}{2}}{\varphi(ar)} \right] \right\}^2 \right)^{1/2} \\ \leq \left(\sum_{r \leq \xi} \frac{d_{k-1}^2(r)}{\varphi(r)} \right)^{1/2} \left(\sum_{s \leq \xi^{1/3}} \varphi(s) \left\{ \left[\pi(\xi; s, -1) - \frac{\text{li } \xi}{\varphi(s)} \right] - \left[\pi\left(\frac{\xi}{2}; s, -1\right) - \frac{\text{li } \frac{\xi}{2}}{\varphi(s)} \right] \right\}^2 \right)^{1/2}.$$

By Lemma 4

$$(18) \quad \left(\sum_{r \leq \xi} \frac{d_{k-1}^2(r)}{\varphi(r)} \right)^{1/2} \ll (\log \xi)^{k^2/2},$$

and by Lemma 2 with $\alpha = 1/3$

$$(19) \quad \varphi(s) \left| \left[\pi(\xi; s, -1) - \frac{\text{li } \xi}{\varphi(s)} \right] - \left[\pi\left(\frac{\xi}{2}; s, -1\right) - \frac{\text{li } \frac{\xi}{2}}{\varphi(s)} \right] \right| \ll \frac{\xi}{\log \xi}.$$

Lemma 3 gives, for any $A > 0$,

$$(20) \quad \sum_{s \leq \xi^{1/3}} \left| \left[\pi(\xi; s, -1) - \frac{\text{li } \xi}{\varphi(s)} \right] - \left[\pi\left(\frac{\xi}{2}; s, -1\right) - \frac{\text{li } \frac{\xi}{2}}{\varphi(s)} \right] \right| \ll \xi (\log \xi)^{-A}.$$

From (17), (18), (19) and (20) we obtain

$$(21) \quad |R_{a,k}(\xi)| \ll (\log \xi)^{k^2/2} \left(\frac{\xi}{\log \xi} \xi (\log \xi)^{-A} \right)^{1/2} \ll \xi (\log \xi)^{-A_1}.$$

For the main term in (16) we have

$$(22) \quad \sum_{r \leq \frac{\xi^{1/3}}{2a}} \frac{f_{k-1}(r, \xi^{1/3})}{\varphi(ar)} \geq \frac{1}{a} \sum_{r \leq \frac{\xi^{1/3}}{2a}} \frac{f_{k-1}(r, \xi^{1/3})}{r} = \frac{1}{a} \sum_{r \leq \frac{\xi^{1/3}}{2a}} \sum_{\substack{x_1 \dots x_{k-1} = r \\ x_i \geq \xi^{2/3^i+1}}} \frac{1}{r} \\ = \frac{1}{a} \sum_{\substack{x_1 \dots x_{k-1} \leq \frac{\xi^{1/3}}{2a} \\ x_i \geq \xi^{2/3^i+1}}} \frac{1}{x_1 \dots x_{k-1}}.$$

Now, if

$$x_i \leq \xi^{\frac{2}{3^{i+1}} + \frac{2}{3^{i+k}}} \quad (i = 1, \dots, k-1),$$

we have

$$x_1 \dots x_{k-1} \leq \xi^{\sum_{i=1}^{k-1} \left(\frac{2}{3^{i+1}} + \frac{2}{3^{i+k}} \right)} = \frac{1}{\xi^3} - \frac{1}{3^{2k-1}} \leq \frac{\xi^{1/3}}{2a},$$

provided $\xi \geq C_{a,k} = (2a)^{3^{2k-1}}$; hence if ξ is large

$$(23) \quad \sum_{\substack{x_1 \dots x_{k-1} \leq \frac{\xi^{1/3}}{2a} \\ x_i \geq \xi^{2/3^i+1}}} \frac{1}{x_1 \dots x_{k-1}} \geq \sum_{\substack{\xi^{2/3^i+1} \leq x_i \leq \xi^{2/3^i+1+2/3^{i+k}}}} \frac{1}{x_1 \dots x_{k-1}} \\ = \prod_{i=1}^{k-1} \sum_{\xi^{2/3^i+1} \leq x_i \leq \xi^{2/3^i+1+2/3^{i+k}}} \frac{1}{x_i} \\ \geq \prod_{i=1}^{k-1} \log \frac{\xi^{2/3^i+1+2/3^{i+k}}}{\xi^{2/3^i+1}} = 2^{k-1} 3^{-\frac{3k(k-1)}{2}} (\log \xi)^{k-1}.$$

Since

$$\text{li } \xi - \text{li } \frac{\xi}{2} \gg \frac{\xi}{\log \xi},$$

we obtain from (22) and (23)

$$(24) \quad \left(\text{li } \xi - \text{li } \frac{\xi}{2} \right) \sum_{r \leq \frac{\xi^{1/3}}{2a}} \frac{f_{k-1}(r, \xi^{1/3})}{\varphi(ar)} \gg_{a,k} \xi (\log \xi)^{k-2}.$$

(16), (21) and (24) together give at once

$$\sum_{p \leq \xi} \omega_{a,k}(p) \gg_{a,k} \xi (\log \xi)^{k-2},$$

which proves the theorem.

5. Proof of Theorem 1. Let $\omega_{a,k}(m)$ be, for any integer $m > 0$, the completely multiplicative function generated by $\omega_{a,k}(p)$. Then

$$\sum_{m \leq \sqrt{N}} \mu^2(m) \prod_{p|m} \frac{\omega_{a,k}(p)}{p - \omega_{a,k}(p)} \geq \sum_{m \leq \sqrt{N}} \mu^2(m) \prod_{p|m} \frac{\omega_{a,k}(p)}{p} = \sum_{m \leq \sqrt{N}} \frac{\mu^2(m) \omega_{a,k}(m)}{m}.$$

Hence, in view of (9), (12) and (14), Theorem 1 is proved if we show that

$$(25) \quad \sum_{m \leq \sqrt{N}} \frac{\mu^2(m) \omega_{a,k}(m)}{m} \gg \exp \{ C(a, k) (\log N)^{1 - \frac{1}{k}} \}.$$

Since $\omega_{a,k}(m) \ll_{\eta} m^{\eta}$ for any $\eta > 0$, the Dirichlet series

$$F_{a,k}(s) = \sum_{m=1}^{\infty} \frac{\mu^2(m) \omega_{a,k}(m)}{m^s}$$

converges for $\operatorname{Re} s > 1$. Also

$$\sum_p \left(\frac{\omega_{a,k}(p)}{p} \right)^2 < \infty;$$

hence, for any $\varepsilon > 0$,

$$\begin{aligned} (26) \quad F_{a,k}(1+\varepsilon) &= \prod_p \left(1 + \frac{\omega_{a,k}(p)}{p^{1+\varepsilon}} \right) \\ &= \exp \left\{ \sum_p \frac{\omega_{a,k}(p)}{p^{1+\varepsilon}} + O(1) \right\} \asymp \exp \sum_p \frac{\omega_{a,k}(p)}{p^{1+\varepsilon}}. \end{aligned}$$

It follows from Theorem 2 that

$$\sum_p \frac{\omega_{a,k}(p)}{p^{1+\varepsilon}} \geq \sum_{p \leq X} \frac{\omega_{a,k}(p)}{p^{1+\varepsilon}} \geq X^{-\varepsilon} \sum_{p \leq X} \frac{\omega_{a,k}(p)}{p} \gg_{a,k} X^{-\varepsilon} (\log X)^{k-1};$$

taking $X = \exp \frac{1}{\varepsilon}$ we obtain

$$(27) \quad \sum_p \frac{\omega_{a,k}(p)}{p^{1+\varepsilon}} \gg_{a,k} \varepsilon^{-(k-1)}.$$

Next

$$\begin{aligned} \sum_p \frac{\omega_{a,k}(p)}{p^{1+\varepsilon}} &= \sum_{n=0}^{\infty} \sum_{2^n \leq p < 2^{n+1}} \frac{\omega_{a,k}(p)}{p^{1+\varepsilon}} \leq \sum_{n=0}^{\infty} 2^{-\varepsilon n} \sum_{2^n \leq p < 2^{n+1}} \frac{\omega_{a,k}(p)}{p} \\ &= \sum_{n=0}^{\infty} (2^{-\varepsilon n} - 2^{-\varepsilon(n+1)}) \sum_{p < 2^{n+1}} \frac{\omega_{a,k}(p)}{p} \\ &= (1 - 2^{-\varepsilon}) \sum_{n=0}^{\infty} 2^{-\varepsilon n} \sum_{p < 2^{n+1}} \frac{\omega_{a,k}(p)}{p}; \end{aligned}$$

hence by Theorem 2

$$\sum_p \frac{\omega_{a,k}(p)}{p^{1+\varepsilon}} \ll (1 - 2^{-\varepsilon}) \sum_{n=0}^{\infty} (n+1)^{k-1} 2^{-\varepsilon n}.$$

Since

$$\sum_{n=0}^{\infty} (n+1)^{k-1} 2^{-\varepsilon n} \ll_k \varepsilon^{-k},$$

we obtain

$$(28) \quad \sum_p \frac{\omega_{a,k}(p)}{p^{1+\varepsilon}} \ll_k \varepsilon^{-(k-1)}.$$

Moreover

$$\begin{aligned} (29) \quad \sum_{m \leq \sqrt{N}} \frac{\mu^2(m) \omega_{a,k}(m)}{m} &\geq \sum_{m \leq \sqrt{N}} \frac{\mu^2(m) \omega_{a,k}(m)}{m^{1+\varepsilon}} \\ &= F_{a,k}(1+\varepsilon) - \sum_{m > \sqrt{N}} \frac{\mu^2(m) \omega_{a,k}(m)}{m^{1+\varepsilon}}, \end{aligned}$$

$$(30) \quad \sum_{m > \sqrt{N}} \frac{\mu^2(m) \omega_{a,k}(m)}{m^{1+\varepsilon}} \leq \sum_{m=1}^{\infty} \left(\frac{m}{\sqrt{N}} \right)^{\varepsilon/2} \frac{\mu^2(m) \omega_{a,k}(m)}{m^{1+\varepsilon}} = N^{-\varepsilon/4} F_{a,k} \left(1 + \frac{\varepsilon}{2} \right);$$

(29) and (30) give, for any $\varepsilon > 0$,

$$(31) \quad \sum_{m \leq \sqrt{N}} \frac{\mu^2(m) \omega_{a,k}(m)}{m} \geq F_{a,k}(1+\varepsilon) - N^{-\varepsilon/4} F_{a,k} \left(1 + \frac{\varepsilon}{2} \right).$$

We deduce from (26), (27), (28) and (31) that, for any sufficiently small ε ,

$$\begin{aligned} \sum_{m \leq \sqrt{N}} \frac{\mu^2(m) \omega_{a,k}(m)}{m} &\geq C_1 \exp(C_2(a, k) \varepsilon^{-(k-1)}) - C_3 N^{-\varepsilon/4} \exp(C_4(k) \varepsilon^{-(k-1)}) \\ &= C_1 \exp(C_2(a, k) \varepsilon^{-(k-1)}) - C_3 \exp \left(-\frac{\varepsilon}{4} \log N + C_4(k) \varepsilon^{-(k-1)} \right). \end{aligned}$$

Putting $\varepsilon = \{4C_4(k)\}^{1/k} \cdot (\log N)^{-1/k}$ we obtain

$$-\frac{\varepsilon}{4} \log N + C_4(k) \varepsilon^{-(k-1)} = 0.$$

Hence the above choice for ε gives, if N is large,

$$\begin{aligned} \sum_{m \leq \sqrt{N}} \frac{\mu^2(m) \omega_{a,k}(m)}{m} &\geq C_1 \exp \{ C(a, k) (\log N)^{1-\frac{1}{k}} \} - C_3 \\ &\gg \exp \{ C(a, k) (\log N)^{1-\frac{1}{k}} \}, \end{aligned}$$

and (25) is proved.

References

- [1] E. Bombieri, *On the large sieve*, *Mathematika* 12 (1965), pp. 201-225.
 [2] P. Erdős, *Some unsolved problems*, *Publ. Math. Inst. Hung. Acad. Sci.* 6 (1961), pp. 221-254.
 [3] — and S. Stein, *Sums of distinct unit fractions*, *Proc. Amer. Math. Soc.* 14 (1963), pp. 126-131.
 [4] R. L. Graham, *On finite sums of unit fractions*, *Proc. London Math. Soc.* 14 (1964), pp. 193-207.
 [5] Leonardo Pisano, *Liber Abaci*, ed. Boncompagni, Roma 1857.
 [6] Ju. V. Linnik, *The dispersion method in binary additive problems*, *Transl. Math. Monographs*, vol. 4, Amer. Math. Soc. (1963).
 [7] H. L. Montgomery, *A note on the large sieve*, *J. London Math. Soc.* 43 (1968), pp. 93-98.
 [8] A. Schinzel, *Sur quelques propriétés des nombres $3|N$ et $4|N$, où N est un nombre impair*, *Mathesis* 65 (1956), pp. 219-222.
 [9] — *Ungelöste Probleme*, *Elem. Math.* 11 (1956), pp. 134-135; *Bemerkung*, *ibidem* 21 (1966), p. 90.
 [10] W. Sierpiński, *Sur les décompositions de nombres rationnels en fractions primaires*, *Mathesis* 65 (1956), pp. 16-32.
 [11] B. M. Stewart and W. A. Webb, *Sums of fractions with bounded numerators*, *Canad. Journ. Math.* 18 (1966), pp. 999-1003.
 [12] R. C. Vaughan, *On a problem of Erdős, Straus and Schinzel*, *Mathematika* 17 (1970), pp. 193-198.

ISTITUTO MATEMATICO
UNIVERSITÀ DI PISA

Received on 19. 1. 1972

(251)

Les volumes IV et suivants sont à obtenir chez	Volumes from IV on are available at	Die Bände IV und folgende sind zu beziehen durch	Томы IV и следу- ющие можно по- лучить через
--	---	--	--

Ars Polona—Ruch, Krakowskie Przedmieście 7, 00-068 Warszawa (Poland)

Les volumes I-III sont à obtenir chez	Volumes I-III are available at	Die Bände I-III sind zu beziehen durch	Томы I-III можно получить через
--	-----------------------------------	---	------------------------------------

Johnson Reprint Corporation, 111 Fifth Ave., New York, N. Y.