ON THE DIRECT PRODUCT OF W*-FACTORS

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The problem in what conditions we can represent a W^*-factor as the direct product of its two subfactors has been considered by Nakamura [3], in which he set the assumption of finite type. In the present note we generalize his result and get the necessary and sufficient condition in general case, together with some applications.

Let M be a W^*-algebra, then M is the conjugate space of M_*, the space of all σ-weakly continuous linear functionals on M. We define the operator L_a and R_a on M_* for $a \varepsilon M$ such as $< x, L_a\varphi > = < ax, \varphi >$ and $< x, R_a\varphi > = < xa, \varphi >$ for all $\varphi \varepsilon M_*$ and $x \varepsilon M$. We call a subspace V of M_* invariant if it is invariant under the operators L_a, R_a for all $a \varepsilon M$. If V is an invariant subspace of M_*, then V^0, the polar of V in M, is a σ-weakly closed ideal of M.

Next, let M be a W^*-factor, M_1, M_2 subfactors of M which commute each other elementwise and φ, ψ linear functionals on M_1 and M_2 respectively, then we define the product functional $\varphi \times \psi$ of φ and ψ on M_0, sub*-algebra of M generated algebraically by M_1 and M_2, as follows

$$< \sum_{i=1}^{n} x_i y_i, \varphi \times \psi > = \sum_{i=1}^{n} < x_i, \varphi > < y_i, \psi > .$$

If $\varphi \times \psi$ is σ-weakly continuous on M_0, then it is uniquely extended on $\overline{M}_0$, σ-weak closure of M_0, by the σ-weak continuity: we denote again this extended functional by $\varphi \times \psi$. Moreover, we mean $V \times W$ the space of all finite linear combinations of $\varphi \times \psi$ $(\varphi \varepsilon V, \ \psi \varepsilon W)$ where V and W are subspaces of M_{1*} and M_{2*} respectively.

THEOREM 1. *A W^*-factor M is isomorphic to the direct product of its two subfactors M_1 and M_2 if and only if the following conditions* 1-3 *are satisfied:*

1. *M_1 and M_2 commute each other elementwise,*
2. *M_1 and M_2 generate M,*
3. *there exist some σ-weakly continuous linear functionals φ_0 and ψ_0 on M_1 and M_2, respectively such that the product functional $\varphi_0 \times \psi_0$ of φ_0 and ψ_0 on M_0 which is algebraically generated sub *-algebra of M by M_1 and M_2 is σ-weakly continuous.*

PROOF. At first, we notice that the space of all σ-weakly continuous linear functionals on a W^*-factor has no non-trivial invariant closed subspace.

Since $(L_a\varphi_0) \times (L_b\psi_0) = L_{ab}(\varphi_0 \times \psi_0)$ on M_0 for all $a \varepsilon M_1$ and $b \varepsilon M_2$, we have $(L_a\varphi_0) \times (L_b\psi_0) \varepsilon M_*$. Similarly, $(R_a \varphi_0) \times (R_b \psi_0) \varepsilon M_*$. Thus, if we put $V = \left[\sum_{i=1}^{n} L_{a_i}R_{b_i}\ \varphi_0 ;\ a_i,\ b_i \in M_1\right]$ and $W = \left[\sum_{i=1}^{n} L_{a_i}R_{b_i}\psi_0 :\ a_i,\ b_i \in M_2\right]$ we have $V \times W \subset M_*$. Since V and W are invariant subspaces of M_{1*} and M_{2*}, V and W are strongly dense in M_{1*} and M_{2*}, respectively. Thus,

$$M_{1*} \times M_{2*} \subset M_*.$$

Since $M_{1*} \times M_{2*}$ is an M_0-invariant subspace of M_* and M_0 is $\sigma(M, M_*)$-dense in $M, M_{1*} \times M_{2*}$ is strongly dense in M_*. Therefore, $\sigma(M, M_*)$-topology and $\sigma(M, M_{1*} \times M_{2*})$-topology coincide on the unit sphere of M.

Next we define a *-isomorphism Φ from $M_1 \odot M_2$, the algebraic direct product of M_1 and M_2, to M_0 as follows

$$\Phi\left(\sum_{i=1}^{n} x_i \otimes y_i\right) = \sum_{i=1}^{n} x_i y_i,$$

where $\sum_{i=1}^{n} x_i \otimes y_i$ is an arbitrary element of $M_1 \odot M_2$. Since the family of all states of type $\varphi \times \psi$, where φ and ψ are normal states on M_1 and M_2 respectively, is total on M by our consideration above, we have

$$||\sum_{i=1}^{n} x_i y_i||^2 = \sup\left\{\frac{<\left(\sum_{k=1}^{m} a_k^* b_k^*\right)\left(\sum_{i=1}^{n} x_i^* y_i^*\right)\left(\sum_{i=1}^{n} x_i y_i\right)\left(\sum_{k=1}^{m} a_k b_k\right),\ \varphi \times \psi >}{<\left(\sum_{k=1}^{m} a_k^* b_k^*\right)\left(\sum_{k=1}^{m} a_k b_k\right), \varphi \times \psi >}\right\}$$

$$= ||\sum_{i=1}^{n} x_i \otimes y_i\ |^2$$

where φ, ψ are all normal states on M_1 and M_2, respectively, and $\sum_{k=1}^{m} a_k b_k$ is any non-zero element of M_0 for an arbitrary element $\sum_{i=1}^{n} x_i \otimes y_i$ of $M_1 \odot M_2$. Hence, Φ is an isometry from $M_1 \odot M_2$ to M_0. Therefore, Φ is uniquely extended to $M_1 \hat{\otimes}_\alpha M_2$ by uniform continuity.

Next, we shall show that Φ is σ-weakly continuous on the unit sphere of $M_1 \odot M_2$. By [6 : Theorem 1] $M_{1*} \odot M_{2*}$ is strongly dense in $(M_1 \otimes M_2)_*$, hence $\sigma(M_1 \otimes M_2, (M_1 \otimes M_2)_*)$-topology and $\sigma(M_1 \otimes M_2,\ M_{1*} \odot M_{2*})$-topology coincide on the unit sphere of $M_1 \otimes M_2$. It is clear that Φ is bicontinuous on $M_1 \odot M_2$ and M_0 with respect to $\sigma(M_1 \odot M_2,\ M_{1*} \odot M_{2*})$-topology and $\sigma(M_0,\ M_{1*} \times M_{2*})$-topology. Therefore, Φ is σ-weakly bicontinuous on the unit sphere of $M_1 \odot M_2$ and M_0. Hence, by Kaplansky's Density Theorem, Φ is uniquely extended to the isomorphism of $M_1 \otimes M_2$ onto M by its σ-weak continuity.

Finally the necessity is easily verified from [6 : Theorem 1].

THEOREM 2. *A W^*-factor M is isomorphic to the direct product of its two* *subfactors* M_1 *and* M_2 *if and only if the conditions* 1, 2 *in Theorem* 1 *and the following condition* 3′ *are satisfied*:

3′. *there exists a* σ*-weakly continuous projection of norm one* θ_0 *from* M *to* M_1 (*or* M_2), *that is,* $\theta_0(x) = x$ *for all* $x \in M_1$ *and* $||\theta_0(x)|| \leqq ||x||$ *for* $x \in M$.[1]

PROOF. By [8: Theorem 1], we have

$$\theta_0(axb) = a\theta_0(x)b \text{ for all } a, b \in M_1 \text{ and } x \in M.$$

Since, by the condition 1, for any $u \in M_2$, $a\theta_0(u) = \theta_0(au) = \theta_0(ua) = \theta_0(u)a$ for all $a \in M_1$, $\theta_0(M_2)$ is included in the center of M_1. As M_1 is a factor, the restriction θ_0 on M_2 induces a σ-weakly continuous linear functional ψ_0 on M_2.

Now, if we denote the transpose of θ_0 by ${}^t\theta_0$, then, for $\varphi \in M_{1*}$, we have

$$< \sum_{i=1}^{n} x_i y_i,\ {}^t\theta_0(\varphi) > = < \theta_0\left(\sum_{i=1}^{n} x_i y_i\right), \varphi > = < \sum_{i=1} x_i\, \theta_0(y_i), \varphi >$$

$$= < \sum_{i=1}^{n} < y_i, \psi_0 > x_i, \varphi >$$

$$= \sum_{i=1}^{n} < x_i,\ \varphi > < y_i, \psi_0 > \ ,$$

where $\sum_{i=1}^{n} x_i y_i$ is an arbitrary element of sub*-algebra of M generated algebraically by M_1 and M_2, i.e. ${}^t\theta_0(\varphi) = \varphi \times \psi_0$. By the σ-weak continuity of θ_0, $\varphi \times \psi_0 \in M_*$. Therefore $M \cong M_1 \otimes M_2$ by Theorem 1.

Conversely, let ψ_0 be a normal state on M_2, we define the linear mapping θ_0' of M_{1*} to M^* as follows

$$\theta_0'(\varphi) = \varphi \otimes \psi_0 \qquad \text{for all } \varphi \in M_{1*}.$$

If we put ${}^t\theta_0' = \theta_0$, θ_0 satisfies the condition 3′.

Theorem 2 is a generalization of the result of [3] which proves the theorem in the finite case. For, in this case, there exists always a linear mapping described in 3′ as an expectation in [9].

Next, we apply Theorem 2 to the problem of non-normality and show an alternative proof for [1: Theorem 3].

THEOREM 3. *Let* M *be a* II_1*-factor, then* M *is non-normal.*

PROOF. There exists an approximately finite sub-factor M_1 in M. If $M_1 = (M_1' \cap M)' \cap M$, then $M \cong M_1 \otimes (M_1' \cap M)$ by Theorem 2. Since M_1 is non-normal one can easily verify that M is also non-normal. On the other

1) Prof. M. Nakamura noticed me that the existence condition of a σ-weakly continuous projection of norm one is relaxed to that of a σ-weakly continuous module linear homomorphism from M to M_1.

hand $M_1 \neq (M_1' \cap M)' \cap M$ implies also the non-normality of M by definition.

Combining the argument in the above proof with that in [5] one can easily verify the following

COROLLARY. *A W^*-factor M on a Hilbert space H is type 1 if and only if ther exists a σ-weakly continuous projection of norm one from $B(H)$, ring of all bounded operators on H, to M.*

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