

On the Dirichlet problem for linear elliptic equations in plane domains with corners*

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Abstract. In this paper we consider the first boundary value problem for linear second order elliptic equations in a plane domain Ω with corners. Conditions sufficient for the solutions to be of class $C^{m+2+\alpha}(\bar{\Omega})$ are given; $m \geq 0$ an integer and $0 < \alpha < 1$.

1. Introduction. In this paper we shall consider the Dirichlet problem for linear second order elliptic equations in domains with sectionally smooth boundaries. In Section 2 we introduce the problem and state some known results. Section 3 contains Theorem 2, which is the main result of this work. In Section 4 we study the problem in a circular sector. The proof of Theorem 2 follows from the sector case and will be given in Section 5.

2. The problem. In a bounded domain $\Omega \subset R^2$ with boundary Γ we consider the Dirichlet problem

$$(2.1) \quad a_{ij}(x)u_{ij} + a_i(x)u_i + a(x)u = f(x),$$

$$(2.2) \quad u = \varphi \quad \text{on } \Gamma.$$

Here $x = (x_1, x_2)$, $u_i = \partial u / \partial x_i$, $u_{ij} = \partial^2 u / \partial x_i \partial x_j$ and we use the summation convention. We make the following assumption.

(A1) The coefficients of (2.1) belong to $C_{m+\alpha}(\bar{\Omega})$, where $m \geq 0$ is an integer, $0 < \alpha < 1$ and $\bar{\Omega}$ is the closure of Ω ; cf. [1] for the definition of $C_{m+\alpha}$.

A known result is as follows (cf. [1]). Let u be a solution of (2.1), (2.2) and let (A1) be satisfied. Furthermore, assume that Γ can be represented parametrically by $C_{m+2+\alpha}$ -functions and $\varphi \in C_{m+2+\alpha}(\Gamma)$. Then $u \in C_{m+2+\alpha}(\bar{\Omega})$.

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If Γ has a corner, this result may not hold. We now state another assumption.

(A2) Γ has a corner at 0 and elsewhere can be represented parametrically by $C_{m+2+\alpha}$ -functions of the arc length s (measured from the corner point), and $\varphi \in C_{m+2+\alpha}(\Gamma \setminus \{0\}) \cap C_0(\Gamma)$.

Under assumptions (A1), (A2), for any solution of (2.1), (2.2) we have $u \in C_{m+2+\alpha}(\Omega_1) \cap C_0(\bar{\Omega})$ (cf. [1]), where Ω_1 is any compact subdomain of $\bar{\Omega}$ with positive distance from the corner. It was proved in [4] that in the neighbourhood of the corner, $u \in C_\nu$; here $\nu = \min(m+2+\alpha, \pi/\omega - \varepsilon)$ with arbitrarily small $\varepsilon > 0$,

$$(2.3) \quad \omega = \arctan \{ [a_{11}(0)a_{22}(0) - a_{12}^2(0)]^{1/2} / [a_{22}(0) \cot \gamma - a_{12}(0)] \}$$

and γ is the angle at the corner. Note that $\omega = \gamma$ if the leading part of (2.1) is the Laplacian. For the Poisson equation analogous results in Sobolev spaces related to $L^p(\Omega)$ for every p were proved in [7]. For more references in the case of boundaries with corners see [7] and the list given at the end of this paper.

3. The main result of the paper. The Dirichlet problem for the Poisson equation has been extensively studied; cf. [5]–[9], [13] and [14]. The following result is known.

THEOREM 1. *Let Ω_{2r_0} be a sector with center at the origin and radius $2r_0$. Assume that this sector has an angle $\omega = \pi/q$, $q \geq 2$ an integer. In Ω_{2r_0} consider the Dirichlet problem*

$$(3.1) \quad \Delta w = F,$$

$$(3.2) \quad w = \varphi \quad \text{on the boundary of } \Omega_{2r_0}.$$

If $F \in C_{m+\alpha}(\bar{\Omega}_{2r_0})$ and on each of the radii bounding the sector Ω_{2r_0} we have $\varphi \in C_{m+2+\alpha}$, then $w \in C_{m+2+\alpha}(\bar{\Omega}_{r_0})$ provided that at the corner the compatibility conditions imposed by (3.1) and (3.2) are satisfied.

We extend this result to the case of (2.1), (2.2), as follows.

THEOREM 2. *Let u be a solution of (2.1), (2.2), and assume that (A1) and (A2) hold. If ω in (2.3) is such that $\pi/\omega = q$, and integer ≥ 2 , and at the corner the functions in (2.1) and (2.2) satisfy the compatibility conditions imposed by (2.1) and (2.2), then $u \in C_{m+2+\alpha}(\bar{\Omega})$.*

From [4] it follows that it is sufficient to consider the case $q \leq m+2$, and the case of a circular sector.

4. The case of a circular sector. We first consider the problem in a circular sector. In Section 5 we shall prove that the general case can be

reduced to the present one by a suitable transformation. We make the following assumptions.

(A3) Let the sector be

$$\Omega_\sigma = \{(r, \theta) \mid r \leq \sigma < 1, \beta \leq \theta \leq \beta + \omega\},$$

where (r, θ) are the polar coordinates of $x = (x_1, x_2)$ and $\beta > 0$ is arbitrarily small, and $\omega = \pi/q$, $2 \leq q \leq m+2$. Let $w(x)$ be a bounded solution of (2.1) in Ω_σ , where the coefficients of (2.1) satisfy (A1) in Ω_σ and $a_{ij}(0) = \delta_{ij}$, the Kronecker delta. The boundary value ψ of w is continuous at the corner, and $\psi \in C_{m+2+\alpha}$ on the lines $\theta = \beta$ and $\theta = \beta + \omega$ ($0 < r \leq \sigma$).

(A4) At the corner, $\psi(r, \theta)$ and $f(x)$ satisfy

$$(4.1) \quad \psi^{(k)}(0, \theta) \equiv d^k \psi / dr^k \big|_{r=0} = 0, \quad k = 0, 1, \dots, p+2,$$

$$(4.2) \quad D^k f(0) = 0, \quad k = 0, 1, \dots, p,$$

where $D^k f$ is any partial derivative of f of order k , and $0 \leq p \leq m$.

We now state a result from [4] which we are going to use.

THEOREM 3. *Let w be a bounded solution of (2.1) in Ω_σ and let assumption (A3) be satisfied. Then*

(a) $w \in C_\nu(\Omega_{r_0})$, where $\nu = \min(m+2+\alpha, \pi/\omega - \varepsilon)$ and $\varepsilon > 0$ is arbitrarily small.

(b) *If for some $\lambda \leq p+2+\alpha$ we have $|w(x)| \leq M_1 r^\lambda$ in Ω_{2r_0} and if (A4) is satisfied, then in Ω_{r_0} we have*

$$|D^k w| \leq M_2 r^{\lambda-k}, \quad k = 0, 1, \dots, p+2.$$

We shall now see that the regularity of $D^q w$ ($q \leq m+2$) can be improved by multiplying $D^q w$ by a suitable function. The details are as follows.

THEOREM 4. *Let w be a bounded solution of (2.1) in Ω_σ and let assumptions (A3), (A4) be satisfied. Suppose that for some integer p , $0 \leq p \leq m$ we have $w \in C_{p+2-\varepsilon}(\Omega_\sigma)$ and $|D^{p+2} w| \leq M_3 r^{-\varepsilon}$, $0 \leq \varepsilon < 1$. Then for any function $h \in C_\delta(\Omega_\sigma)$, $\varepsilon \leq \delta \leq 1$, vanishing at the corner point, we have $h D^{p+2} w \in C_\mu(\Omega_{r_0})$, where*

$$\mu = \begin{cases} \delta - \varepsilon & \text{if } p < m, \\ \min(\alpha, \delta - \varepsilon) & \text{if } p = m. \end{cases}$$

Proof. Consider any two points $P(r_1, \theta_1)$ and $Q(r_2, \theta_2)$ in Ω_{r_0} and suppose that $0 \leq r_2 \leq r_1 \leq r_0$. If $r_2 \leq r_1/2$, then $\overline{PQ} \geq r_1/2$ and

$$(4.5) \quad |h(P) D^{p+2} w(P) - h(Q) D^{p+2} w(Q)| / \overline{PQ}^\mu \leq 2M_3 M_4 r_1^{\delta-\varepsilon} / (r_1/2)^\mu \leq M_5,$$

since $|h(x)| \leq M_4 r^\delta$. We turn to the case $r_2 > r_1/2$. We first prove that $|D^{p+2} w(P) - D^{p+2} w(Q)| / \overline{PQ}^\mu \leq M_6 r_1^{-\mu-\varepsilon}$; from this we shall then readily

derive an inequality of the form (4.5) [(4.12), below]. Consider the transformation

$$(4.6) \quad x_i = 2r_1 y_i / r_0, \quad i = 1, 2.$$

This transformation takes

$$\Omega_0 = \{(r, \theta) \mid r_1/2 \leq r \leq r_1, \beta \leq \theta \leq \beta + \omega\}$$

and

$$\Omega'_0 = \{(r, \theta) \mid r_1/4 \leq r \leq 2r_1, \beta \leq \theta \leq \beta + \omega\}$$

to

$$\Omega_1 = \{(\varrho, \theta) \mid r_0/4 \leq \varrho \leq r_0/2, \beta \leq \theta \leq \beta + \omega\}$$

and

$$\Omega'_1 = \{(\varrho, \theta) \mid r_0/8 \leq \varrho \leq r_0, \beta \leq \theta \leq \beta + \omega\},$$

respectively, where $\varrho = r_0 r / 2r_1$. In Ω'_1 the function $v(y) = w(2r_1 y / r_0)$ satisfies the elliptic equation

$$(4.7) \quad b_{ij}(y) v_{ij} + (2r_1/r_0) b_i(y) v_i + (2r_1/r_0)^2 b(y) v = (2r_1/r_0)^2 g(y),$$

where the coefficients of (4.7) are those of (2.1) after the transformation (4.6). On the two straight line segments Γ'_1 of the boundary of Ω'_1 , the function $v(y)$ coincides with $\chi(\varrho, \theta) = \psi(2r_1 \varrho / r_0, \theta)$. In Ω_1 and Ω'_1 , Schauder's inequality yields

$$(4.8) \quad \|v\|_{p+2+\mu}^{\Omega_1} \leq c [\|v\|_0^{\Omega_1} + (2r_1/r_0)^2 \|g\|_{p+\mu}^{\Omega_1} + \|\chi\|_{p+2+\alpha}^{\Gamma'_1}].$$

Now,

$$(4.9) \quad \|v\|_0^{\Omega_1} = \|w\|_0^{\Omega'_0} \leq M_7 r_1^{p+2-\varepsilon},$$

$$\|g\|_{p+\mu}^{\Omega_1} = \sum_{0 \leq k \leq p} \|D_1^k g(y)\|_0^{\Omega_1} + H_\mu^{\Omega_1}(D_1^p g),$$

where D_1^k is the derivative in the (y_1, y_2) -plane corresponding to D^k and $H_\mu^{\Omega_1}(D_1^p g)$ is the Hölder coefficient of $D_1^p g$ with exponent μ in Ω_1 . From (4.6) and the definition $g(y) = f(2r_1 y / r_0)$ we have

$$D_1^k g(y) = (2r_1/r_0)^k D^k f(x), \quad k = 0, 1, \dots, p.$$

From (4.2) it also follows that in Ω'_0

$$|D^k f(x)| \leq M_8 r_1^{-k+\mu}, \quad k = 0, 1, \dots, p.$$

Thus in Ω'_1 ,

$$|D_1^k g(y)| \leq M_9 r_1^{p+\mu}.$$

Using the definition of $H_\mu^{\Omega_1}(D_1^p g)$ and a similar argument, we also obtain

$$(4.10) \quad H_\mu^{\Omega_1}(D_1^p g) \leq M_{10} r_1^{p+\alpha}.$$

Together,

$$(4.11) \quad \|g\|_{p+\mu}^{\Omega_1} \leq M_{11} r_1^{p+\alpha}.$$

In a similar way, by (4.3), we conclude that

$$\|\chi\|_{p+2+\mu}^{\Gamma_1} \leq M_{12} r_1^{p+2+\alpha}.$$

From (4.8), (4.9) and (4.11) we obtain $\|v\|_{p+2+\mu}^{\Omega_1} \leq c_0 r_1^{p+2-\epsilon}$. We now return to the x -plane. Noting that in Ω_0 ,

$$(2r_1/r_0)^{p+2} |D^{p+2} w| = |D_1^{p+2} v| \leq \|v\|_{p+2+\mu}^{\Omega_1}$$

and

$$(2r_1/r_0)^{p+2+\mu} H_\mu^{\Omega_0}(D^{p+2} w) = H_\mu^{\Omega_1}(D_1^{p+2} v) \leq \|v\|_{p+2+\mu}^{\Omega_1},$$

we see that in Ω_0

$$H_\mu^{\Omega_0}(D^{p+2} w) \leq M_6 r_1^{-\epsilon-\mu}.$$

Using these estimates as well as the fact that, in the present case, $r_2 > r_1/2$, we finally arrive at the desired inequality

$$(4.12) \quad \begin{aligned} & |h(P) D^{p+2} w(P) - h(Q) D^{p+2} w(Q)| / \overline{PQ}^\mu \\ & \leq |h(P)| |D^{p+2} w(P) - D^{p+2} w(Q)| / \overline{PQ}^\mu + \\ & \quad + |D^{p+2} w(Q)| (|h(P) - h(Q)| / \overline{PQ}^\delta)^{\mu/\delta} (|h(P)| + |h(Q)|)^{\epsilon/\delta} \\ & \leq M_4 r_1^\delta M_6 r_1^{-\epsilon-\mu} + M_3 r_2^{-\epsilon} M_4 (r_1^\delta + r_2^\delta)^{\epsilon/\delta} \leq M_{15}. \end{aligned}$$

This completes the proof of Theorem 4.

We now prove Theorem 2 for a sector, in which case it takes the following form.

THEOREM 5. *Let w be a solution of (2.1) in Ω_σ and suppose that (A3) holds. Let $\pi/\omega = q$ be an integer and $2 \leq q \leq m+2$. Assume that the compatibility conditions at the corner are satisfied. Then $w \in C_{m+2+\alpha}$.*

Proof. Let $m \geq 0$. If $q = 2$, then from Theorem 3(a) it follows that $w \in C_{2-\epsilon}(\Omega_{r_0})$. Consider the function $v = w - Y$, where

$$(4.13) \quad \begin{aligned} Y(x) &= \psi_\beta + \sum_{k=1}^2 (x_1 \cos \beta + x_2 \sin \beta)^k \psi_\beta^k / k! + \\ &+ \sum_{k=1}^2 \sum_{j=1}^k [\psi_{\beta+\omega}^k / j! (k-j)!] (x_1 \cos \beta + x_2 \sin \beta)^{k-j} (-x_1 \sin \beta + x_2 \cos \beta)^j. \end{aligned}$$

Here $\psi_\beta = \psi(0, \beta) = \psi(0, \beta + \omega)$ and $\psi'_\beta = d^k \psi(r, \beta) / dr^k |_{r=0}$ and $\psi'_{\omega+\beta}$ is defined in a similar way. In Ω_σ the function v satisfies

$$(4.14) \quad Lv = f_1 \equiv f - LY$$

and on the lines $\theta = \beta$ and $\theta = \omega + \beta$ it coincides with $\psi_1(r, \theta)$. The function ψ_1 vanishes at the corner point, together with its first and second derivatives in the directions of $\theta = \beta$ and $\theta = \beta + \omega$. It also follows from (4.14) that the compatibility conditions at the corner point give $f_1(0, 0) = 0$. Thus condition (A4) is satisfied with $p = 0$. For the simplicity of writing we shall still use the functions w , f and ψ , assuming (A4) with $p = 0$ to be satisfied. From Theorem 3(b) it then follows that in Ω_{r_0} we have $|D^2 w| \leq M_2 r^{-\epsilon}$. We now write (2.1) in the form

$$(4.15) \quad \Delta w = F(x) = f(x) - aw - a_i w_i - (a_{ij} - \delta_{ij}) w_{ij}.$$

To prove that $w \in C_{2+\epsilon}(\Omega_{r_0})$, it is sufficient to show that $F \in C_\alpha$. Here C_α means $C_\alpha(\Omega_{r_0})$ and similarly we omit Ω_{r_0} until the end of this proof. The first three terms in F belong to C_α . Put $h(x) = a_{ij}(x) - \delta_{ij}$. Clearly $h \in C_\alpha$ and $h(0) = 0$. Using Theorem 4, we get $F \in C_{\alpha-\epsilon}$. Thus $w \in C_{2+\alpha-\epsilon}$, where $\epsilon > 0$ is arbitrarily small. Before proceeding we note the following. If w has been shown to belong to $C_{p+2+\eta}$, $0 \leq p \leq m$, $0 \leq \eta < 1$, then the function $z = w - T_{p+2}$ satisfies in Ω_σ the equation

$$(4.16) \quad Lz = f_1 \equiv f - LT_{p+2}.$$

Here, T_{p+2} is the sum of the first terms of the Maclaurin expansion of w , up to and including terms of order $p+2$. The function z vanishes at the corner together with all its partial derivatives of order not exceeding $p+2$. Thus it follows that the boundary value ψ_1 of z on the lines $\theta = \beta$ and $\theta = \beta + \omega$ satisfies (4.1). It also follows from (4.1) that f_1 satisfies (4.2). Without loss of generality and for the sake of simplicity in writing, whenever w has been proved to belong to $C_{p+2+\eta}$ we shall assume that (A4) is satisfied and that $|D^k w| \leq M r^{p+2-k+\eta}$, $k = 0, \dots, p+2$. We now show that from $w \in C_{2+\alpha-\epsilon}$ it follows that $w \in C_{2+\alpha}$. Take $h = a_{ij} - \delta_{ij} \in C_\alpha$. Since (A4) with $p = 0$ is satisfied and $|D^2 w| \leq A_1 r^{\alpha-\epsilon} \leq A_2$, Theorem 4 with $\delta = \alpha$ and $\epsilon = 0$ gives $(a_{ij} - \delta_{ij}) w_{ij} \in C_\alpha$. Using Theorem 1 we obtain $w \in C_{2+\alpha}$. If $m = 0$; then $q = 2$. This case has been discussed. Let $m > 0$ and $q \geq 2$. Suppose that it has been shown that $w \in C_{p+2-\epsilon}$, where $0 \leq p \leq m$ and $\epsilon = 1 - \alpha$ if $q < p+2$ while ϵ is arbitrarily small if $q = p+2$. Since the coefficients of (2.1) belong to $C_{m+\alpha}$, it follows that $f - aw - a_i w_i \in C_{p+\alpha}$. To show that $(a_{ij} - \delta_{ij}) w_{ij} \in C_{p+\mu}$ ($\mu \leq \alpha$), it is sufficient to prove that $(a_{ij} - \delta_{ij}) D^{p+2} w \in C_\mu$. As it was mentioned above, we may assume that $D^k w(0) = 0$, $k = 0, 1, \dots, p+1$, and $|D^k w(x)| \leq A_3 r^{p+2-k-\epsilon}$, $k = 0, 1, \dots, p+1$. We also assume that (A4) with p replaced by $p-1$ is satisfied. Thus Theorem 3(b) gives $|D^{p+2} w(x)| \leq A_4 r^{-\epsilon}$. Using Theorem 4

with $h = a_{ij} - \delta_{ij} \in C_1$ we finally conclude that $(a_{ij} - \delta_{ij}) D^{p+2} w \in C_\alpha$. This completes the proof of Theorem 5.

We now prove Theorem 2 by showing that the general case may be reduced to the case just studied.

5. Proof of Theorem 2. To prove the theorem it is sufficient to show that $u \in C_{m+2+\alpha}(N)$, where

$$N = \{(x_1, x_2) \mid (x_1, x_2) \in \bar{\Omega}, x_1^2 + x_2^2 \leq \sigma_0^2\}.$$

This follows from the known result that $u \in C_{m+2+\alpha}(\bar{\Omega} \setminus N)$. Without loss of generality, assume that the corner point is at the origin and the two curves bounding the corner are represented by $x_1 = g_2(x_2)$ and $x_2 = g_1(x_1)$, where $g_1(0) = g_2(0) = g_1'(0) = 0$ and $g_2'(0) = \cot \gamma$. We transform the equation

$$(5.1) \quad a_{ij}(0) u_{ij} = 0$$

to canonical form. The new angle after the transformation is independent of the transformation used and is given by

$$(5.2) \quad \omega = \arctan \{ [a_{11}(0) a_{22}(0) - a_{12}^2(0)]^{1/2} / [a_{22}(0) \cot \gamma - a_{12}(0)] \}.$$

This transformation is of class C_∞ (cf. (3.5)–(3.6) in [4]) and transforms the domain N to a domain N_0 bounded by two straight-line segments Γ_1 and Γ_2 making angles β and $\beta + \omega$ with the horizontal line, and by a curve joining the two non-coinciding end points of these segments. In this domain the transformed function w is a solution of an equation of the form (2.1) with all the conditions of Theorem 5 being satisfied. Thus in a subdomain Ω_{r_0} of N_0 we have $w \in C_{m+2+\alpha}$. Noting that the transformation used is of class C_∞ and its Jacobian at $(0, 0)$ has the value $[a_{11}(0) a_{22}(0) - a_{12}^2(0)]^{-1}$ (cf. [4]), we conclude that in a subdomain $N \subset \bar{\Omega}$ we have $u \in C_{m+2+\alpha}$. This completes the proof of Theorem 2.

We conclude this section by a theorem which follows from [1], [4] and Theorem 2 of this paper.

THEOREM 6. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain whose boundary Γ consists of a finite number of curves $\Gamma_1, \dots, \Gamma_k$, $k \geq 2$. Let $\Gamma_i \in C_{m+2+\alpha}$, $i = 1, 2, \dots, k$, and suppose that Γ_i and Γ_{i+1} intersect at O_i making an angle γ_i , $0 < \gamma_i < 2\pi$. Assume that u satisfies (2.1) in Ω and on Γ coincides with $\varphi \in C_{m+2+\alpha}(\Gamma \setminus \bigcup O_i) \cap C_0(\Gamma)$. If (A1) is satisfied and the necessary compatibility conditions at the corners hold, then $u \in C_{m+2+\alpha}(\Omega_1)$, where Ω_1 is a compact subdomain of $\bar{\Omega}$ with positive distance from those corners satisfying neither (i) nor (ii):

(i) $\pi/\omega_i > m+2+\alpha$.

(ii) π/ω_i is an integer.

In the neighbourhood of such an “excluded” corner we have $u \in C_{\pi/\omega_i - \varepsilon}$, where $\varepsilon > 0$ is arbitrarily small.

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