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On the Dissipation of Primordial Turbulence in the Expanding Universe

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The decay laws of primordial turbulence and the heating rates by its dissipation are derived in the expanding medium, and it is shown that the matter in the expanding universe cannot be heated and kept at temperatures higher than 10^5 °K which are necessary for the galaxy formation by thermal instability. Moreover the effects of its dissipation on hydrodynamic instability are discussed.

§ 1. Introduction

The growth of primordial density contrasts and their separation from the general expansion of the universe are the first step in the course of galaxy formation, which has been attempted to describe by various mechanisms. The epoch of the separation depends on the amount of density contrasts at some epoch, which must be more than several billion years ago. In the case of gravitational instability there arises some lower limit for the initial density contrast which cannot be explained by the statistical origin.[1] It has been expected, on the other hand, that thermal instability may play an important role at an early stage of the growing of the density contrasts.[2],[3] This mechanism can be effective, only if heating and cooling balance each other so as to keep matter at high temperature (at least higher than 10^6 °K) at the pregalactic stage. However, if no heating source of matter exists, the matter temperature T_m downs faster than the radiation temperature T_r after the epoch of the decoupling at $T_r \simeq 4000$°K.[4]

The rotational and peculiar motions of the galaxies in the present state suggest us a possibility that enormous turbulent motions have existed at the pregalactic stage. Weizsäcker[5] and Gamow[6] insisted upon its importance in the problem of galaxy formation. To meet with this, the theory of turbulence in the expanding universe has been developed by one of the authors (H. N.).[7],*) On the basis of a more realistic picture for the hot universe motivated by the discovery of cosmic black-body radiation, Ozernoi and Chernin[8] have recently

*) This paper is referred to as [N] in the following.

analyzed the significance of primordial turbulence. As a possible heating source, therefore, we can consider the dissipation of energies of large-scale turbulent motions which would have arisen at the epoch of the big-bang.

In this paper we investigate the thermal history of the universe in the presence of this heating source. For this purpose, in § 2 we shall review fluid dynamics and the theory of subsonic turbulence in the expanding universe. In § 3 we shall analyze the physical properties of eddies which vary in time. In § 4 the decay laws of subsonic turbulence are derived and in § 5 the change of T_m with time is studied under the condition that the distortions in the black-body radiation spectrum is not appreciable. For the dissipation of super-sonic turbulence which arises after the decoupling of matter and radiation, a simplified model is assumed for its estimation. Detailed derivations for equations in these sections are found in the Appendices.

§ 2. Fluid dynamics and theory of turbulence in the expanding universe

(a) *Fluid dynamical equations*

The equations of motion of the viscous fluid consisting of coupled matter and radiation are summarized in this subsection. The condition that matter and radiation can be regarded as a fluid will be examined in the next section.

Equations of motion of viscous fluid in the expanding universe have been derived in [N] at first. Here we present them in a generalized form, which is applicable even at the radiation dominant stage. First, we consider local fluid motions which do not disturb the space-time of the isotropic and homogeneous background universe and for which an effect of the spatial curvature of the background is negligible. Then the space-time can be expressed approximately by the line-element

$$ds^2 = c^2dt^2 - a^2(t)\,(dx^2 + dy^2 + dz^2)\,. \tag{2·1}$$

Here $a(t)$ is an expansion scale factor of the universe, and the four velocity is given by $U^\mu = dx^\mu/ds$, while the physical fluid velocity is defined by $v^i \equiv au^i$ ($u^i \equiv U^i/U^0$).*) Next, we assume $v^2 = \sum_i (v^i)^2 \ll c^2$ and take the non-relativistic approximation. Then the generalized Navier-Stokes equation and the equation of continuity are obtained as shown in Appendix A.

$$\dot{\boldsymbol{u}} + \left[(\boldsymbol{u}\cdot\boldsymbol{\nabla}) + \frac{1}{\varepsilon}\,\mathrm{div}\,(\varepsilon\boldsymbol{u}) + (\varepsilon a^5)^{\cdot}/(\varepsilon a^5)\right]\boldsymbol{u}$$
$$= -a^{-2}\boldsymbol{\nabla} p/\varepsilon + a^{-2}\nu\left[\boldsymbol{\nabla}^2\boldsymbol{u} + \frac{1}{3}\boldsymbol{\nabla}\,(\mathrm{div}\,\boldsymbol{u})\right], \tag{2·2}$$

and

$$\dot{\varepsilon} + \frac{3\dot{a}}{a}\varepsilon + \mathrm{div}\,(\varepsilon\boldsymbol{u}) = \dot{p}/c^2. \tag{2·3}$$

*) The Greek indices take the values 0, 1, 2, 3 and Latin indices 1, 2, 3.

Here the notations should not be confused; ρ_m denotes matter mass density including internal energy, ρ_r radiation mass density, p_m matter pressure, p_r radiation pressure, $\varepsilon \equiv \rho_m + \rho_r + p/c^2$ inertial mass density, p total pressure, $\nu = \mu/\varepsilon$ kinematic viscosity, and a dot a time derivative.

Now let us assume the condition of quasi-incompressibility[7] $\mathrm{div}\,\boldsymbol{u} = 0$ and $\varepsilon = \varepsilon(t)$, so that we can describe subsonic and vortical motions, and then we get

$$\dot{\boldsymbol{u}} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -a^{-2}\nabla p/\varepsilon + a^{-2}\nu\nabla^2\boldsymbol{u} - \{(\varepsilon a^5)^{\cdot}/(\varepsilon a^5)\}\boldsymbol{u}\,, \tag{2·4}$$

$$\rho_m \propto a^{-3}, \qquad \rho_r \propto a^{-4}, \tag{2·5}$$

assuming that $p_m \ll p_r = \rho_r c^2/3$ and ρ_m, ρ_r depends only on t as well as ε.

In particular the velocity of vortical motions in inviscid fluid is described by the theorem of circulation[9]

$$(\varepsilon a^3) \oint v_i dl^i = \text{const}\,, \tag{2·6}$$

where $l^i (\equiv a x^i)$ denotes a proper length. The integral of Eq. (2·6) leads to

$$\varepsilon a^4 v = \text{const}\,. \tag{2·7}$$

In the case when matter and radiation have already decoupled, the motion of matter can be described by the above expressions without ρ_r and p_r.

(b) *Turbulent energy and its spectrum*

In order to study the statistical properties of a turbulent fluid such that the velocity field is at random at any point in the space-time, let us consider the correlations of various quantities between two different points $\boldsymbol{x}$ and $\boldsymbol{x}'(=\boldsymbol{x}+\boldsymbol{r})$. First, the velocity correlation is defined by

$$R_{ij}(\boldsymbol{r}) = \overline{v_i v_j'} = a^2 \overline{u_i u_j'} = R_{ji}(-\boldsymbol{r})\,.$$

Here, the dashes denote the quantities at $\boldsymbol{x}'$. Similarly the following correlation tensors are defined:

$$T_{ij}(\boldsymbol{r}) = a^2 \frac{\partial}{\partial r_k}(\overline{u_i u_k u_j'} - \overline{u_i u_k' u_j'})\,,$$

$$P_{ij}(\boldsymbol{r}) = \frac{1}{\varepsilon}\left(\frac{\partial}{\partial r_i}\overline{p u_j'} - \frac{\partial}{\partial r_j}\overline{p' u_i}\right).$$

Using these correlations we can analyze turbulent motions in a form parallel with an ordinary theory of homogeneous turbulence.[10] If we write the three dimensional Fourier transforms of R_{ij}, T_{ij} and P_{ij} as Φ_{ij}, Γ_{ij} and Π_{ij}, respectively, i.e.

$$R_{ij} = \int \Phi_{ij}(\boldsymbol{k}) \exp(i\boldsymbol{k}\boldsymbol{r})\, d\boldsymbol{k}\,,$$

for example, quasi-incompressibility leads to the conditions

$$k_i\Phi_{ij}(\boldsymbol{k})=k_j\Phi_{ij}(\boldsymbol{k})=0\,,$$
$$\Pi_{ii}(\boldsymbol{k})=0\,,\qquad \int\Gamma_{ij}(\boldsymbol{k})\,d\boldsymbol{k}=0\,, \tag{2·8}$$

and from the Navier-Stokes equation (2·4) we get

$$\frac{\partial\Phi_{ij}}{\partial t}=\Gamma_{ij}+\Pi_{ij}-2\left\{\frac{\nu k^2}{a^2}+\frac{(\varepsilon a^4)^{\cdot}}{\varepsilon a^4}\right\}\Phi_{ij}\,. \tag{2·9}$$

The contraction of the above equation with respect to suffices i, j and the integration over all direction around the origin in the k space enable us to derive

$$-\frac{\partial E(k)}{\partial t}=T(k)+2\left\{\frac{\nu k^2}{a^2}+\frac{(\varepsilon a^4)^{\cdot}}{\varepsilon a^4}\right\}E(k), \tag{2·10}$$

where

$$E(k)=\frac{1}{2}\int\Phi_{ii}(\boldsymbol{k})\,dA(\boldsymbol{k}),\qquad T(k)=-\frac{1}{2}\int\Gamma_{ii}(\boldsymbol{k})\,dA(\boldsymbol{k}),$$

and $dA(\boldsymbol{k})$ denotes an areal element, i.e. $d\boldsymbol{k}=dk\,dA(\boldsymbol{k})$. Equation (2·10) shows how the kinetic energy $E(k)$ in the eddies with k is lost through the inertial term T, by viscous dissipation, and by cosmic expansion. If $k_{\max}$ and $k_{\min}$ are the maximum and minimum wave numbers of the eddies, the total kinetic turbulent energy $(v_t)^2/2$ is defined by

$$(v_t)^2/2=\frac{1}{2}\,a^2\overline{u_iu_i}=\int_{k_{\min}}^{k_{\max}}E(k)\,dk\,, \tag{2·11}$$

and we obtain from Eq. (2·10)

$$-\frac{1}{(v_t)^2}\frac{d(v_t)^2}{dt}=\frac{10\nu}{(a\lambda)^2}+2\frac{(\varepsilon a^4)^{\cdot}}{\varepsilon a^4}\,, \tag{2·12}$$

where $a\lambda$ is Taylor's micro-scale defined by

$$(a\lambda)^2=5a^2\int_{k_{\min}}^{k_{\max}}Edk\bigg/\int_{k_{\min}}^{k_{\max}}k^2Edk\,.$$

In the case $\rho_m\gg\rho_r$, Eq. (2·12) is reduced to Eq. (53) in [N].

§ 3. Basic properties of the turbulent eddies in the expanding universe

Fluid dynamical description for the motion of matter and radiation is possible when the size L exceeds the mean free path of interactions between the constituting particles. At the early stage of cosmic expansion when the matter is fully ionized, the interaction by electron scattering is dominant and the mean free path is given by $l_{\rm re}=m_p/\rho_m\sigma_T$. Here $\sigma_T(=6.65\times10^{-25}{\rm cm}^2)$ stands for Thomson's cross section. If $L>l_{\rm re}$, matter and radiation are a mixed fluid with viscosity $\nu_{\rm re}=4/15(m_pc/\sigma_T)(\rho_r/\rho_m\varepsilon)$ and the sound velocity of long wavelength $\lambda>l_{\rm re}$ is given by $c_s\simeq(4p_r/3\varepsilon)^{1/2}$ because of $p_r\gg p_m$. If $L<l_{\rm re}$, the matter behaves independently

of the radiation, so that we have the viscosity $\nu_{\text{ion}} \simeq 10^{-16}\ T_m^{5/2}/\rho_m$ (through the coulomb interaction between ionized particles) and the sound velocity $c_s \simeq (kT_m/m_p)^{1/2}$ of short wavelength $\lambda < l_{\text{re}}$. Once the ionized matter recombines, the interaction between matter and radiation ceases and the viscosity of the neutral matter becomes $\nu_{\text{neu}} = 8\times 10^{-6} T_m^{1/2}/\rho_m$ and the sound velocity is also $c_s \simeq (kT_m/m_p)^{1/2}$.

Now let us consider a turbulent eddy with velocity v such that the size L is longer than l_{re}. If $L/t > v$, the changes of the velocity field due to the non-linear inertial term in the Navier-Stokes equation is not effective and the eddy motion is frozen in the expanding medium. If $L/t < v$, the eddy decays to smaller eddies in time and the eddies with $L \sim \sqrt{\nu t}\,(<vt)$ contribute mostly to the dissipation into thermal energy. Relative orders of sizes l_{re}, vt, $\sqrt{\nu_{\text{re}} t}$ and ct have been shown separately in the figure[11] in terms of masses M_l, M_{in}, M_{vis} and M_{hor}, respectively, which are the masses included within the spheres with their sizes as the radii. Here $M < M_{\text{hor}}$ represents the non-relativistic condition $v < c$. In that figure $z_D = 10^{3.1}$ has been taken as the red-shift at the decoupling epoch by assuming $T_m/T_r = 1$. If $T_m/T_r > 1$, however, the decoupling may be delayed.

Moreover it should be noticed that a characteristic peak appears at the epoch $t = t_*$ when $\rho_r = \rho_m$. If $M >$ [the maximum of M_{in}], the eddy does not decay forever. If otherwise, it decays when $M < M_{\text{in}}$, and the smaller M is, the faster it decays. When $M \sim M_{\text{in}}$, the time scale of the decay is comparable with the expansion time t. These enable us to assume that at the epoch $t = t_*$ the

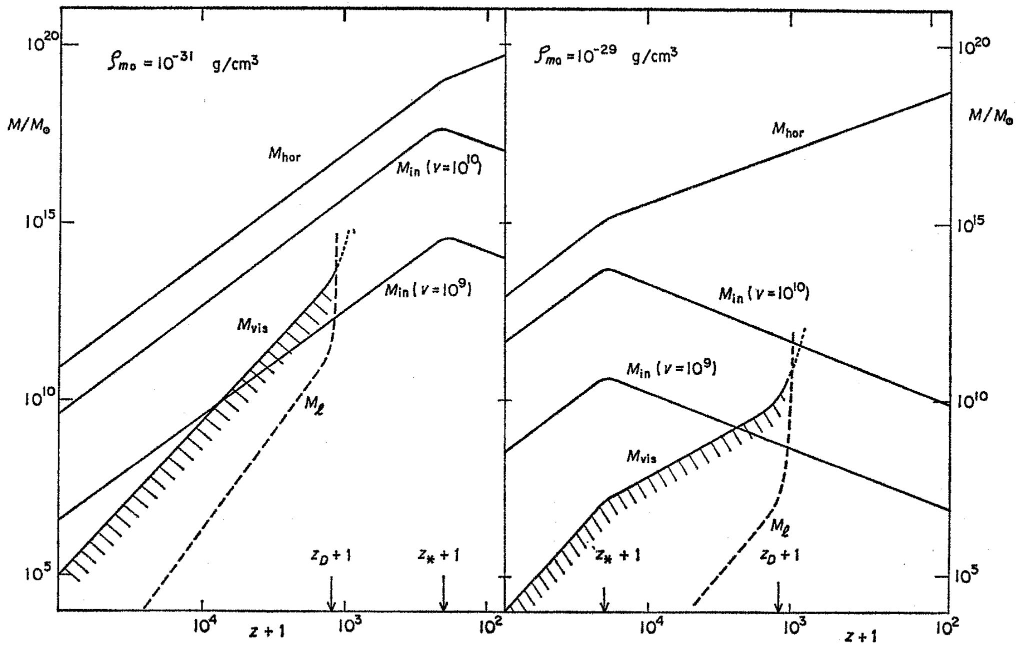

Fig. 1. For the two models of the present matter density $\rho_{m0} = 10^{-31}$ g/cm^3 and 10^{-29} g/cm^3, we show the variations of M_{hor}, M_{in}, M_{vis} and M_l, which are defined as $4\pi\rho_m(ct)^3/3$, $4\pi\rho_m(vt)^3/3$, $4\pi\rho_m(\nu_{\text{re}}t)^{3/2}/3$ and $4\pi\rho_m l_{\text{re}}^3/3$, respectively. M_{in} is given for $v=10^9$ cm/sec and 10^{10} cm/sec at $t=t_*$ and v is assumed to change without decay, following Eq. (2·7). At the stage of $z < z_D$, M_{vis} is smaller than M_l and the treatment of the dissipation by the viscosity becomes meaningless.

eddies with $L \ll vt$ have already decayed and their energy dissipated into thermal energy, and that only the eddies with $L \sim vt$ can decay mainly after that epoch. But the situation must be divided into two cases according to the universe models characterized by Ω, which is the ratio of the present matter density to that of the flat model, i.e. $10^{-29}\,\mathrm{g/cm^3}$. In the case $\Omega > 0.05$, the peak appears at the stage when matter and radiation closely couple together and the motions of eddies are subsonic, even if v is as large as $c/10$. In the case $\Omega < 0.05$, the residual eddies at the peak consist only of matter and most of their motions are supersonic, because c_s is extremely small.

§ 4. Decay laws of subsonic turbulence in the expanding universe

In quasi-stationary situation in a laboratory, the decay of subsonic turbulence can be described by

$$-(v_t)^{-2} d(v_t)^2/dt = Av_t/l_t. \qquad (A = \mathrm{const} \sim 1) \qquad (4\cdot1)$$

Here l_t is a typical length of energy-containing eddies which is related to λ by

$$l_t = (A/10)\,(a\lambda)^2 v_t/\nu \qquad (4\cdot2)$$

from the comparison with Eq. (2·12). Then we may define the Reynolds number by

$$R \equiv l_t v_t/\nu = (A/10)\,(a\lambda v_t/\nu)^2. \qquad (4\cdot3)$$

Moreover, according to the experiments of steady flows behind a grid, the decay of turbulence at the early stage follows a simple law such as[10]

$$(v_t)^{-2} \propto t + \mathrm{const}, \qquad (4\cdot4)$$

which reflects quasi-stationarity of $R = \mathrm{const}$. On the other hand, the above law has been derived theoretically by an application of the principle of similarity. Another theoretical laws have been derived from Loitsyanskii's law[12] which means the conservation of angular momenta of the largest eddies, that is to say,

$$(v_t)^{-2} \propto (t + \mathrm{const})^n, \qquad (4\cdot5)$$

where n is 10/7 (at the early stage) or 5/2 (at the later stage).

Taking these results for the static medium into account, let us derive the decay law in the expanding medium. Since the velocity of vortical motions is proportional inversely to εa^4 (cf. Eq. (2·7)), the change of $\varepsilon a^4 v_t$ with time does not depend much on the expansion but on the decay of turbulence itself. Therefore the decay law in the expanding medium is roughly given by

$$(v_t)^{-2} \propto (\varepsilon a^4)^2 (t + \mathrm{const})^n, \qquad (4\cdot6)$$

where $n = 1$, 10/7 or 5/2 (cf. Eqs. (4·4), (4·5)). In fact these laws will be derived in Appendix B theoretically on the basis of the similarity principle or the Loitsyanskii's law. Inserting Eq. (4·6) into Eq. (2·12) and representing const. in Eq. (4·6) by quantities such as the micro-scale, etc., at an initial epoch t_i, we get

$$(v_t)^2 = \left(\frac{(\varepsilon a^4)_i}{\varepsilon a^4}\right)^2 \left(\frac{\beta_i}{t/t_i - 1 + \beta_i}\right)^n (v_t)_i^2, \tag{4·7}$$

where

$$\beta \equiv \frac{n}{10}\frac{(a\lambda)^2}{\nu t} = \frac{n}{A}\left(\frac{l_t}{v_t t}\right),$$

$$\lambda^2 = \frac{10}{n}(\nu/a^2)\{t - t_i(1-\beta_i)\}.$$

For the motion of eddies such that their non-linear interactions dominate the effect of expansion, we have the condition $\beta < 1$.

Furthermore, the rate of the net dissipation to thermal energy, ε_d, is given by the first term on the right-hand side in Eq. (2·12)

$$\varepsilon_d = \frac{10\nu}{(a\lambda)^2}\frac{(v_t)^2}{2}.$$

Substituting Eq. (4·7) in the above, we have

$$\varepsilon_d = \frac{1}{2}\left(\frac{(\varepsilon a^4)_i}{\varepsilon a^4}\right)^2 \frac{n\beta_i^n (v_t)_i^2/t_i}{(t/t_i - 1 + \beta_i)^{n+1}}. \tag{4·8}$$

Here we examine the fraction of the turbulent energy which is transformed into thermal energy by the dissipation. Its fraction is given by

$$\zeta \equiv \int_{t_i}^{\infty} \varepsilon_d dt / \{(v_t)_i^2/2\},$$

and we find that $\zeta \simeq 1$ for small eddies such as $\beta_i \ll 1$ and $\zeta \simeq 1/2$ for large eddies such as $\beta_i = 1$.

In an application in the next section, the decay law in the case $n=1$ will be used, because it seems to be simplest and most appropriate. For l_t and R we have Eqs. (4·2) and (4·3) also in the expanding medium by extending naturally Eq. (4·1) as

$$-(v_t)^{-2} d(v_t)^2/dt = Av_t/l_t + (\varepsilon a^4)^{\cdot}/(\varepsilon a^4). \tag{4·1′}$$

In the case $n=1$, R decreases with time contrary to the case of the static medium, but it does not mean any inconsistency, because the idea of quasi-stationarity should be modified by the over-all expansion.

In the case $n=1$, the energy spectrum of turbulence has the following form, which is derived from the similarity principle (See Appendix B):

$$E(k,t) = (v_t)^2 \lambda F(\xi), \qquad \xi = \lambda k, \tag{4·9}$$

with

$$\xi\frac{dF}{d\xi} - F + \frac{2}{5}\xi^2 F + Q(\xi) = 0. \tag{4·10}$$

Here F, Q, ξ are dimensionless and Q is proportional to the inertial term T in Eq. (2·10). If we assume Heisenberg's eddy viscosity hypothesis[10] for T or Q, Eq. (4·10) is reduced to Chandrasekhar's equation[13] and we find from his solution that, if $R \gg 1$, the Kolmogoroff spectrum $E \propto k^{-5/3}$ appears in the middle region of k, while $E \propto k^{-7}$ for $k \to \infty$ in general. On the other hand, $R \sim (l_t/\sqrt{\nu t})^2 (vt/l_t) \gg 1$ in the region $l_t \sim v_t t$ before the decoupling of matter and radiation (cf. § 3). Therefore we can expect the Kolmogoroff spectrum for the eddies with $\beta \sim 1$ and the steeper spectrum for the smaller eddies.

§ 5. Time variation of matter temperature T_m

In this section, we clarify how the matter is heated by the dissipation of the turbulent energy at the matter dominant stage. Since the radiation has a heat capacity much larger (by a factor $\sim 10^8$) than matter, $T_r (\propto a^{-1}(t))$ depends scarcely on the heating, while T_m depends sensitively. However, if the ratio T_m/T_r is large, the interaction between matter and radiation may distort the short wave-length part of the radiation spectrum, because the heat capacity of such a part is comparable with that of matter.[4),14)] If we put the condition that the radiation spectrum does not deviate so much from Planck's spectrum, we have the following condition in Appendix C:

$$(1 \leq)\ (T_m/T_r)_* < 2,\ 136 \quad \text{for } \Omega = 1,\ 10^{-1}, \text{ respectively.} \tag{5·1}$$

In the following, the low density models with $\Omega < 0.05$ are omitted from our consideration.

Now at the stage when the matter is fully ionized, we have

$$\dot{T}_m = \frac{4}{3}(T_r - T_m)/\tau_r - T_m \frac{(a^2)^{\cdot}}{a^2} + \frac{m_p}{3k}\varepsilon_d \tag{5·2}$$

from the first law of thermodynamics. Here $\tau_r \equiv m_e c/(\sigma_T b T_r^4)$ represents the characteristic time of the Compton scattering, b is the Stefan-Boltzmann constant. At the stage when the interaction is very effective, i.e. $t/\tau_r \gg 1$, Eq. (5·2) can be integrated approximately, as shown in Appendix D, provided that

$$|d \ln (a^6 \varepsilon_d)/d \ln a| \ll t/\tau_r\,, \tag{5·3}$$

and we obtain

$$T_m - T_r - \frac{m_p}{4k}\tau_r \varepsilon_d = \left(T_m - T_r - \frac{m_p}{4k}\tau_r \varepsilon_d\right)_i \left(\frac{z+1}{z_i+1}\right)^2 \exp(-B(F_i - F)) \tag{5·4}$$

with $B \equiv 4/(3H_0(\tau_r)_0) = 5.35 \times 10^{-3}$.*) Here $F(t)$ is defined in Appendix D. For the stage $1+z > |1-\Omega|/\Omega$, $BF \simeq t/\tau_r \simeq 2.1 \times 10^{-3} \Omega^{-1/2} (1+z)^{5/2}$ which is larger than 10 at the stage such as

$$1 + z \gtrsim 29.4 \Omega^{1/2}. \tag{5·5}$$

*) The suffix 0 denotes the present epoch. As the Hubble constant we take $H_0 = 75$ km/sec/Mpc.

In such cases, the right-hand side of Eq. (5·4) can be neglected and it is reduced to

$$T_m/T_r = 1 + \frac{m_p}{4kT_r}\tau_r\varepsilon_d\,. \tag{5·6}$$

On the other hand, in the neighbourhood of the epoch $t=t_*$, we have

$$B(F(z_*) - F(z_* + \Delta z)) \simeq 4.3\times 10^8 \Omega^2 (\Delta z/z_*)$$

for $\Omega > 10^{-2}$. So, even if T_m/T_r is different from Eq. (5·6) at t_*, T_m/T_r arrives at that ratio promptly after the epoch t_*.

(a) *The early stage when matter and radiation can be regarded as a mixed fluid.* At this stage the dissipation rate given by Eq. (4·8) satisfies the condition (5·3) and hence Eq. (5·6) holds. For the ratio T_m/T_r at the initial epoch $t_i = t_*$, we obtain from Eqs. (4·7), (4·8) and (5·6)

$$\begin{aligned}(T_m/T_r)_* &= 1 + \frac{3}{4\beta_i}\left(\frac{m_p v_t^2}{6kT_r}\right)_* (\tau_r/t)_*\,,\\ &= 1 + 7.3\times 10^{-2}\Omega^{-3}\beta_i^{-1}(v_t/c)_*^2,\end{aligned} \tag{5·7}$$

which satisfies the condition of Eq. (5·1) for $\beta_i \approx 1$, so that any influence on the radiation spectrum is negligible. Equation (5·7) can also be rewritten as

$$(v_t/v_{\rm th})_*^2 = [1 - (T_r/T_m)_*]\,(t/\tau_r)_*\,\frac{4\beta_i}{3}\,,$$

where $v_{\rm th}$ is a thermal velocity defined by $\frac{1}{2}m_p v_{\rm th}^2 = 3kT_m$. After the initial epoch, we get from Eqs. (4·7) with $n=1$, (5·6) and (5·7)

$$T_m/T_r = 1 + \{(T_m/T_r)_* - 1\}\beta_i^2\left(\frac{t/t_*}{t/t_* - 1 + \beta_i}\right)^2 \alpha(t)\,, \tag{5·8}$$

where $\alpha(t) \equiv (a/\varepsilon t^2)/(a/\varepsilon t^2)_*$ is nearly equal to unity.*) As already stated in § 3, it is mainly the eddies with $\beta_i \simeq (l_t/v_t t)_i \simeq 1$ that dissipate after the initial epoch $t_i = t_*$. Accordingly let us set $\beta_i = 1$, and then we find that the ratio T_m/T_r remains constant and T_m decreases with time. At the decoupling epoch $z_D = 10^3$, we obtain

$$T_m = 2700\{1 + 7.3\times 10^{-2}\Omega^{-3}(v_t/c)_*^2\}$$

from Eqs. (5·7) and (5·8) with $\beta_i = 1$. This temperature does not exceed 10^5 °K. If z_D is taken to be smaller than 10^3 (cf. § 3), T_m is also smaller than 10^5 °K.

(b) *After the decoupling epoch.* At this stage the matter is neutral or, if otherwise, the mean free path of a photon is so long that radiation and matter cannot be regarded as a one fluid. Then, most of residual turbulent motions are highly supersonic, since the sound velocity c_s is very small. Although we

*) If $\rho_m > \rho_r$ and $a/a_0 \ll 1$, we may assume $\varepsilon \propto a^{-3}$ and $a \propto t^{2/3}$. Then $\alpha(t) = \text{const} \sim 1$.

have no reliable theory of supersonic turbulence, the influence on T_m can be estimated according to a simplified model.

Let us assume that the loss of turbulent energy except the one due to the expansion, $\varepsilon_d \equiv -(2a^2)^{-1}d(av_t)^2/dt$, is transformed into thermal energy. In the case (I) when that time scale is comparable with t, i.e. the rate is given by $\varepsilon_d = \gamma(v_t)^2/t$ (γ=const ~ 1), we obtain $(v_t)^2 = (a_D/a)^2(t_D/t)^\gamma(v_t)_D^2$ and $\varepsilon_d = \gamma(a_D/a)^2(t_D/t)^{\gamma+1}\{(v_t)^2/t\}_D$. In the case (II) when the turbulent energy is lost on a time scale t_d much smaller than t, i.e. the rate is given by $\varepsilon_d = (v_t)^2/t_d$, we have $(v_t)^2 = \exp(-(t-t_D)/t_d)(v_t)_D^2$ and $\varepsilon_d = \exp(-(t-t_D)/t_d)(v_t)_D^2/t_d$.

If the matter remains neutral at this stage, it can be heated up to $T_m = 10^4$ °K, above which temperature a collisional ionization begins.[15] If the matter is heated above 10^4 °K, the cooling due to the Compton scattering becomes effective because of $t/\tau_r \gg 1$. In the case (I), the matter temperature T_m will converge to the value given by Eq. (5·6) and T_m/T_r increases (or decreases) with time according to $\gamma > 1$ (or $\gamma < 1$). However, since the turbulent energy will be soon consumed, T_m falls below 10^4 °K or even below T_r by the cosmic expansion.

Now let us explain the above situations numerically. At the decoupling epoch such as $z_D = 10^3$, the residual turbulent energy is given by

$$\tfrac{1}{2}(v_t)_D^2 = 3\times 10^{-5}\Omega^{-7/2}\,\tfrac{1}{2}(v_t)_*^2,$$

which is smaller than $10^{14}\ \Omega^{-7/2}$ ergs/g for $(v_t)_* < c/10$. Here Eq. (4·7) with $\beta_i = 1$ and $n=1$ has been used. If all of this energy were transformed directly to thermal energy, T_m would rise to $3\times 10^7\Omega^{-7/2}((v_t)^*/c)^2$ °K for $\Omega > 0.05$. This is not realistic, however. If that energy is transformed during a time scale of the order of $t_d \sim t$, the temperature rise is slowed down by the cooling as long as matter is ionized, and we get from Eq. (5·6)

$$T_m - T_r \simeq \frac{m_p}{8k}(v_t)^2\tau_r/t_d\,.$$

Since $(t/\tau_r)_D \simeq 10^5\sqrt{\Omega}$, only the part of the order of 10^{-5} in $(v_t)_D^2/2$ can be transformed into thermal energy and it is difficult that T_m exceeds 10^5 °K.

Moreover, in the case (II), T_m may rise suddenly in terms of the sharp dissipation.*) However, the turbulent energy is consumed exponentially during the time interval $\sim t_d \ll t$. Thereafter the matter is cooled by Compton scattering with no heating and T_m falls soon below 10^4 °K and further.

Incidentally let us examine the turbulent motions which survive till the cooling by the Compton scattering becomes ineffective, i.e. $\tau_r > t$. Even if the initial turbulent energy is as large as $(v_t)_* = c/10$, we have $(v_t)^2/2 = 10^{10}\Omega^{-2}$ erg/g at an epoch as $z=10$ when $t/\tau_r \sim 1$. The matter can be heated by that energy to $T_m \simeq 30\Omega^{-2}$ °K at most.

*) If $t_d \sim \tau_r$, the condition (5·3) does not hold and Eq. (5·6) is not applicable.

§ 6. Conclusion and discussion

From the foregoing analysis it has been shown that the dissipation of primordial turbulence cannot heat and keep the matter to such a high temperature that thermal instability may contribute to the growth of density contrast. This conclusion has been derived from the facts that at the matter dominant stage there is no supply of the turbulent energy and that the excess thermal energy relative to the radiative energy is carried away rapidly through Compton scattering. Although the foregoing discussions are limited to the stage $\rho_m > \rho_r$, the above conclusion may be correct. Our conclusion is consistent with Zel'dovich and Sunyaev's[14] assertion that, even in the presence of turbulent heating, the matter has not been kept in a fully ionized state throughout the cosmic expansion. However, our conclusion is incompatible with the assumption of an enormous heating by the turbulence.[3]

As an another possible mechanism for the galaxy formation, we should notice the hydrodynamic instability introduced by Ozernoi and Chernin.[8] In this case, turbulent eddies with such a high velocity as $v > c/30$ are required at the stage $\rho_r > \rho_m$ to find a mass range such that $M_{\rm in} \gtrsim M \gtrsim M_{\rm vis}$ at $t = t_D$.[11] However, if the decrease of turbulent energy due to the decay is effective as well as that due to the expansion, the eddies with a further higher velocity must exist at that early stage for the formation of galaxies. Moreover, if we have $T_m/T_r \gg 1$ at $t \leq t_D$ (cf. Eq. (5·8)), the decoupling will be somewhat delayed after the epoch of $z = 10^3$, since t_D is specified by the collisional ionization temperature $T_m \simeq 10^4$ °K rather than the photoionization temperature $T_r \simeq 4 \times 10^3$ °K. Hence the mass range ($M_{\rm in} \gtrsim M \gtrsim M_{\rm vis}$) at $t = t_D$ becomes narrower than that evaluated at $z = 10^3$. Accordingly the hydrodynamic instability may arise in a situation in which the dissipation of primordial turbulence is ineffective.

Appendix A

Fluid dynamics in the expanding universe

Fluid dynamical equations in the space-time with a given metric are derived from the energy-momentum conservation law, which is expressed as

$$T^{\alpha\beta}{}_{;\beta} = 0\,. \tag{A1}$$

Here $T^{\alpha\beta}$ for viscous fluid is given by

$$T^{\alpha\beta} = \varepsilon U^\alpha U^\beta - g^{\alpha\beta} p/c^2 + \tau^{\alpha\beta}, \tag{A2}$$

and the viscous tensor $\tau^{\alpha\beta}$ is expressed as $\tau^{\alpha\beta} = -\mu\sigma^{\alpha\beta}$ by shear tensor $\sigma^{\alpha\beta}$ and the coefficient of viscosity μ, where $\sigma_{\alpha\beta} \equiv \frac{1}{2}(U_{\alpha;\beta} + U_{\beta;\alpha}) - \frac{1}{2}(U_{\alpha;\gamma} U_\beta + U_{\beta;\gamma} U_\alpha) U^\gamma - \frac{1}{3}\theta h_{\alpha\beta}$, $\theta \equiv U^\beta{}_{;\beta}$ and $h_{\alpha\beta} \equiv g_{\alpha\beta} - U_\alpha U_\beta$.

For the metric (2·1) we obtain Eqs. (2·2) and (2·3) from Eqs. (A1) and (A2) in the non-relativistic approximation.

Appendix B

Derivation of decay rates

(a) *Decay rate due to the similarity principle*

In a static medium all kinds of dimensions are expressed by two independent parameters. In the expanding medium we must take into account not only the two parameters but also such an independent parameter as the expansion rate or cosmic time t, from which a dimensionless quantity can be composed, for example, the Reynolds number.

Taking $(v_t)^2/2$ and λ as the two parameters, we have the following dimensionless quantities

$$\xi=\lambda k\,,\qquad H=E(k,t)/(v_t)^2\lambda\,,\qquad G=T(k,t)/(v_t)^3,$$

where H and G are functions of ξ and R (or t) in general. If the turbulent motions satisfy a similarity law, we may put

$$H=h(t)F(\xi)\,,\qquad G=g(t)Q(\xi)\,. \tag{B1}$$

Inserting Eq. (B1) into Eq. (2·11) we get

$$h(t)\int_{\xi_{\min}}^{\xi_{\max}}F(\xi)\,d\xi=1/2\,. \tag{B2}$$

Here $\xi_{\max}$, $\xi_{\min}$ show the range of sizes of eddies, and can be regarded as 0, ∞, respectively, approximately, because they are far from the energy-containing part of the spectrum. Hence $h(t)$ can be reduced to a constant. From Eqs. (B1) and (2·10) we obtain

$$-[2(av_t)^{\cdot}/(av_t)+\dot{\lambda}/\lambda]F(\xi)-\frac{\dot{\lambda}}{\lambda}\xi dF(\xi)/d\xi$$
$$=a^2g(t)\,(v_t)^3Q(\xi)+2\nu/(a\lambda)^2\cdot\xi^2F(\xi)\,, \tag{B3}$$

which is consistent only when

$$(av_t)^{\cdot}/(av_t)\propto\dot{\lambda}/\lambda\propto\nu/(a\lambda)^2\propto a^2g(t)\,(v_t)^3. \tag{B4}$$

By the help of Eq. (2·12) we get

$$\lambda^2=\lambda_i^2+\alpha\int_{t_i}^{t}(\nu/a^2)\,dt \tag{B5}$$

from Eq. (B4), and, by taking a constant α as 10 so as to fit the decay law in the static case (cf. Eq. (4·4)), we obtain

$$(v_t)^2=\left(\frac{a_i}{a}\right)^2\frac{\{(\lambda_i)^2/10\}\,(v_t)_i^2}{\int_{t_i}^{t}(\nu/a^2)\,dt+(\lambda_i)^2/10}\,. \tag{B6}$$

Since the viscosity ν_{re} due to electron scattering is proportional to a^2 at the matter dominant stage, Eq. (B6) reduces to Eq. (4·7) with $n=1$.

Moreover we get $g(t) \propto (\nu/a^2)/(a\lambda)^2$ from Eq. (B4) and Eq. (B3) leads to Eq. (4·10) for the spectrum of the turbulent energy.

Remark. Once three decay laws were derived in [N] by dividing the cases according as $\beta<1$, $=1$ or >1. In the case $\beta=1$, $\tilde{\varepsilon}\,(\equiv -\frac{1}{2}d(v_t)^2/dt)$ and $\tilde{t}_d\,(\equiv \frac{1}{2}(v_t)^2/\tilde{\varepsilon})$ have been taken as two independent parameters and $\tilde{t}_d$ identified with t. As a result the decay law is different from the above one and is applicable in the narrow time interval. However if we distinguish between t and $\tilde{t}_d$ and take into account that dimensionless quantities can depend on t also in the forms

$$k(\tilde{\varepsilon}\,\tilde{t}_d^{\,3})^{1/2} = f(t)\,\xi\,,$$

$$E(\tilde{\varepsilon}^3\,\tilde{t}_d^{\,5})^{-1/2} = g(t)\,F(\xi)\,,$$

we can get the same result as above.

(b) *Decay rate due to Loitsyanskii's law*

For big eddies the spectrum tensor Φ_{ij} can be expanded in power series of k as

$$\Phi_{ij}(k,t) = k_l k_m C_{ijlm}(t) + 0(k^3) \tag{B7}$$

and $C_{ijlm} \propto a^{-2}(t)$ in an expanding medium, as was shown in [N]. Since $(\partial\Phi_{ij}/\partial k_l \partial k_m)_{k=0} = -(8\pi^3)^{-1}\int x_l x_m R_{ij}(x,t)\,dx$, we find

$$\bar{r}^5 v^2 \propto (\varepsilon a^4)^{-2} \quad \text{or} \quad l^5 v^2 \propto (\varepsilon^2 a^3)^{-1} \tag{B8}$$

from Eq. (B7). Here $l \equiv a\bar{r}$ represents the physical size of an energy-containing eddy, and its behavior changes in the course of decay. At the early stage when the decay due to the inertial term is dominant, we have

$$l \propto a^m v\,(t + \text{const}) \tag{B9}$$

and at the later stage when the dissipation due to viscosity is dominant,

$$l \propto a^{m'} \{\nu\,(t + \text{const})\}^{1/2}. \tag{B10}$$

Here a^m, $a^{m'}$ reflect the effect of expansion and are determined in such a way that $v^2 \propto (\varepsilon a^4)^{-2}$ if we omit the turbulent dissipation. Then we obtain Eq. (4·6) with $n = 10/7$, $5/2$ at the early and later stages, respectively.

Appendix C

Distortion of the background radiation spectrum

In the presence of the interaction by Compton scattering between radiation and ionized matter, the behavior of the photons occupation number $n_\nu = (c^3/8\pi h\nu^3) \times dE_r/d\nu$ corresponding to radiation energy density $E_r(\nu)$ is described in non-relativistic approximation by

$$\frac{\partial n_\nu}{\partial y}=x^{-2}\frac{\partial}{\partial x}\left[x^4\left(\frac{\partial n_\nu}{\partial x}+n_\nu+n_\nu^2\right)\right], \tag{C1}$$

where $x\equiv h\nu/kT_m$ and the effective scattering thickness

$$y\equiv\Omega n_c\sigma_T cH_0^{-1}\int_z^{z_i}\frac{kT_m}{m_e c^2}\frac{1+z'}{\sqrt{1+\Omega z'}}dz'. \tag{C2}$$

Following Zel'dovich and Sunyaev's procedure,[14] let us derive the small difference Δn_ν of n_ν from Planck's one $\bar{n}_\nu=1/(\exp(\bar{x})-1)$, where $\bar{x}=h\nu/kT_r$ is assumed to be independent of t. For a small distortion such as $y<1$

$$\begin{aligned}\left(\frac{\Delta n_\nu}{n_\nu}\right)_{RJ}&=\left(1-\frac{T_r}{T_m}\right)\frac{\bar{x}ye^{\bar{x}}}{e^{\bar{x}}-1}\{\bar{x}/\mathrm{th}(\bar{x}/2)-2\},\\ \left(\frac{\Delta T_r}{T_r}\right)_{RJ}&=\frac{d\ln T_r}{d\ln n_\nu}\left(\frac{\Delta n_\nu}{n_\nu}\right)_{RJ}=\left(1-\frac{T_r}{T_m}\right)y\{\bar{x}/\mathrm{th}(\bar{x}/2)-2\}\end{aligned} \tag{C3}$$

by adjusting $\Delta n_\nu/n_\nu$ so as to vanish in the Rayleigh-Jeans region. These expressions are different a little from Zel'dovich and Sunyaev's one with respect to the factor $(1-T_r/T_m)$.

If we assume that only a small distortion such as $(\Delta T_r/T_r)_{RJ}<0.05$ in the short wavelength region ($\lambda\sim0.3$ cm) of the cosmic black-body radiation is consistent with the present observations,[14] we have, from Eq. (C3), a condition such as

$$y(1-T_r/T_m)<0.15\,. \tag{C4}$$

Assuming that T_m/T_r remains constant around z_i, we can obtain from Eq. (C2)

$$y=1.1\Omega^3(T_m/T_r)\qquad\text{for } z_i=z_*$$

and (C5)

$$y=0.13\Omega^{3/2}(T_m/T_r)\qquad\text{for } z_i=10^4.$$

In order for the distortion of the spectrum to be small, Eq. (C4) must be satisfied if $y<1$. Therefore, we have the restrictions on the heating such as $T_m/T_r<2$ for $\Omega=1$ and $T_m/T_r<136$ for $\Omega=10^{-1}$.

Appendix D

The integration of the equation for T_m

Let us rewrite Eq. (5·2) by use of $y(t)=a/a_0$ for t. At the matter dominant stage, $y(t)$ may be described approximately by

$$\frac{dy}{dt}=H_0y^{-1/2}\{\Omega+(1-\Omega)y\}^{1/2}$$

in Friedman's pressure-free model. Hence we have

$$y^{-1/2}\{\Omega+(1-\Omega)y\}^{1/2}d(T_m y^2)/dy+By^{-4}(T_m y^2)=By^{-2}(T_r+sy^4\varepsilon_d),$$

where we put $s=m_p/(3kH_0B)$ for brevity. Integrating this we get

$$T_m y^2\exp(-BF)-(T_m y^2\exp(-BF))_i=-\int_{F_i}^{F}dF\exp(-BF)By(T_{r0}+sy^5\varepsilon_d). \tag{D1}$$

Here we used for T_r the present radiation temperature $T_{r0}=yT_r$ and put as follows:

$$F(y)\equiv-\int^{y}dyy^{-7/2}\{\Omega+(1-\Omega)y\}^{-1/2},$$

$$=\frac{2}{15}(3-10q+15q^2)q^{-5/2}(1-\Omega)^{5/2}/\Omega^3,$$

$$q(y)\equiv(1-\Omega)y/\{\Omega+(1-\Omega)y\}.$$

The integral in Eq. (D1) is performed partially as

$$\Big[\exp(-BF)y(T_{r0}+sy^5\varepsilon_d)+\frac{d\ln y}{dF}\frac{\exp(-BF)}{B}y\{T_{r0}+s\,d(y^6\varepsilon_d)/dy\}\Big]_{y_i}^{y}$$

$$-\int_{F_i}^{F}dF\frac{\exp(-BF)}{B}\Big[s(dy/dF)^2d^2(y^6\varepsilon_d)/dy^2$$

$$+\{T_{r0}+sd(y^6\varepsilon_d)/dy\}d^2y/dF^2\Big].$$

Repeating partial integrations furthermore, we obtain a power series of $1/(BF)$ and $|d\ln(y^6\varepsilon_d)/d\ln y|/(BF)$. Therefore we can arrive at the approximate integral (5·4), if these terms are small.

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