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ON THE DISTANCE BETWEEN ZEROES

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ABSTRACT. For the equation $x'' + q(t)x = 0$, let $x(t)$ be a solution with consecutive zeroes at $t = a$ and $t = b$. A simple inequality is proven that relates not only a and b to the integral of $q^+(t)$ but also any point $c \in (a, b)$ where $|x(t)|$ is maximized. As a corollary, it is shown that if the above equation is oscillatory and if $q^+(t) \in L^p[0, \infty)$, $1 \leq p < \infty$, then the distance between consecutive zeroes must become unbounded.

Consider the following second order linear differential equation:

(1) $$x''(t) + q(t)x(t) = 0,$$

where $q(t)$ is continuous on some appropriate t interval. Let $q^+(t) \equiv \max[q(t), 0]$. Pertaining to (1), the following theorem of Hartman [3, p. 345] is known.

Theorem 1. *Let $q(t)$ be real-valued and continuous for $a \leq t \leq b$. If $x(t)$ is a solution of* (1) *with two zeroes in $[a, b]$, then*

(2) $$\int_a^b (t - a)(b - t)q^+(t)\,dt > (b - a).$$

Since $(b - a)^2/4 \geq (t - a)(b - t)$ for $t \in (a, b)$, equation (2) $\Longrightarrow$ *that*

(3) $$\frac{(b - a)^2}{4} \int_a^b q^+(t)\,dt > (b - a),$$

or

(4) $$\int_a^b q^+(t)\,dt > \frac{4}{b - a}.$$

Thus Theorem 1 has as a corollary the following condition of Lyapunov. Again, see Hartman [3, p. 345].

Corollary 1. *A necessary condition for any solution $x(t)$ of* (1) *to have two zeroes in $[a, b]$ is that $\int_a^b q^+(t)\,dt > 4/(b - a)$.*

The lemma that we would like to present is the following.

Lemma 1. *Let $x(t)$ be a solution of* (1), *where $x(a) = x(b) = 0$, and*

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$x(t) \neq 0$, $t \in (a, b)$. Let c be a point in (a, b) where $|x(t)|$ is maximized. Then

(i) $\int_a^c q^+(t)\,dt > 1/(c-a)$,

(ii) $\int_c^b q^+(t)\,dt > 1/(b-c)$,

(iii) $\int_a^b q^+(t)\,dt > (b-a)/[(b-c)(c-a)]$.

Proof. Integrating (1) yields

$$x'(t) - x'(c) = \int_c^t q^-(s)x(s)\,ds - \int_c^t q^+(s)x(s)\,ds.$$

Note that $x'(c) = 0$. Another integration gives

$$x(t) - x(c) = \int_c^t (t-s)q^-(s)x(s)\,ds - \int_c^t (t-s)q^+(s)x(s)\,ds. \tag{5}$$

Let $t = b$, so that $x(b) = 0$. Equation (5) implies that

$$x(b) - x(c) = \int_c^b (b-s)q^-(s)x(s)\,ds - \int_c^b (b-s)q^+(s)x(s)\,ds,$$

or

$$x(c) + \int_c^b (b-s)q^-(s)x(s)\,ds = \int_c^b (b-s)q^+(s)x(s)\,ds.$$

W.L.O.G., we may assume $x(t) \geq 0$, $t \in [a, b]$. Thus we have

$$x(c) \leq \int_c^b (b-s)q^+(s)x(s)\,ds < (b-c)\int_c^b q^+(s)x(s)\,ds$$

$$\Longrightarrow 1 < (b-c)\int_c^b q^+(s)\,ds, \quad \text{since } x(s) \leq x(c), \text{ if } s \in [a, b],$$

$$\Longrightarrow \int_c^b q^+(t)\,dt > \frac{1}{b-c}.$$

This proves part (ii). Part (i) follows in a similar fashion, except that in equation (5), one now replaces t by a. The sum of (i) and (ii) yields part (iii), which completes the lemma.

One way to view Lemma 1 is that it imposes some restrictions on the location of the point c and thus the maximum of $|x(t)|$ in $[a, b]$. That is, $\int_a^b q^+(t)\,dt$ is a finite number. But

$$\lim_{c \to a^+} \frac{b-a}{(b-c)(c-a)} = \lim_{c \to b^-} \frac{b-a}{(b-c)(c-a)} = \infty.$$

Thus c cannot be "too close" to a or b. Also, it is interesting to note that $(b-a)/[(b-c)(c-a)] \geq 4/(b-a)$. This means that under the hypotheses of Lemma 1, Corollary 1 follows from Lemma 1.

As a consequence of Lemma 1 (also Theorem 1 or Corollary 1), we have

Theorem 2. *Suppose* $q^+(t) \in L^p[0, \infty)$, $1 \le p < \infty$. *If* (1) *is oscillatory and if* $x(t)$ *is any solution, then the distance between consecutive zeroes of* $x(t)$ *must become infinite.*

Proof. Suppose not. Then there exists a solution $x(t)$ with its sequence of zeroes $\{t_n\}$, which sequence has a subsequence $\{t_{n_k}\}$ such that $|t_{n_k+1} - t_{n_k}| \le M < \infty$ $\forall k$. Let s_{n_k} be a point in (t_{n_k}, t_{n_k+1}) where $|x(t)|$ is maximized. Then $|s_{n_k} - t_{n_k}| < M$, for all k. Since $q^+(t) \in L^p[0, \infty)$, $1 \le p < \infty$, choose k so large that

$$\left(\int_{t_{n_k}}^{\infty} q^+(t)^p \, dt\right)^{1/p} \le M^{-1-1/r}, \quad \text{where } \frac{1}{p} + \frac{1}{r} = 1.$$

From Lemma 1, part (i), we have

$$\int_{t_{n_k}}^{s_{n_k}} q^+(t)\, dt > \frac{1}{s_{n_k} - t_{n_k}}.$$

Thus

$$\begin{aligned} 1 &< (s_{n_k} - t_{n_k}) \int_{t_{n_k}}^{s_{n_k}} q^+(t)\, dt \\ &< (s_{n_k} - t_{n_k}) \left(\int_{t_{n_k}}^{s_{n_k}} q^+(t)^p \, dt\right)^{1/p} (s_{n_k} - t_{n_k})^{1/r} \\ &< (s_{n_k} - t_{n_k})^{1+1/r} \left(\int_{t_{n_k}}^{\infty} q^+(t)^p \, dt\right)^{1/p} \\ &< M^{1+1/r} \cdot M^{-1-1/r} \implies 1 < 1, \end{aligned}$$

a contradiction. This completes the theorem.

Pertaining to (1), there is the following oscillation theorem of Wintner [5].

If $\lim_{t\to\infty} \int_0^t q(s)\, ds = \infty$, then (1) is oscillatory.

The above condition enables us to construct some simple examples. Consider the equation

$$(6) \qquad x'' + (1 + t)^{-1} x = 0, \qquad t \ge 0.$$

The Wintner condition guarantees that (6) is oscillatory. Since $1/(1 + t) \in L^2[0, \infty)$, Theorem 2 asserts that the distance between zeroes of any solution must become unbounded.

As another example, let

$$q(t) = 1/(n+1), \qquad n + 1/n^2 \le t \le (n+1) - 1/(n+1)^2, \qquad n \ge 2;$$

$$= (n)^{1/4}, \qquad t = n, \qquad n \ge 2;$$

$= $ the line segment joining $(n - 1/n^2, 1/n)$ to $(n, n^{1/4})$

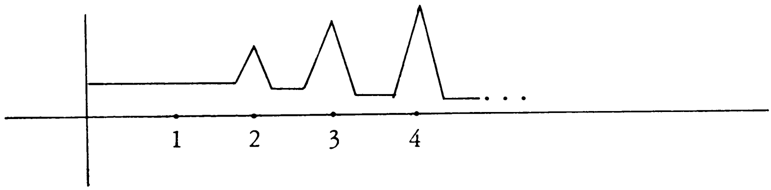
for $n - 1/n^2 \le t \le n, \quad n \ge 2;$

$= $ the line segment joining $(n, n^{1/4})$ to $(n + 1/n^2, 1/(n+1))$

for $n \le t \le n + 1/n^2, \quad n \ge 2;$

$= 1/2$ for $0 \le t \le 7/4$.

So $q(t)$ has the following appearance.



It is easy to verify that $\int_0^\infty q(t)\, dt = \infty$, but $\int_0^\infty q(t)^2\, dt < \infty$. The Wintner condition again implies that (1) is oscillatory, while Theorem 2 implies that the distance between zeroes is unbounded.

Theorem 2 can also be used to derive a known limit point result (Patula and Wong [4, p. 10, Corollary]). Note that for $t \ge 0$, equation (1) is called limit point, L.P., if at least one solution $x(t) \notin L^2[0, \infty)$. If any two linearly independent (and thus all) solutions are square integrable, (1) is called limit circle, L.C. See Coddington and Levinson [1, p. 225].

The following lemma is known (Patula and Wong [4, p. 11]).

Lemma 2. *If equation* (1) *is L.C., then* (1) *is oscillatory, and the distance between consecutive zeroes of any solution tends to zero, as* $t \to \infty$.

We can now prove the following limit point result.

Corollary 2. *If* $q^+(t) \in L^p[0, \infty)$, $1 \le p < \infty$, *then* (1) *is in the limit point classification.*

Proof. Suppose not. Then (1) is L.C. Let $x(t)$ be any solution of (1). By Lemma 2, $x(t)$ oscillates and the distance between consecutive zeroes of any solution tends to zero, as $t \to \infty$. However, Theorem 2 maintains that if $x(t)$ oscillates, the distance between consecutive zeroes must become unbounded, a contradiction. Thus the equation must be limit point.

It should be noted that Theorem 2 does not hold for $p = \infty$, as evidenced by the simple example $x'' = 0$. However, it would be interesting to know if Theorem 2 is true for $0 < p < 1$. If it were, then Corollary 2 could also be extended to the case $0 < p < 1$. This would answer a question posed by Everitt, Giertz, and Weidmann [2, p. 346] as to whether or not (1) is limit point for $q^+(t) \in L^p[0, \infty)$, $0 < p < 1$.

Note added in proof. Lemma 1 is also contained in a paper by J. H. E. Cohn, *Consecutive zeroes of solutions of ordinary second order differential equations,* J. London Math. Soc. (2) **5** (1972), 465–468.

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