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Carlitz–Uchiyama [3] due to Serre [14] (it has been adapted for duals of BCH codes in [6] and [12]).

Theorem 1: If $|q/2 - i| > 2(t - 1)2^{m/2}$, $i \neq 0, q$, then $B_i = 0$. \square

The next result deals with divisibility properties and is based on the Ax theorem [2], see [7], [11], [13], and [16].

Theorem 2: Let a be the smallest positive integer $\geq m/[\log_2 2t]$. If i is not a multiple of 2^a then $B_{q/2-i} = 0$. \square

Apart from some particular cases, namely $t = 1, 2, 3$, when all the values of the distribution were computed explicitly, to the extent of our knowledge, no general estimates of B_i 's were published.

In this correspondence we derive upper bounds on B_i 's. Roughly speaking, these bounds show that the distance distribution can be upper-bounded by the corresponding normal distribution. To derive the bounds we use the linear programming approach along with some estimates on the magnitude of Krawtchouk polynomials of fixed degree in a vicinity of $q/2$.

II. PRELIMINARIES

Let $F = \mathbf{F}_q$ be the finite field of $q = 2^m$ elements and Tr denote the trace function from F to \mathbf{F}_2 . Let \mathcal{G}_t be an additive subgroup of $F[x]$

$$\mathcal{G}_t = \left\{ G(x) = \sum_{i=1}^t a_i x^{2^i-1} : a_i \in F \right\}.$$

Let α be a primitive element in F . For every $G(x) \in \mathcal{G}_t$ and $\epsilon \in \mathbf{F}_2$ we define a vector in \mathbf{F}_2^q

$$\mathbf{c}(G, \epsilon) = (\text{Tr}(G(0)) + \epsilon, \text{Tr}(G(1)) + \epsilon, \dots, \text{Tr}(G(\alpha^{q-2})) + \epsilon).$$

When $G(x)$ runs over \mathcal{G}_t , the set of vectors $\mathbf{c}(G, \epsilon)$ constitute the code dual to the extended BCH codes of length q and with minimum distance $2t + 2$, see, e.g., [1], [10], and [15]. Let $w(\mathbf{c}(G, \epsilon))$ stand for the number of nonzero coordinates in $\mathbf{c}(G, \epsilon)$. For $i \in [0, q]$

$$B_i = |\{G(x) \in \mathcal{G}_t, \epsilon \in \mathbf{F}_2 : w(\mathbf{c}(G, \epsilon)) = i\}|.$$

It is easy to check that $B_0 = 1$ and $\sum_{i=0}^q B_i = 2|\mathcal{G}_t| = 2q^t$. By the MacWilliams identity

$$\sum_{j=0}^q B_j P_i(j) = \begin{cases} 2q^t, & i = 0 \\ 0, & 1 \leq i < 2t + 2. \end{cases} \quad (1)$$

Here $P_i(j)$ are Krawtchouk polynomials (orthogonal on the interval $[0, q]$ with weight $\binom{q}{j}$) defined by the following recurrence (for their properties see, e.g., [5], and [8]–[10]):

$$\begin{aligned} (k+1)P_{k+1}(x) &= (q-2x)P_k(x) - (q-k+1)P_{k-1}(x) \\ P_0(x) &= 1 \quad P_1(x) = q-2x. \end{aligned} \quad (2)$$

We need the following facts about Krawtchouk polynomials:

Orthogonality Relation:

$$\sum_{i=0}^q \binom{q}{i} P_\ell(i) P_k(i) = \delta_{\ell,k} 2^q \binom{q}{\ell}.$$

On the Distance Distribution of Duals of BCH Codes

Ilia Krasikov and Simon Litsyn, *Member, IEEE*

Abstract—We derive upper bounds on the components of the distance distribution of duals of BCH codes.

Index Terms—BCH codes, distance distribution.

I. INTRODUCTION

Let C be the code dual to the extended t -error correcting Bose-Chaudhuri-Hocquenghem (BCH) code of length $q = 2^m$, and let $B = (B_0, \dots, B_q)$ stand for the distance distribution of C . Our aim is to derive upper bounds on B_i 's. The following theorems summarize our present knowledge.

The first one shows that outside a certain interval B_i 's vanish. This is a refinement of the celebrated result by Weil [18] and

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I. Krasikov is with the School of Mathematical Sciences, Tel-Aviv University, Ramat-Aviv 69978, Tel-Aviv, Israel. He is also with the Beit-Berl College, Kfar-Sava, Israel.

S. Litsyn is with the Center for Discrete Mathematics, Rutgers University, Piscataway, NJ 08854 USA, on leave from the Department of Electrical Engineering–Systems, Tel-Aviv University, Ramat-Aviv 69978, Tel-Aviv, Israel (e-mail: litsyn@eng.tau.ac.il).

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Expansion in the Basis of Krawtchouk Polynomials: For a polynomial $\alpha(x) = \sum_{i=0}^r \alpha_i P_i(x)$

$$\alpha_i = 2^{-q} \sum_{j=0}^q P_j(i) \alpha(j). \tag{3}$$

The Christoffel–Darboux Formula:

$$\binom{q}{t} \sum_{i=0}^t \frac{P_i(x)P_i(y)}{\binom{n}{i}} = \frac{t+1}{2(y-x)} (P_{t+1}(x)P_t(y) - P_t(x)P_{t+1}(y)).$$

Letting $y \rightarrow x$ and taking the limit, we get

$$\binom{q}{t} \sum_{i=0}^t \frac{(P_i(x))^2}{\binom{n}{i}} = \frac{t+1}{2} (P_{t+1}(x)P'_t(x) - P_t(x)P'_{t+1}(x)).$$

The following lemma is crucial in our considerations, and is a version of a result implicitly appearing in the thesis by Delsarte [4].

Lemma 1: Let

$$\alpha(x) = \sum_{i=0}^r \alpha_i P_i(x), \quad 0 \leq r < 2t + 2$$

then

$$2q^t \alpha_0 = \sum_{j=0}^q \alpha(j) B_j. \tag{4}$$

Proof: Calculating $2q^t \sum_{i=0}^r \alpha_i B'_i$, and taking into account that $\alpha_i = 0$ for $i > r$, we get the claim from (1). \square

To obtain a bound on B_k , choose in the previous lemma as $\alpha(x)$ a nonnegative polynomial of degree less than $2t + 2$. It yields

$$B_k \leq 2q^t \frac{\alpha_0}{\alpha(k)}. \tag{5}$$

The following lemma gives a polynomial minimizing the right-hand side of this inequality under an extra condition $\alpha(x) = \beta(x)^2$ for some polynomial $\beta(x)$.

Lemma 2: For k given, an optimal polynomial is

$$\begin{aligned} \alpha(x) &= \left(\sum_{i=0}^t \frac{P_i(x)P_i(k)}{\binom{q}{i}} \right)^2 \\ &= \frac{(t+1)^2}{4 \binom{q}{t}^2 (k-x)^2} (P_{t+1}(k)P_t(x) - P_t(k)P_{t+1}(x))^2 \end{aligned} \tag{6}$$

yielding

$$B_k \leq \frac{4 \binom{q}{t} q^t}{(t+1)(P_{t+1}(k)P'_t(k) - P_t(k)P'_{t+1}(k))}.$$

Proof: Let $\beta(x) = \sum_{j=0}^t \beta_j P_j(x)$ and $\alpha(x) = \beta^2(x)$. Then

$$\begin{aligned} \alpha_0 &= \frac{1}{2^q} \sum_{i=0}^q \binom{q}{i} \alpha(i) \\ &= \frac{1}{2^q} \sum_{i=0}^q \binom{q}{i} \left(\sum_{j=0}^t \beta_j P_j(i) \right)^2 \\ &= \frac{1}{2^q} \sum_{j,\ell=0}^t \beta_j \beta_\ell \sum_{i=0}^q \binom{q}{i} P_j(i) P_\ell(i) \end{aligned}$$

by orthogonality of Krawtchouk polynomials

$$\begin{aligned} &= \frac{1}{2^q} \sum_{j,\ell=0}^t \beta_j \beta_\ell \delta_{j,\ell} \binom{q}{j} 2^q \\ &= \sum_{j=0}^t \beta_j^2 \binom{q}{j}. \end{aligned}$$

Thus for k given

$$\begin{aligned} \max_{\beta} \frac{\alpha(k)}{\alpha_0} &= \max_{\beta} \frac{\left(\sum_{j=0}^t \beta_j P_j(k) \right)^2}{\sum_{j=0}^t \beta_j^2 \binom{q}{j}} \\ &= \max_{\beta} \frac{\left(\sum_{j=0}^t (\beta_j \sqrt{\binom{q}{j}}) \left(P_j(k) / \sqrt{\binom{q}{j}} \right) \right)^2}{\sum_{j=0}^t \beta_j^2 \binom{q}{j}} \end{aligned}$$

by Cauchy–Schwartz inequality

$$\leq \sum_{j=0}^t \frac{P_j^2(k)}{\binom{q}{j}}$$

by the Christoffel–Darboux formula

$$= \frac{t+1}{2 \binom{q}{t}} (P_{t+1}(k)P'_t(k) - P_t(k)P'_{t+1}(k)).$$

This bound is clearly achieved for $\beta_j = (P_j(k))/\binom{q}{j}$, that is, the optimal choice for a given k is

$$\begin{aligned} \alpha(x) &= \left(\sum_{j=0}^t \frac{P_j(k)P_j(x)}{\binom{q}{j}} \right)^2 \\ &= \frac{(t+1)^2}{4 \binom{q}{t}^2 (k-x)^2} (P_{t+1}(k)P_t(x) - P_t(k)P_{t+1}(x))^2. \end{aligned}$$

Then the second claim follows from (5). \square

III. ESTIMATES OF B_k

To use the bound of Lemma 2 one needs a lower estimate for the Christoffel–Darboux kernel $P_{t+1}(k)P_t(x) - P_t(k)P_{t+1}(x)$. Assume that q is sufficiently large and t is fixed. In this situation, a classical connection (see, e.g., [17, eq. (2.82.7)]) between Krawtchouk and Hermite polynomials can be employed. However, we need somehow more involved estimates for the accuracy of approximation of Krawtchouk polynomials by Hermite polynomials.

The Hermite polynomials $H_k(x)$ are defined by the recurrence relation

$$\begin{aligned} H_{k+1}(x) &= 2xH_k(x) - 2kH_{k-1}(x) \\ H_0(x) &= 1 \quad H_1(x) = 2x. \end{aligned} \tag{7}$$

Let ε_t stand for the largest root of $H_t(x)$.

Lemma 3:

$$\begin{aligned} P_k \left(\frac{q - \sqrt{2q}y}{2} \right) &= \frac{1}{k! 2^{k/2}} \left(q^{k/2} H_k(y) + 4q^{(k-2)/2} \right) \binom{k}{3} H_{k-2}(y) \\ &\quad + 2 \binom{k}{4} H_{k-4}(y) + q^{(k-4)/2} R_k(y) \end{aligned} \tag{8}$$

where $R_0(y) = R_1(y) = 0$, and

$$\begin{aligned} R_{k+1}(y) &= 2yR_k(y) - \frac{2k(q-k+1)}{q} R_{k-1}(y) \\ &\quad + 8(k-1) \binom{k}{4} (3H_{k-3}(y) + 2yH_{k-4}(y)). \end{aligned} \tag{9}$$

In particular, for fixed k and y

$$P_k \left(\frac{q - \sqrt{2q}y}{2} \right) = \frac{q^{k/2}}{k! 2^{k/2}} H_k(y) + O\left(\frac{1}{q}\right). \tag{10}$$

Proof: Relations (8) and (9) are verified just by substitution into (2) and using (7). \square

In what follows we use the prime sign to denote the derivative in y .

Corollary 1: For k and y fixed and $x = (q - \sqrt{2qy})/2$

$$\frac{d}{dx} P_k(x) = -\frac{q^{(k-1)/2}}{2^{(k-1)/2} k!} H'_k(y) + O\left(\frac{1}{q}\right). \quad \square$$

Using these approximations we get the following.

Lemma 4: For fixed y and $x = (q - \sqrt{2qy})/2$

$$\begin{aligned} P_{t+1}(x) \frac{d}{dx} P_t(x) - P_t(x) \frac{d}{dx} P_{t+1}(x) \\ = \frac{q^t}{2^{t+1}((t+1)!)^2} ((H'_{t+1}(y))^2 - H_{t+1}(y)H''_{t+1}(y)) + O(q^{t-1}). \end{aligned} \quad (11)$$

Proof: With accuracy up to $O(1/q)$ we have from Lemma 3

$$\begin{aligned} P_{t+1}(x) \frac{d}{dx} P_t(x) - P_t(x) \frac{d}{dx} P_{t+1}(x) \\ = \frac{q^t}{2^t t! (t+1)!} (H_t(y)H'_{t+1}(y) - H_{t+1}(y)H'_t(y)) \end{aligned}$$

and using $H'_{t+1}(x) = 2(t+1)H_t(x)$ (see, e.g., [17, p. 106]) we get the claim. \square

Now we are in a position to translate the derived estimates to bounds for B_k .

Theorem 3: For fixed y and $k = (q - \sqrt{2qy})/2$

$$B_k \leq \frac{q^t (t+1)! 2^{t+3}}{(H'_{t+1}(y))^2 - H_{t+1}(y)H''_{t+1}(y)} \left(1 + O\left(\frac{1}{q}\right)\right). \quad (12)$$

To use this expression we need estimates for Hermite polynomials when $y < \sqrt{2t}$.

The denominator of (12) can be easily computed if x does not belong to the interval where the roots of $P_t(x)$ are located (or, which is asymptotically the same, $|y| > \varepsilon_t$). Indeed, by (2), $P_t(x)$ is a polynomial of degree t in q , and

$$\sum_{i=0}^t \frac{(P_i(x))^2}{\binom{q}{i}} = \frac{(P_t(x))^2}{\binom{q}{t}} \left(1 + O\left(\frac{1}{q}\right)\right).$$

In this case, we have

$$\frac{P_t^2(x)}{\binom{q}{t}} \approx \frac{q^t H_t^2(y)}{2^t (t!)^2 \binom{q}{t}} \approx \frac{(H_t(y))^2}{2^t t!}.$$

Theorem 4: Let $y = \frac{q-2k}{\sqrt{2q}}$. For $|\frac{q}{2} - k| > \frac{(t-1)\sqrt{q}}{\sqrt{t+2}}$

$$B_k \leq \frac{q^t t! 2^{t+1}}{(H_t(y))^2} \left(1 + O\left(\frac{1}{q}\right)\right).$$

Proof: Follows from the estimate on the largest root of $H_t(y)$ due to Laguerre, see [17, p. 120]

$$\varepsilon_t \leq \frac{\sqrt{2}(t-1)}{\sqrt{t+2}} \quad (13)$$

and $y = O(t)$ by Theorem 1. \square

To apply this estimate one needs asymptotics for Hermite polynomials. For the interval under consideration it is well known and can be found, e.g., in [17, p. 200]. When y belongs to the interval where the roots of $H_t(y)$ exist, another approach should be employed.

Lemma 5: Let

$$W_t(y) = (H'_t(y))^2 - H_t(y)H''_t(y).$$

Then

$$W_t(0) = \begin{cases} 2t \binom{t}{t/2} t!, & \text{for } t \text{ even} \\ 4t \binom{t-1}{(t-1)/2} t!, & \text{otherwise} \end{cases}$$

and

$$\begin{aligned} W_t(y) &\geq e^{y^2} \frac{\sqrt{2t} - |y|}{\sqrt{2t}} W_t(0), \quad |y| \leq \sqrt{2t} \\ W_t(y) &\leq e^{y^2} \frac{\sqrt{2t} + |y|}{\sqrt{2t}} W_t(0). \end{aligned}$$

Proof: We start with calculating $W_t(0)$. It is known that

$$H_t(0) = \begin{cases} (-1)^{t/2} \frac{t!}{(t/2)!}, & \text{for } t \text{ even} \\ 0, & \text{otherwise.} \end{cases}$$

From the differential equation for Hermite polynomials

$$H''_t(y) = 2yH'_t(y) - 2tH_t(y)$$

and

$$H'_t(y) = 2tH_{t-1}(y) \quad (14)$$

we get for t even

$$W_t(0) = 2t(H_t(0))^2 = 2t \binom{t}{t/2} t!.$$

For t odd

$$W_t(0) = 4t^2(H_{t-1}(0))^2 = 4t \binom{t-1}{(t-1)/2} t!.$$

Notice that $W_t(y)$ is strictly positive. Indeed, let y_i stand for the i th root of $H_t(y)$. Then

$$H_t(y) = 2^t \prod_{i=1}^t (y - y_i)$$

and differentiating it we get

$$\begin{aligned} H'_t(y) &= H_t(y) \sum_{i=1}^t \frac{1}{y - y_i}, \\ H''_t(y) &= H'_t(y) \sum_{i=1}^t \frac{1}{y - y_i} - H_t(y) \sum_{i=1}^t \frac{1}{(y - y_i)^2} \\ &= H_t(y) \left(\sum_{i=1}^t \frac{1}{y - y_i} \right)^2 - \sum_{i=1}^t \frac{1}{(y - y_i)^2}. \end{aligned}$$

Thus

$$W_t(y) = (H_t(y))^2 \sum_{i=1}^t \frac{1}{(y - y_i)^2} > 0.$$

Without loss of generality we assume y is nonnegative. Using (14) we obtain

$$\begin{aligned} W_t(y) &= 2t(H_t(y))^2 - 2yH_t(y)H'_t(y) + (H'_t(y))^2 \\ W'_t(y) &= 4t y(H_t(y))^2 - 2(1 + 2y^2)H_t(y)H'_t(y) + 2y(H'_t(y))^2. \end{aligned}$$

Denoting $t = \mu^2/2$, we get

$$\begin{aligned} W'_t(y) + \frac{1 - 2\mu y + 2y^2}{\mu - y} W_t(y) &= \frac{(\mu H_t(y) - H'_t(y))^2}{\mu - y} \\ W'_t(y) - \frac{1 + 2\mu y + 2y^2}{\mu + y} W_t(y) &= -\frac{(\mu H_t(y) + H'_t(y))^2}{\mu + y}. \end{aligned}$$

From the first equality, for $0 \leq y < \mu$, and taking into account that $W_t(y) > 0$, we conclude

$$\frac{W'_t(y)}{W_t(y)} \geq -\frac{1 - 2\mu y + 2y^2}{\mu - y}. \tag{15}$$

On the other hand, from the second equality

$$\frac{W'_t(y)}{W_t(y)} \leq \frac{1 + 2\mu y + 2y^2}{\mu + y}. \tag{16}$$

Integrating (15), we obtain

$$\int_0^y \frac{W'_t(z)}{W_t(z)} dz = \ln \frac{W_t(y)}{W_t(0)} \geq y^2 + \ln \frac{\mu - y}{\mu}$$

thus proving the lower bound on $W_t(y)$. Similarly, integrating (16), we get the claimed upper bound. \square

Notice, that the estimates of the lemma are quite accurate for $y < \sqrt{2t}$. Indeed, the maximum of the function

$$e^{y^2} \frac{\sqrt{2t} - |y|}{\sqrt{2t}}$$

is achieved at

$$|y| = \frac{\sqrt{t} + \sqrt{t-1}}{\sqrt{2}} \approx \sqrt{2t} - \frac{1}{\sqrt{8t}} > \varepsilon_t$$

i.e., almost at the end of the interval $|y| < \sqrt{2t}$. Even at this point the ratio between the upper and lower bound is less than $8t$, and all the roots of $H_t(y)$ are within this interval.

Numerical evidence suggests that (11) still gives an accurate approximation in a much wider interval of t and y . It is tempting to conjecture that actually the Christoffel–Darboux kernel can be well approximated by Hermite polynomials for all $t = o(\sqrt{q})$.

Now we can give an upper bound on B_k for the interval containing zeroes of $H_t(y)$.

Theorem 5: Let $\left| \frac{q}{2-k} \right| < \sqrt{(t+1)q}$, then

$$B_k \leq \frac{\sqrt{q} q^t 2^{t+4}}{\sqrt{t+1} |2\sqrt{q(t+1)} - q + 2k| \binom{t+1}{(t+1)/2}} \cdot e^{-((q-2k)^2/sq)} \left(1 + O\left(\frac{1}{q}\right) \right), \quad \text{for } t \text{ odd}$$

$$B_k \leq \frac{\sqrt{q} q^t 2^{t+3}}{\sqrt{t+1} |2\sqrt{q(t+1)} - q + 2k| \binom{t}{t/2}} \cdot e^{-((q-2k)^2/sq)} \left(1 + O\left(\frac{1}{q}\right) \right), \quad \text{for } t \text{ even}$$

$$B_k \leq \frac{4\sqrt{2\pi} q^t}{\left| 2\sqrt{q(t+1)} - q + 2k \right|} \cdot e^{-((q-2k)^2/q)} \left(1 + O\left(\frac{1}{t}\right) \right), \quad \text{for sufficiently large } t.$$

\square

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