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Pavel Křivka; Nenad Trinajstić

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ON THE DISTANCE POLYNOMIAL OF A GRAPH

PAVEL KŘIVKA, NENAD TRINAJSTIĆ

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The distance matrix, which is a structure less common than the adjacency matrix [1], has been increasingly used in recent years in several different areas such as anthropology [2], [3], graph theory [4]–[6], theory of communication [7], chemistry [8]–[10], history [11], etc.

The distance matrix (which is sometimes also called the metrics matrix) is, in a sense, a more complicated and also a richer structure than the adjacency matrix. While each (0, 1) symmetric matrix in which each diagonal entry is zero corresponds to a unique graph or digraph, this is not the case with the distance matrix (which is a symmetric matrix with diagonal entries zero, but not (0, 1)). We will consider finite connected graphs without loops. Graphs will be symbolized by G. The distance matrix $\mathbf{D}$ has the entries $d_{ii} = 0$ and d_{ij} = the length of the shortest path between the vertices i and j. The distance polynomial of G is defined as $\det |x\mathbf{I} - \mathbf{D}|$ where $\mathbf{I}$ is the unit matrix.

Proposition 1. *Every root of the distance polynomial is real.*

Proof. The eigenvalues of a symmetric matrix are always real [12].

Proposition 2. *The coefficient a_{n-1} at x^{n-1} is always zero.*

Proof. Follows from the development of the determinant.

Proposition 3. *The distance polynomial of a complete graph with n vertices equals $(x + 1)^{n-1} (x - n + 1)$.*

Proof. For complete graphs the distance matrix coincides with the adjacent matrix.

Proposition 4. *The sum of the roots of the distance polynomial is zero.*

Proof. The sum of roots is equal to the trace of $\mathbf{D}$ which is zero since G has no loops.

Proposition 5. *The sum of the squares of the roots of the distance polynomial is equal to* $\sum d_{ij}^2$ *(the sum is through all pairs i, j and each pairs is taken twice —* (i, j) *and* (j, i)*).*

Proof. The sum of the squares of the roots is equal to the trace of $\mathbf{D}^2$.

Proposition 6. *If there are two vertices with the same neighbourhood in G then one root of the distance polynomial is either* -1 *(if the vertices are adjacent) or* -2 *(if the two vertices are not adjacent).*

Proof. If the two vertices are adjacent, the distance between them is 1. If the two vertices are not adjacent, the distance between them is 2, otherwise the corresponding rows in $\mathbf{D}$ are the same. After subtracting these two rows, we obtain before the determinant either $(x + 1)$ or $(x + 2)$.

Proposition 7. *If G is a path on* $n + 1$ *vertices* $(n \geqq 3)$, *then* $a_0 = (-1)\, n\, 2^{n-1}$, *where* a_0 *is a constant coefficient, i.e.* $a_0 = \det |-\mathbf{D}|$.

Proof. By directly calculating the determinant

$$(1) \qquad \det |\mathbf{D}| = \begin{vmatrix} 0 & 1 & 2 & 3 & \dots & n \\ 1 & 0 & 1 & 2 & \dots & n-1 \\ 2 & 1 & 0 & 1 & \dots & n-2 \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ n & n-1 & \cdot & \cdot & \dots & 0 \end{vmatrix}$$

First we subtract the n-th column from the $(n + 1)$-st and afterwards we add the last row to all the precedings ones, so that in the last column we obtain all zeros except the entry $a_{n+1,n+1} = -1$.

$$(2) \qquad \begin{vmatrix} n & n & n & \dots & n & 0 \\ n+1 & n-1 & n-1 & \dots & n-1 & 0 \\ n+2 & n & n-2 & \dots & n-2 & 0 \\ \vdots & \cdot & & & & \cdot \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ 2n-1 & \cdot & \cdot & & \cdot & 0 \\ n & \cdot & & & \cdot & -1 \end{vmatrix} =$$

$$= (-1)\, n \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ n+1 & n-1 & n-1 & \dots & n-1 \\ n+2 & n & n-2 & \dots & n-2 \\ \cdot & \cdot & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ 2n-1 & \cdot & \cdot & & 1 \end{vmatrix}$$

The resulting determinant can be expanded with respect to the last column and n put in front of the determinant. Next we subtract the last column from all the preceding ones and expand with respect to the first row:

(3)
$$(-1)\, n(-1)^{n+1} \begin{vmatrix} 2 & 0 & 0 & . & . & 0 \\ 4 & 2 & 0 & . & . & 0 \\ 6 & 4 & 2 & . & . & . \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & & & & & \cdot \\ 2n-2 & 2n-4 & . & . & . & 2 \end{vmatrix} = (-1)_{n+2}\, n\, 2^{n-1}$$

Because $\det|-\mathbf{D}| = (-1)^{n+1} \det|\mathbf{D}|$, we have $a_c = (-1)^{2n+3}\, n\, 2^{n-1} = -n \,.\, 2^{n-1}$, q.e.d.

Proposition 8. *For complete bipartite graphs $K_{m,n}$, the roots are* -2 ($m + n - 2$ times) *and the remaining two roots are given by the following equations*:

$$x_1^2 + x_2^2 = 2(2m^2 - 4m + 2n^2 - 4n + mn + 4)$$

$$x_1 + x_2 = 2(m + n - 2)$$

Proof. We can write the distance polynomial of $K_{m,n}$ in the form

(4)
$$\det|x\mathbf{I} - \mathbf{D}| = \left|\begin{array}{cccc:cccc} x & \dots & & -2 & & & & \\ \cdot & \cdot & & \cdot & & & & \\ \cdot & & \cdot & \cdot & & \mathbf{B} & & \\ \cdot & & \cdot & \cdot & & & & \\ -2 & \dots & & x & & & & \\ \hdashline & & & & x & \dots & & -2 \\ & \mathbf{B}^{\mathrm{T}} & & & \cdot & \cdot & & \cdot \\ & & & & \cdot & & \cdot & \cdot \\ & & & & \cdot & & \cdot & \cdot \\ & & & & -2 & \dots & & x \end{array}\right|$$

where $\mathbf{B}$ denotes the $(m \times n)$ matrix consisting entirely of -1's. By subtracting the second row the first, the third from the second, until m-th from the $(m - 1)$-st, and the $(m + 2)$-nd from the $(m + 1)$-st, etc., we obtain the following determinant

(5)

$$\left|\begin{array}{ccccc|ccccc} x+2 & -(x+2) & 0 & \dots & 0 & & & & & \\ 0 & x+2 & -(x+2) & \dots & 0 & & & 0 & & \\ \cdot & \cdot & \cdot & & \cdot & & & & & \\ \cdot & & & & & & & & & \\ \cdot & & & & & & & & & \\ \cdot & & & & & & & & & \\ -2 & \cdot & \cdot & \dots\ -2 & 0 & -1 & -1 & 0 & \dots & -1 \\ \hline & & & & & x+2 & -(x+2) & 0 & \dots & 0 \\ & & & & & 0 & x+2 & -(x+2) & \dots & 0 \\ & & & & & \cdot & & & & \\ & & 0 & & & \cdot & & & & \\ & & & & & \cdot & & & & \\ & & & & & \cdot & & & & \\ -1 & \cdot & \cdot & & -1 & -2 & \cdot & \cdot & -2 & x \end{array}\right|$$

We immediately see that $m + n - 2$ eigenvalues are -2. According to Proposition 5, we have $\sum x_i^2 = \sum_{i,j} d_{ij}^2 = 4m(m - 1) + 4n(n - 1) + 2mn = 4m^2 - 4m + 4n^2 - 4n + 4mn$ and as $\sum_{i=3}^{m+n} x_i^2 = 4(m + n - 2)$, we have $x_1^2 + x_2^2 = 2(2m^2 + 2n^2 - 4m - 4n + mn + 4)$. According to Proposition 4, we have $x_1 + x_2 - 2(m + n - 2) = 0$.

Proposition 9. *If G is a star with n vertices, then*

$$\det |x\mathbf{I} - \mathbf{D}| = (x + 2)^{n-2} (x^2 - 1 - (n - 2)(2x + 1)).$$

The distance polynomial of the star graph will be denoted by S_n.

Proof. The distance polynomial of a star is given by

$$(6) \qquad S_n = \begin{vmatrix} x & -1 & -1 & \dots & -1 \\ -1 & x & -2 & \dots & -2 \\ -1 & -2 & x & \dots & -2 \\ \cdot & & & & \\ \cdot & & & & \\ -1 & -2 & -2 & \dots & x \end{vmatrix}$$

By subtracting the $(n - 1)$-st row from the n-th row, we can put the factor $(x + 2)$ (from the last row) in front of the determinant and expand it with respect to the last row:

$$(7)\qquad S_n = (x+2)\left(\begin{vmatrix} x & -1 & -1 & \dots & -1 \\ -1 & x & -2 & \dots & -2 \\ -1 & -2 & x & \dots & -2 \\ \cdot & & & & \\ \cdot & & & & \\ -1 & -2 & -2 & \dots & -2 \end{vmatrix} + \begin{vmatrix} x & -1 & -1 & \dots & -1 \\ -1 & x & -2 & \dots & -2 \\ -1 & -2 & x & \dots & -2 \\ \cdot & & & & \\ \cdot & & & & \\ -1 & -2 & -2 & \dots & x \end{vmatrix}\right)$$

The second determinant evidently corresponds to S_{n-1}. The first determinant differs from the second one only in the entry $d_{n-1,n-1} = -2$. If we denote this one by $\bar{S}_{n-1}$, we obtain the recurrent formula

$$(8)\qquad S_n = (x+2)(\bar{S}_{n-1} + S_{n-1}).$$

Now, if we consider the determinant $\bar{S}_n$, we obtain by the same procedure the formula

$$(9)\qquad \bar{S}_n = (x+2)\bar{S}_{n-1}.$$

Combining (8) and (9) we obtain

$$(10)\qquad S_n = (x+2)^{n-2}(S_2 + (n-2)\bar{S}_2)$$

and

$$(11)\qquad S_2 = \begin{vmatrix} x & -1 \\ -1 & x \end{vmatrix} = x^2 - 1,$$

$$(12)\qquad \bar{S}_2 = \begin{vmatrix} x & -1 \\ -1 & -2 \end{vmatrix} = -2x - 1, \quad \text{q.e.d.}$$

Proposition 10. *If G is an even cycle, then at least one root of the distance polynomial is zero.*

Proof. It suffices to prove that $\det|D| = 0$.

$$(13)\qquad \det|\mathbf{D}| = \begin{vmatrix} 0 & 1 & 2 & 3 & \dots & n/2 & \dots & 1 \\ 1 & 0 & 1 & 2 & \dots & (n/2)-1 & \dots & 2 \\ 2 & 1 & 0 & 1 & \dots & (n/2)-2 & \dots & 3 \\ 3 & 2 & 1 & 0 & \dots & \cdot & & \cdot \\ \cdot & & & & & & & \\ \cdot & & & & & & & \\ n/2 & (n/2)-1 & (n/2)-2 & (n/2)-3 & \dots & 0 & \dots & (n/2)-1 \\ \cdot & \cdot & \cdot & \cdot & & & & \\ \cdot & & & & & & & \\ \cdot & & & & & & & \\ \cdot & & & & & \cdot & & 1 \\ 1 & 2 & 3 & \cdot & \dots & n/2 & \dots\ 1 & 0 \end{vmatrix}$$

By adding the first row to the $(n/2+1)$-st one and the second row to the $(n/2+2)$-nd one, we get two identical rows, namely the $(n/2+1)$-st and the $(n/2+2)$-nd, q.e.d.

Comment. Since the distance matrix for cycles is a circulant matrix, we can obtain its roots by means of rules for the corresponding determinant [13].

Examples.

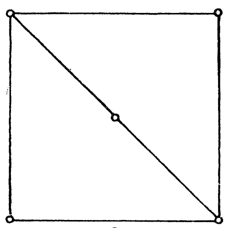
a) C_3 – cycle

$$\det |x\mathbf{I} - \mathbf{D}| = (x - 2)(x + 1)^2$$

b) C_4-cycle

$$\det |x\mathbf{I} - \mathbf{D}| = (x + 2)^2 (x - 4) x$$

c)



$$\det |x\mathbf{I} - \mathbf{D}| = (x + 2)^3 (x^2 - 6x + 2)$$

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Note added in proof. Proposition 7 holds generally for any tree as was first proved in [7]. By an accident we overlooked a paper [a] which contains without proofs some theorems related to our work. Some other related references can be found in the second edition of the book [b] that came only recently into our hands.

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Souhrn

O DISTANČNÍM POLYNOMU GRAFU

PAVEL KŘIVKA, NENAD TRINAJSTIĆ

Článek se zabývá některými vlastnostmi distančního polynomu některých typů grafů, speciálně cest, bipartitních grafů, cyklů a hvězd.

Authors' addresses: RNDr. *Pavel Křivka*, CSc., Vysoká škola chemickotechnologická, Leninovo nám. 565, 532 10 Pardubice; Prof. *Nenad Trinajstić*, Rugjer Bošković Institute, P.O.B. 1016, 41001 Zagreb, Croatia Yugoslavia.