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ON THE DISTRIBUTION OF BROWNIAN AREAS

BY MIHAEL PERMAN AND JON A. WELLNER[1]

University of Ljubljana and University of Washington

We study the distributions of the areas under the positive parts of a Brownian motion process B and a Brownian bridge process U: with $A^+ = \int_0^1 B^+(t)\,dt$ and $A_0^+ = \int_0^1 U^+(t)\,dt$, we use excursion theory to show that the Laplace transforms $\Psi^+(s) = E\exp(-sA^+)$ and $\Psi_0^+(s) = E\exp(-sA_0^+)$ of A^+ and A_0^+ satisfy

$$\int_0^\infty e^{-\lambda s}\Psi^+\big(\sqrt{2}\,s^{3/2}\big)\,ds = \frac{\lambda^{-1/2}Ai(\lambda) + \big(1/3 - \int_0^\lambda Ai(t)\,dt\big)}{\sqrt{\lambda}\,Ai(\lambda) - Ai'(\lambda)}$$

and

$$\int_0^\infty \frac{e^{-\lambda s}}{\sqrt{s}}\Psi_0^+\big(\sqrt{2}\,s^{3/2}\big)\,ds = 2\sqrt{\pi}\,\frac{Ai(\lambda)}{\sqrt{\lambda}\,Ai(\lambda) - Ai'(\lambda)},$$

where Ai is Airy's function. At the same time, our approach via excursion theory unifies previous calculations of this type due to Kac, Groeneboom, Louchard, Shepp and Takács for other Brownian areas. Similarly, we use excursion theory to obtain recursion formulas for the moments of the "positive part" areas. We have not yet succeeded in inverting the double Laplace transforms because of the structure of the function appearing in the denominators, namely, $\sqrt{\lambda}\,Ai(\lambda) - Ai'(\lambda)$.

1. Introduction. Our goal in this paper is to study the distributions of the random areas

$$A^+(t) \equiv \int_0^t B^+(s)\,ds \quad\text{and}\quad A_0^+ \equiv \int_0^1 U^+(t)\,dt, \tag{1.1}$$

where B is a standard Brownian motion process, U is a Brownian bridge process and f^+ denotes the positive part of any real-valued function f on $[0,1]$: $f^+(t) \equiv f(t) \vee 0$. We also compare our calculations for A^+ and A_0^+ with similar calculations for the related Brownian areas

$$A(t) \equiv \int_0^t |B(s)|\,ds, \qquad A_0 \equiv \int_0^1 |U(t)|\,dt, \tag{1.2}$$

$$A_{\text{excur}} \equiv \int_0^1 e(t)\,dt, \qquad A_{\text{mean}} \equiv \int_0^1 d(t)\,dt, \tag{1.3}$$

where $e(t)$ is a Brownian excursion process and $d(t)$ is a Brownian meander process; see, for example, Durrett and Iglehart (1977). We show how excur-

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sion theory leads to a common structure for calculations for all of these Brownian areas.

The "double Laplace transform" of the distribution of A_0 was found by Cifarelli (1975) and independently by Shepp (1982). The first level of inversion of this transform was accomplished by Rice (1982), and the second level was carried out by Johnson and Killeen (1983). In the case of A, the Laplace transform was found by Kac (1946), and this has been inverted and the density function tabled by Takács (1993a).

The double Laplace transform of A_{excur} was computed independently by Louchard (1984a) and Groeneboom (1989) (Groeneboom's paper was written in 1984), and the first stage of inversion was accomplished by Louchard (1984b). Takács (1992b) carried out the second stage of inversion to obtain the distribution function explicitly. Takács also gave recursion formulas for the moments and pointed out an interesting connection between the distribution of A_{excur} and the supremum of a certain Gaussian process which was studied by Darling (1983). See Borodin (1984) for related results.

Despite the considerable knowledge of the distributions of A, A_0, A_{excur} and A_{mean}, the distributions of A^+ and A_0^+ are apparently unknown.

This paper presents two approaches to computing the double Laplace transforms. The first one uses random scaling and the master formulas from excursion theory and uses known results derived by Kac (1951) and Shepp (1982) for the double Laplace transforms of the random variables A and A_0. This approach offers some new insight into the structure of such transforms. The second approach is the same as that used by Shepp (1982) and Louchard (1984a, b): we use Kac's formula [Kac (1951) and Itô and McKean (1974)], or an appropriate conditioned version in the case of Brownian bridge U, to find formulas for the double Laplace transform of the general additive functionals

$$K(t) \equiv \int_0^t k(B(s))\,ds \tag{1.4}$$

and

$$K_0(t) \equiv \int_0^t k(U(s))\,ds, \tag{1.5}$$

where

$$k(x) \equiv \beta x^+ + \gamma x^- = \beta x 1_{[0,\infty)}(x) - \gamma x 1_{(-\infty,0)}(x) \tag{1.6}$$

for $\beta,\ \gamma \geq 0$. Specializing these formulas to the case $\gamma = \beta$ gives back the double transforms found by Kac (1946) and Shepp (1982), while taking the limit as $\gamma \to 0$ yields the desired double transforms for the distributions of A^+ and A_0^+.

We then use these double transforms to derive recurrence formulas for the moments of A^+ and A_0^+, and these in turn yield expansions for the distributions of A^+ and A_0^+ in terms of Laguerre series much as in Takács (1993a).

We have not yet succeeded in direct inversion of the double Laplace transforms. This requires more detailed knowledge about the function

$$\sqrt{u}\,Ai(u) - Ai'(u)$$

in both cases.

2. Statistical background. Suppose that $X_1, \dots, X_n$ are i.i.d. with distribution function F, and let F_0 be a fixed continuous distribution function. Consider testing

$$H\colon F = F_0 \quad \text{versus} \quad K\colon F \prec_s F_0;$$

here $F \prec_s F_0$ means $F(x) \ge F_0(x)$ for all x and $F(x) > F_0(x)$ for some x. Under H the variables $U_i = F_0(X_i)$ are i.i.d. Uniform(0, 1), while under K the U_i's have a distribution function G given by

$$G(x) = P(U_1 \le x) = F\big(F_0^{-1}(x)\big), \qquad 0 < x < 1.$$

Note that $G \prec_s G_0$, where G_0 is the Uniform(0, 1) distribution function.

One simple statistic for testing H versus K is

$$T_n^* = -\sqrt{n}\,\big(\overline{U}_n - 1/2\big) = \int_0^1 \sqrt{n}\,(\mathbb{G}_n(t) - t)\,dt = \int_0^1 U_n(t)\,dt,$$

where $\mathbb{G}_n(t) = n^{-1}\sum_{i=1}^n 1_{[0,t]}(U_i)$ is the empirical distribution function of the U_i's, and $U_n(t) = \sqrt{n}\,(\mathbb{G}_n(t) - t)$ is the uniform empirical process (under H). The statistic T_n^* was apparently proposed by L. Moses; see Chapman (1958) and Birnbaum and Tang (1964). Under the null hypothesis H we have $T_n^* \to_{\mathscr{D}} N(0, 1/12)$, and for small sample sizes the distribution can even be calculated exactly; see, for example, Feller (1971), Theorem 1a, page 28. Of course, another formulation of the limiting distribution is that

$$T_n^* = \int_0^1 U_n(t)\,dt \to_{\mathscr{D}} \int_0^1 U(t)\,dt \sim N(0, 1/12);$$

here U is a standard Brownian bridge process on $[0, 1]$. This follows from standard weak convergence arguments.

Another appealing statistic for testing H versus K is

$$\begin{aligned} T_n^+ &= \int_0^1 \sqrt{n}\,(\mathbb{G}_n(t) - t)^+\,dt = \int_0^1 U_n^+(t)\,dt \\ &\to_{\mathscr{D}} \int_0^1 U^+(t)\,dt = A_0^+. \end{aligned}$$

Thus the limiting distribution of T_n^+ is not normal, and is, in fact, unknown. This is part of the motivation for studying the distribution of A_0^+. Another statistic related to T_n^+ is the statistic S_n^+ proposed by Riedwyl (1967):

$$S_n^+ = \frac{1}{\sqrt{n}} \sum_{i=1}^n \left(\mathbb{G}_n\left(\frac{i}{n}\right) - \frac{i}{n}\right)^+ = \frac{1}{n} \sum_{i=1}^n U_n^+\left(\frac{i}{n}\right).$$

Riedwyl tabulated the null distribution of S_n^+ for $1 \le n \le 12$. It is easily seen that the asymptotic distribution of S_n^+ under H is the same as T_n^+, namely, that of A_0^+. Riedwyl also considered the two-sided statistic S_n defined by

$$S_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left| \mathbb{G}_n\left(\frac{i}{n}\right) - \frac{i}{n} \right| = \frac{1}{n} \sum_{i=1}^{n} \left| U_n\left(\frac{i}{n}\right) \right|,$$

and tabulated its null distribution for $1 \le n \le 12$. Here S_n has the same asymptotic limiting null distribution as the two-sided test statistic

$$T_n = \int_0^1 |\sqrt{n}\,(\mathbb{G}_n(t) - t)|\, dt = \int_0^1 |U_n(t)|\, dt,$$

namely, the distribution of $A_0 = \int_0^1 |U(t)|\, dt$.

3. Double Laplace transforms via excursions. Define Ψ_0^k and Ψ^k to be the Laplace transforms of the random variables $K_0(1)$ and $K(1)$ defined in (1.4) and (1.5) for some $\beta, \gamma \ge 0$, $k(x) = \beta x^+ + \gamma x^-$:

$$\Psi_0^k(s) \equiv E \exp\left(-s \int_0^1 k(U(t))\, dt \right), \tag{3.7}$$

$$\Psi^k(s) \equiv E \exp\left(-s \int_0^1 k(B(u))\, du \right), \tag{3.8}$$

where B is standard BM and U is a standard Brownian bridge on $[0, 1]$. This section will be concerned with the computation of the double Laplace transforms:

$$\int_0^\infty \frac{e^{-\lambda s}}{\sqrt{2\pi s}} \Psi_0^k(s^{3/2})\, ds, \tag{3.9}$$

$$\int_0^\infty e^{-\lambda s} \Psi^k(s^{3/2})\, ds. \tag{3.10}$$

The approach will be based on excursion theory and properties of Poisson point processes. Theorems 3.3 and 3.5 give suitable conditional versions of (3.9) and (3.10) from which the above transformations will follow by integration.

First some preliminary facts about excursions need to be established. Throughout this section let B be standard Brownian motion and let $(l_t: t \ge 0)$ stand for its local time process at level 0 in the standard normalization such that $M = |B| - l$ is a martingale. Furthermore, let $g_t = \sup\{u \le t: B_u = 0\}$ denote the last exit time from 0 of B before time t. The following lemma is well known [see Lévy (1948), Dynkin (1961) and Barlow, Pitman and Yor (1989)].

LEMMA 3.1. *The distribution of* $g_1 = \sup\{u \le 1: B_u = 0\}$ *is* Beta$(1/2, 1/2)$. *Given* g_1, *the process* $(B_t: 0 \le t \le g_1)$ *is a Brownian bridge of length* g_1, *and* *the rescaled process* $(B(g_1 u)/\sqrt{g_1}: 0 \le u \le 1)$ *is a Brownian bridge independent of* g_1.

Let S_λ be an exponential random variable with parameter λ independent of B. It is clear from Lemma 3.1 that the process $(B_u: 0 \le u \le g_{S_\lambda})$ is a Brownian bridge randomly rescaled to the interval $[0, g_{S_\lambda}]$ by Brownian scaling. On the other hand, it is well known that the excursion of B that straddles S_λ can be thought of as the first marked excursion in the Poisson process of excursions with marks assigned independently with probability $1 - e^{-\lambda R(e)}$, where $R(e)$ is the duration of the excursion e. The results on marked excursions are well known; for a detailed treatment, see Rogers and Williams (1987), page 418. The precise statement of the assertions is as follows.

THEOREM 3.1. *Let* $(e_s: s \ge 0)$ *be the excursion process of* B *in the sense of Itô and let* S_λ *be an exponential random variable with parameter* λ *independent of* B.

(i) *The local time* l_{S_λ} *during the excursion straddling* S_λ *has exponential distribution with parameter* $\sqrt{2\lambda}$ *and is independent of* B_{S_λ}. *Moreover,* l_{S_λ} *is independent of the excursion* $e^* = e_{l(S_\lambda)}$.

(ii) *Given* $l_{S_\lambda} = l$, *the process of excursions* $(e_s: 0 < s < l)$ *is conditionally a Poisson process with Itô excursion law* m *given by* $m(de) = e^{-\lambda R(e)} n(de)$, *where* n *is Itô's excursion law for* B *and* $R(e)$ *denotes the duration of the excursion. Moreover,* $(e_s: 0 < s < l)$ *is independent of* $e^* = e_{l(S_\lambda)}$.

(iii) *The law* n^* *of* $e^* = e_{l(S_\lambda)}$ *is given by*

$$n^*(e^* \in de) = (1 - e^{-\lambda R(e)}) n(de)/\sqrt{2\lambda}.$$

Let f be a measurable nonnegative function and define the additive functional F as

$$F(t) = \int_0^t f(B_u)\, du.$$

The following proposition can be derived from last exit theorems [see Getoor (1979)]. For an alternative discussion of the distribution of the two pairs of random variables defined in (3.11), see Biane and Yor (1988). Here a proof based on Theorem 3.1 will be given.

PROPOSITION 3.1. *The two pairs of random variables*

$$\big(F(g_{S_\lambda}), l(g_{S_\lambda})\big) \quad \text{and} \quad \big(F(S_\lambda) - F(g_{S_\lambda}), B_{S_\lambda}\big) \tag{3.11}$$

are independent.

PROOF. A simple calculation shows that g_{S_λ} and $S_\lambda - g_{S_\lambda}$ are independent. By Lemma 3.1, given g_{S_λ}, the process $(B_t: 0 \le t \le g_{S_\lambda})$ is conditionally a Brownian bridge on $[0, g_{S_\lambda}]$. On the other hand, the random pair on the right is a functional of e^*, as defined in Theorem 3.1, and $S_\lambda - g_{S_\lambda}$. To conclude, we need to argue that g_{S_λ} is independent of e^*. However, this follows from the independence of l_{S_λ} and e^* and the conditional independence of $(e_s: 0 < s < l)$ and e^*, given $l_{S_\lambda} = l$. □

The master formulas for Poisson processes can now be applied to the process $(e_s: 0 < s < l(g_{S_\lambda}))$ conditioned on $l(g_{S_\lambda})$. See Revuz and Yor (1994), page 452, for details. Denote $l_{g(S_\lambda)}$ by l_λ. Using Theorem 3.1(ii),

$$\begin{aligned} E\Big(\exp\big(-F(g_{S_\lambda})\big)|l_\lambda = l\Big) &= \exp\Big(-l\int(1-e^{-F(u)})m(du)\Big) \\ &= \exp\Big(-l\int(1-e^{-F(u)})e^{-\lambda R(u)}n(du)\Big), \end{aligned} \tag{3.12}$$

where $F(u) = \int_0^{R(u)} f(u(s))\,ds$ is the integral over the lifetime of the excursion u and the integral on the right-hand side of (3.12) is over the space of excursions. In the sequel (3.12) will be applied with the function f equal to $k(x) = \beta x^+ + \gamma x^-$. It will be convenient to introduce the function

$$h(\beta,\lambda) = \int(1-e^{-\beta A(u)})e^{-\lambda R(u)}n(du),$$

where now $A(u) = \int_0^{R(u)}|u(s)|\,ds$ is the absolute area of the excursion u. By scaling properties of Itô's excursion law, it is easy to see that h has the scaling property

$$h(\beta,\lambda) = h(c^{3/2}\beta, c\lambda)/\sqrt{c} \quad \text{for } c > 0. \tag{3.13}$$

To compute h explicitly, note that the approach based on Kac's formula in Section 3 gives the double Laplace transform of A_0, which agrees with the formula given by Shepp (1982).

THEOREM 3.2.

$$\int_0^\infty \frac{e^{-\lambda s}}{\sqrt{s}} E\Big(\exp\big(-\sqrt{2}\,s^{3/2}A_0\big)\Big)\,ds = -\sqrt{\pi}\,\frac{Ai(\lambda)}{Ai'(\lambda)}, \tag{3.14}$$

where Ai is the Airy function [*see Abramowitz and Stegun* (1965), *pages* 446–451].

On the other hand, by Lemma 3.1 the process $(B_t: 0 \le t \le g_{S_\lambda})$ is just a randomly rescaled Brownian bridge and the process

$$\Big(B\big(tg_{S_\lambda}\big)/\sqrt{g_{S_\lambda}} : 0 \le t \le 1\Big)$$

is a Brownian bridge on $[0,1]$ independent of g_{S_λ}. Therefore,

$$A(g_{S_\lambda}) =_{\mathscr{D}} g_{S_\lambda}^{3/2} A_0, \tag{3.15}$$

where A_0 and $A(t)$ are defined in (1.1) and (1.2), and A_0 is independent of g_{S_λ}. This equality in law allows us to compute h explicitly.

PROPOSITION 3.2. *The function h is given by*

$$h(\beta, \lambda) = -(2\beta)^{1/3} \frac{Ai'(\lambda 2^{1/3}/\beta^{2/3})}{Ai(\lambda 2^{1/3}/\beta^{2/3})} - \sqrt{2\lambda}.$$

PROOF. An elementary computation shows that the distribution of g_{S_λ} is $\Gamma(1/2, \lambda)$. By multiplying (3.14) by $\sqrt{\lambda}$ and dividing by $\sqrt{\pi}$, the left-hand side becomes the Laplace transform of $\sqrt{2} g_{S_\lambda}^{3/2} A_0$. By (3.15), then,

$$E\Big(\exp\big(-\sqrt{2}\, A(g_{S_\lambda})\big)\Big) = -\frac{\sqrt{\lambda}\, Ai(\lambda)}{Ai'(\lambda)}.$$

Formula (3.12) applied to A gives that the conditional double Laplace transform (3.14) equals

$$E\Big(\exp\big(-\sqrt{2}\, A(g_{S_\lambda})\big) | l_\lambda = l\Big) = \exp\big(-lh(\sqrt{2}, \lambda)\big).$$

By Theorem 3.1(i) the distribution of $l_\lambda = l_{S_\lambda}$ is exponential with parameter $\sqrt{2\lambda}$, and integration gives

$$E\Big(\exp\big(-\sqrt{2}\, A(g_{S_\lambda})\big)\Big) = \frac{\sqrt{2\lambda}}{\sqrt{2\lambda} + h(\sqrt{2}, \lambda)}.$$

This identity, the scaling property (3.13) of h and some straightforward calculations conclude the proof. □

The properties of Poisson point processes can now be used to derive expressions for the Laplace transform of $A^+(g_{S_\lambda})$ and $A^-(g_{S_\lambda})$ or the joint Laplace transform of these two. Recall that $l_\lambda = l_{S_\lambda}$, where S_λ is an exponential random variable with parameter λ independent of B. Further, recall the definitions of $K(t)$ and $K_0(t)$ in (1.5) and (1.4).

THEOREM 3.3. *For $\beta, \gamma > 0$ let $k(x) = \beta x^+ + \gamma x^-$. Then, for $\lambda > 0$,*

$$E\Big(\exp\big(-K(g_{S_\lambda})\big) | l_\lambda = l\Big) = \exp\big(-l(h(\beta, \lambda)/2 + h(\gamma, \lambda)/2)\big) \tag{3.16}$$

and, consequently,

$$\int_0^\infty \frac{e^{-\lambda t}}{\sqrt{2\pi t}} \Psi_0^k(t^{3/2})\, dt = \frac{1}{\sqrt{2\lambda} + h(\beta, \lambda)/2 + h(\gamma, \lambda)/2}. \tag{3.17}$$

PROOF. By Theorem 3.1 conditionally on $l_\lambda = l$, the positive and negative excursions of $(e_s\colon 0 < s < l)$ are independent Poisson processes with excursion law $m/2$. Formula (3.16) now follows from (3.12). The second assertion follows by integration noting that, by Brownian scaling and Lemma 3.1, $K(g_{S_\lambda}) =_{\mathscr{D}} g_{S_\lambda}^{3/2} K_0$, where g_{S_λ} and K_0 are independent. □

We now turn to the computation of the double Laplace transform (3.10). A classical result by Kac (1946), which also appears as a special case in the proof given in Section 3, states the following.

THEOREM 3.4. *For* $\lambda > 0$ *one has*

$$E\Big(\exp\big(-\sqrt{2}\,A_{S_\lambda}\big)\Big) = \frac{\lambda\big(3\int_0^\lambda Ai(t)\,dt - 1\big)}{3Ai'(\lambda)}.$$

Based on this result, an analogous formula for the double Laplace transform of $K(t)$ is given by the following result.

THEOREM 3.5. *Let* $\beta, \gamma > 0$ *and let* $k(x) = \beta x^+ + \gamma x^-$. *For* $\lambda > 0$ *let*

$$\phi(\lambda) = -\frac{\sqrt{\lambda}\,\big(3\int_0^\lambda Ai(s)\,ds - 1\big)}{3Ai(\lambda)} \quad \text{and} \quad \tilde{\phi}(x) = \phi\left(\frac{2^{1/3}\lambda}{x^{2/3}}\right).$$

Then

$$E(\exp(-K(S_\lambda))) = E\big(\exp\big(-K(g_{S_\lambda})\big)\big)\big(\tilde{\phi}(\beta)/2 + \tilde{\phi}(\gamma)/2\big). \tag{3.18}$$

Consequently,

$$\begin{aligned}&\int_0^\infty e^{-\lambda t}\Psi^k(t^{3/2})\,dt\\ &\quad = \left(\frac{\sqrt{2/\lambda}}{\sqrt{2\lambda} + h(\beta,\lambda)/2 + h(\gamma,\lambda)/2}\right)\left(\frac{\tilde{\phi}(\beta)}{2} + \frac{\tilde{\phi}(\gamma)}{2}\right).\end{aligned} \tag{3.19}$$

PROOF. By (3.1) the random variables $K(S_\lambda) - K(g_{S_\lambda})$ and $K(g_{S_\lambda})$ are independent. The Laplace transform of the second is given in Theorem 3.3. The compute the Laplace transform of the first term, note that the sign of the excursion straddling S_λ is independent of its absolute area and hence

$$\begin{aligned}&E\big(\exp\big(-K(S_\lambda) - K(g_{S_\lambda})\big)\big)\\ &\qquad = E\Big(\exp\Big(-\beta\big(A(S_\lambda) - A(g_{S_\lambda})\big)\Big)\Big)/2\\ &\qquad\quad + E\Big(\exp\Big(-\gamma\big(A(S_\lambda) - A(g_{S_\lambda})\big)\Big)\Big)/2.\end{aligned} \tag{3.20}$$

By Proposition 3.1 again

$$\begin{aligned}&E(\exp(-\beta A(S_\lambda)))\\ &\quad = E\Big[\exp\Big(-\beta\big(A(S_\lambda) - A(g_{S_\lambda})\big)\Big)\Big]E\big(\exp\big(-\beta A(g_{S_\lambda})\big)\big).\end{aligned} \tag{3.21}$$

The left-hand side of (3.21) is given in Theorem 3.4 and the second term in the product on the right-hand side is given in Theorem 3.3. Dividing, it follows, for $\beta = \sqrt{2}$,

$$E\Big(\exp\big(-\sqrt{2}\,(A(S_\lambda) - A(g_{S_\lambda}))\big)\Big) = -\frac{\sqrt{\lambda}\,\big(3\int_0^\lambda Ai(s)\,ds - 1\big)}{3Ai(\lambda)} = \phi(\lambda).$$

However, by the scaling properties of BM,

$$\begin{aligned} &E\Big(\exp\big(-\beta\big(A(S_\lambda) - A(g_{S_\lambda})\big)\big)\Big) \\ &\quad = E\Big(\exp\big(-\sqrt{2}\,(A(S_{\lambda'}) - A(g_{S_{\lambda'}}))\big)\Big) = \phi(\lambda'), \end{aligned} \tag{3.22}$$

where $\lambda' = 2^{1/3}\lambda/\beta^{2/3}$. The formula for the integral (3.19) follows by scaling because $K(t) =_{\mathscr{D}} t^{3/2}K(1)$. □

REMARK 3.1. Note that $\int_0^\infty Ai(s)\,ds = 1/3$ [Abramowitz and Stegun (1965), page 448] and hence the function ϕ can also be expressed as

$$\phi(\lambda) = \frac{\sqrt{\lambda}\int_\lambda^\infty Ai(u)\,du}{Ai(u)}.$$

The above results have a few simple corollaries.

COROLLARY 3.1. *When $\beta = 1$ and $\gamma = 0$ [so $k(x) = x^+$], we have*

$$\int_0^\infty \frac{e^{-\lambda s}}{\sqrt{s}}\Psi_0^+(\sqrt{2}\,s^{3/2})\,ds = 2\sqrt{\pi}\,\frac{Ai(\lambda)}{\sqrt{\lambda}\,Ai(\lambda) - Ai'(\lambda)} \tag{3.23}$$

and

$$\int_0^\infty e^{-\lambda s}\Psi^+(\sqrt{2}\,s^{3/2})\,ds = \frac{\lambda^{-1/2}Ai(\lambda) + \big(1/3 - \int_0^\lambda Ai(t)\,dt\big)}{\sqrt{\lambda}\,Ai(\lambda) - Ai'(\lambda)}. \tag{3.24}$$

PROOF. Equation (3.23) follows from the master formula (3.12) because the positive excursions are a Poisson point process with mean measure $m/2$ and hence the second term on the right-hand side of (3.16) disappears. The rest follows just as in Theorem 3.3.

The second identity follows from (3.18). For $\gamma = 0$ one has

$$E\Big(\exp\big(-K(g_{S_\lambda})\big)\Big) = \frac{\sqrt{2\lambda}}{\sqrt{2\lambda} + h(\beta,\lambda)/2},$$

which follows easily from (3.16), and

$$E\Big(\exp\big(-K(S_\lambda) - K(g_{S_\lambda})\big)\Big) = \tilde{\phi}(\beta)/2 + 1/2,$$

which follows from (3.20). Now take $\beta = \sqrt{2}$. □

The following corollary is apparently known to Takács [see Takács (1993b)], but we do not know of any published derivation.

COROLLARY 3.2. *Let A_{mean} be the area of the absolute Brownian meander as defined in* (1.3), *and let* $\Psi_{\mathrm{mean}}(s) = E\exp(-sA_{\mathrm{mean}})$. *For* $\lambda > 0$,

$$\int_0^\infty \frac{e^{-\lambda s}}{\sqrt{\pi s}} \Psi_{\mathrm{mean}}\big(\sqrt{2}\, s^{3/2}\big)\, ds = \frac{\int_\lambda^\infty Ai(t)\, dt}{Ai(\lambda)}. \tag{3.25}$$

PROOF. One needs to notice that, by the scaling properties of BM,

$$A(S_\lambda) - A(g_{S_\lambda}) =_{\mathscr{D}} g_{S_\lambda}^{3/2} A_{\mathrm{mean}},$$

where the two random variables on the right-hand side are independent. Now (3.25) follows from (3.22). □

For comparison, we also state the result of Louchard (1984a) and Groeneboom (1989) for A_{excur} here. For an equivalent statement with a different proof, see Biane and Yor (1987), page 75. Takács (1992a) has inverted these transforms and computed $P(A_{\mathrm{excur}} \le x)$.

COROLLARY 3.3. *Let*

$$\Psi_e(s) = E\left(\exp\left(-s\int_0^1 e(u)\, du\right)\right),$$

where e is the standard Brownian excursion on $[0,1]$. *For* $\lambda > 0$,

$$\frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{(1 - e^{-\lambda s})}{s^{3/2}} \Psi_e\big(\sqrt{2}\, s^{3/2}\big)\, ds = -\sqrt{2}\left(\frac{Ai'(\lambda)}{Ai(\lambda)} - \frac{Ai'(0)}{Ai(0)}\right). \tag{3.26}$$

PROOF.

$$\begin{aligned}
&\frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{(1 - e^{-\lambda s})}{s^{3/2}} \Psi_e\big(\sqrt{2}\, s^{3/2}\big)\, ds \\
&\quad = \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{(1 - e^{-\lambda s})}{s^{3/2}} E\big[\exp\big(-\sqrt{2}\, s^{3/2} A(e)\big)\big]\, ds \\
&\quad = \int_0^\infty E\left[(1 - e^{-\lambda s}) \frac{\exp\big(-\sqrt{2}\, s^{3/2} A(e)\big)}{\sqrt{2\pi s^3}}\right] ds \\
&\quad = \int (1 - e^{-\lambda R(u)}) e^{-\sqrt{2} A(u)} n(du) \\
&\qquad\qquad \text{by scaling and } n(R(e) \in ds) = \frac{ds}{\sqrt{2\pi s^3}} \\
&\quad = \int \Big[(1 - e^{-\sqrt{2} A(u)}) e^{-\lambda R(u)} - (1 - e^{-\sqrt{2} A(u)}) + (1 - e^{-\lambda R(u)})\Big] n(du) \\
&\quad = h\big(\sqrt{2}, \lambda\big) - h\big(\sqrt{2}, 0\big) + \sqrt{2\lambda}.
\end{aligned}$$

All the changes in the order of integration are justified by Fubini. The last line follows from the identity $\int(1-e^{-\lambda R(e)})n(de)=\sqrt{2\lambda}$, which is contained in Theorem 3.1(iii). Substituting for h, one obtains (3.26). $\square$

For $\beta=\gamma=1$ and hence $k(x)=|x|$, the double Laplace transform for A_0 has been inverted in terms of Airy functions in Rice (1982) and Johnson and Killeen (1983); the double Laplace transform for $A(1)$ has been inverted by Kac (1946) and Takács (1992a, b); and the transform (3.26) has been inverted by Louchard (1984b) and Takács (1992b). We have not yet accomplished a similar inversion for the case $\gamma=0$, but we do use the transforms in Section 4 to develop recursions for the moment sequences of A_0^+ and A^+.

4. The double Laplace transforms via Kac's formula. Theorems 3.3 and 3.5 can also be derived via Kac's formula by solving a differential equation. For completeness, the proof of the two main results in the previous section is repeated here.

PROOF OF THEOREMS 3.3 AND 3.5. First, writing E_x for expectation conditional on the process B starting at x at $t=0$, Kac's formula [see, e.g., Itô and McKean (1974), page 54] says that

$$u(x)=E_x\left\{\int_0^\infty e^{-\lambda t}\exp\left(-\int_0^t k(B(s))\,ds\right)f(B(t))\,dt\right\}\tag{4.27}$$

is the bounded solution of

$$(\lambda-D_k)u=f,\tag{4.28}$$

where D_k is the differential operator

$$D_k u(x)\equiv\tfrac{1}{2}u''(x)-k(x)u(x).\tag{4.29}$$

Hence, letting $0<g_1\nearrow$ and $0<g_2\searrow$ be two independent solutions of the homogeneous equation

$$(\lambda-D_k)u=0,\tag{4.30}$$

and writing $W\equiv g_1'g_2-g_1g_2'$ for the Wronskian,

$$G(a,b)=g_1(a\wedge b)g_2(a\vee b)/W\tag{4.31}$$

for the Green's function, it is classical that the solution of the inhomogeneous equation (4.28) is given by

$$u(a)=2\int G(a,b)f(b)\,db.\tag{4.32}$$

Hence, in particular, when $f\equiv 1$ and $a=0$ we have

$$u(0)=\frac{2}{W}\left\{g_2(0)\int_{-\infty}^0 g_1(b)\,db+g_1(0)\int_0^\infty g_2(b)\,db\right\}.\tag{4.33}$$

By arguing as in Shepp (1982), it is easily seen that the corresponding result for $\int_0^1 k(U(s))\,ds$ is as follows:

$$E_0\left\{\int_0^\infty \frac{e^{-\lambda t}}{\sqrt{2\pi t}}\exp\left(-t\int_0^1 k\left(\sqrt{t}\,U(s)\right)ds\right)\right\} = \frac{2}{W}g_1(0)g_2(0). \tag{4.34}$$

We now use (4.33) and (4.34) for the particular k given in (1.6). For this k the homogeneous differential equation (4.30) becomes

$$\tfrac{1}{2}u''(x) - (\lambda + \beta x^+ + \gamma x^-)u(x) = 0. \tag{4.35}$$

It is easily verified that the two solutions g_1 and g_2 that we seek are given by $g_1(x) = g_2(-x; \gamma, \beta)$, where $g_2 \equiv g_2(\cdot; \beta, \gamma)$ is defined by

$$g_2(x) \equiv \begin{cases} Ai\left((2\beta)^{1/3}(x + \lambda/\beta)\right), & x \geq 0, \\ C_1 Ai\left((2\gamma)^{1/3}(-x + \lambda/\gamma)\right) & \\ \qquad + C_2 Bi\left((2\gamma)^{1/3}(-x + \lambda/\gamma)\right), & x < 0, \end{cases} \tag{4.36}$$

where Ai and Bi are the two standard independent solutions of $w''(z) - zw(z) = 0$ with Ai decreasing and Bi increasing on $(0, \infty)$; see Abramowitz and Stegun (1965), page 446. Here C_1 and C_2 are constants chosen so that $g_2(0+) = g_2(0-)$ and $g_2'(0+) = g_2'(0-)$. It follows by straightforward calculation that

$$\begin{aligned} C_1 = \Big[& Ai\left((2\beta)^{1/3}\lambda/\beta\right)Bi'\left((2\lambda)^{1/3}\lambda/\gamma\right) \\ & + (\beta/\gamma)^{1/3}Ai'\left((2\beta)^{1/3}\lambda/\beta\right)Bi\left((2\lambda)^{1/3}\lambda/\gamma\right)\Big] \\ \times \Big[& Ai\left((2\gamma)^{1/3}\lambda/\gamma\right)Bi'\left((2\gamma)^{1/3}\lambda/\gamma\right) \\ & - Ai'\left((2\gamma)^{1/3}\lambda/\gamma\right)Bi\left((2\gamma)^{1/3}\lambda/\gamma\right)\Big]^{-1} \end{aligned}$$

and

$$\begin{aligned} C_2 = \Big[& -(\beta/\gamma)^{1/3}Ai\left((2\gamma)^{1/3}\lambda/\gamma\right)Ai'\left((2\beta)^{1/3}\lambda/\beta\right) \\ & - Ai\left((2\beta)^{1/3}\lambda/\beta\right)Ai'\left((2\lambda)^{1/3}\lambda/\gamma\right)\Big] \\ \times \Big[& Ai\left((2\gamma)^{1/3}\lambda/\gamma\right)Bi'\left((2\gamma)^{1/3}\lambda/\gamma\right) \\ & - Ai'\left((2\gamma)^{1/3}\lambda/\gamma\right)Bi\left((2\gamma)^{1/3}\lambda/\gamma\right)\Big]^{-1}. \end{aligned}$$

Substitution of these into (4.33) and (4.34) concludes the proof. $\square$

5. Moments of A_0^+ and A^+. Now we use the methods developed in Shepp (1982) and Takács (1993a) to obtain recursion formulas for the moments

$$\mu_k^+ \equiv E(A_0^+)^k, \qquad \nu_k^+ \equiv E(A^+)^k, \qquad k = 1, 2 \dots,$$

of A_0^+ and A^+. For comparison, we first state the recursion relations found by Shepp (1982) and Takács (1993a) for

$$\mu_k \equiv E(A_0^k), \qquad \nu_k \equiv E(A^k), \qquad k = 0, 1, 2, \ldots .$$

For $n = 0, 1, 2, \ldots,$ define

$$L_n \equiv \mu_n \Gamma\left(\frac{3n+1}{2}\right) \frac{(\sqrt{2})^n}{n!\Gamma(1/2)} \tag{5.37}$$

and

$$K_n \equiv \nu_n \Gamma\left(\frac{3n+2}{2}\right) \frac{(\sqrt{2})^n}{n!} \tag{5.38}$$

and set $\gamma_0 \equiv 1$, $\beta_0 \equiv 1$ and

$$\gamma_n \equiv \frac{\Gamma(3n+1/2)}{\Gamma(n+1/2)} \frac{1}{(36)^n n!} \tag{5.39}$$

and

$$\beta_n \equiv \gamma_n + \tfrac{3}{4}(2n-1)\beta_{n-1} \tag{5.40}$$

for $n = 0, 1, 2, \ldots .$

THEOREM 5.1 [Shepp (1982) and Takács (1993a)]. *For* $n = 1, 2, \ldots,$

$$L_n = \gamma_n + \sum_{k=1}^{n} L_{n-k} \frac{6k+1}{6k-1} \gamma_k \tag{5.41}$$

and

$$K_n = \beta_n + \sum_{k=1}^{n} K_{n-k} \frac{6k+1}{6k-1} \gamma_k . \tag{5.42}$$

PROOF. The recursion for $\{L_n\}$ follows by rewriting Shepp (1982), formula (1.8): divide both sides of Shepp's (1.8) by $n!(36)^n$ and rearrange to obtain (5.41). The recursion for $\{K_n\}$ is exactly Takács (1993a), formula (24). $\square$

A recursion for the moments of A_{excur} is given by Takács (1992b).

Now, for $n = 0, 1, 2, \ldots,$ define

$$L_n^+ \equiv \mu_n^+ \Gamma\left(\frac{3n+1}{2}\right) \frac{(\sqrt{2})^n}{n!}, \tag{5.43}$$

$$K_n^+ \equiv \nu_n^+ \Gamma\left(\frac{3n+2}{2}\right) \frac{(\sqrt{2})^n}{n!}. \tag{5.44}$$

In the sequel we will also need the moments of the Brownian meander area:

$$\rho_n \equiv E(A_{\text{mean}}^n) \quad \text{and} \quad R_n \equiv \rho_n \Gamma\left(\frac{3n+1}{2}\right) (\sqrt{2})^n / n!\Gamma(1/2) \tag{5.45}$$

for $n = 1, 2, \dots$. Now let S_λ be an exponential random variable independent of the Brownian motion B, and define $A(S_\lambda)$, $A(g_{S_\lambda})$, $A^+(S_\lambda)$ and $A^+(g_{S_\lambda})$ just as in Section 2. The quantities L_n, K_n, L_n^+ and K_n^+ can also be rewritten using the scaling properties of BM and the independence property from Proposition 3.1 as

$$L_n = E\left(A(g_{S_1})^n\right)\frac{(\sqrt{2})^n}{n!}, \tag{5.46}$$

$$\begin{aligned} K_n &= E\left(\left(A(S_1) - A(g_{S_1}) + A(g_{S_1})\right)^n\right)\frac{(\sqrt{2})^n}{n!} \\ &= \sum_{k=0}^{n} L_k R_{n-k} \end{aligned} \tag{5.47}$$

and

$$L_n^+ = E\left(A^+(g_{S_1})^n\right)\frac{(\sqrt{2})^n}{n!}, \tag{5.48}$$

$$\begin{aligned} K_n^+ &= E\left(A^+(S_1)^n\right)\frac{(\sqrt{2})^n}{n!} \\ &= E\left(\left(A^+(S_1) - A^+(g_{S_1}) + A^+(g_{S_1})\right)^n\right)\frac{(\sqrt{2})^n}{n!} \\ &= \frac{1}{2}\left(E\left(\left(A(S_1) - A(g_{S_1}) + A^+(g_{S_1})\right)^n\right)\right)\frac{(\sqrt{2})^n}{n!} \\ &\quad + \frac{1}{2}E\left(A^+(g_{S_1})^n\right)\frac{(\sqrt{2})^n}{n!}, \end{aligned} \tag{5.49}$$

$$= \sum_{k=0}^{n} L_k^+ R_{n-k}/2 + L_n^+/2 \tag{5.50}$$

and

$$R_n = E\left(A(S_1) - A(g_{S_1})\right)\frac{(\sqrt{2})^n}{n!}. \tag{5.51}$$

The identity (5.49) follows from the fact that the sign of the excursion straddling S_1 is independent of $A^+(g_{S_1})$ and of $A(S_1) - A(g_{S_1})$. To derive recurrence formulas similar to those in Theorem 5.1, note the following result.

PROPOSITION 5.1. *Let f_1, f_2 be the densities of $A(g_{S_\lambda})$ and $A^+(g_{S_\lambda})$, respectively. The densities satisfy the equation*

$$f_2 = 2f_1 - f_1 * f_2, \tag{5.52}$$

where $*$ denotes convolution.

PROOF. From Theorem 3.3 we know that the Laplace transforms $\hat{f}_1$ and $\hat{f}_2$ are

$$\hat{f}_1(\beta) = \frac{\sqrt{2\lambda}}{\sqrt{2\lambda} + h(\beta,\lambda)} \tag{5.53}$$
$$\hat{f}_2(\beta) = \frac{\sqrt{2\lambda}}{\sqrt{2\lambda} + h(\beta,\lambda)/2}.$$

For (5.52) to hold the Laplace transforms would have to satisfy the relation

$$\hat{f}_2 = 2\hat{f}_1 - \hat{f}_1\hat{f}_2.$$

That they do is checked by straightforward calculation. □

COROLLARY 5.1. *For* $n = 1, 2, \dots,$

$$L_n^+ = L_n - \sum_{k=1}^{n} L_k^+ L_{n-k}. \tag{5.54}$$

PROOF. Multiplying (5.52) by x^n and integrating, one obtains

$$\begin{aligned} E\left(A^+(g_{S_1})^n\right) &= 2E\left(A(g_{S_1})^n\right) - \sum_{k=0}^{n}\binom{n}{k}E\left(A^+(g_{S_1})^k\right)E\left(A(g_{S_1})^{n-k}\right)\\ &= E\left(A(g_{S_1})^n\right) - \sum_{k=1}^{n}\binom{n}{k}E\left(A^+(g_{S_1})^k\right)E\left(A(g_{S_1})^{n-k}\right). \end{aligned}$$

Substituting the expressions for L_n and L_n^+ gives (5.54). □

For two sequences $a_0, a_1, \dots$ and $b_0, b_1, \dots$ we will write $a * b$ for the convolution: $(a*b)_n = \sum_{k=0}^{n} a_k b_{n-k}$. In this notation, setting $L_0 \equiv 1$ and $L_0^+ \equiv 1$, the identity (5.54) can also be written as

$$L^+ = 2L - L * L^+. \tag{5.55}$$

COROLLARY 5.2. *For* $n = 0, 1, 2, \dots,$

$$L_n^+ = \gamma_n + \sum_{k=1}^{n} L_{n-k}^+ \frac{1}{6k-1}\gamma_k, \tag{5.56}$$

$$K_n^+ = \frac{1}{2}(\gamma_n + \beta_n) + \sum_{k=1}^{n} K_{n-k}^+ \frac{1}{6k-1}\gamma_k \tag{5.57}$$

and

$$R_n = \beta_n - \sum_{k=1}^{n} \gamma_k R_{n-k}. \tag{5.58}$$

PROOF. For $k = 0, 1, \dots$ denote $\tilde{\gamma}_k = 2\gamma_k/(6k-1)$. The recurrence formula (5.41) can be rewritten as

$$0 = \gamma + L * \gamma + L * \tilde{\gamma}. \tag{5.59}$$

From (5.55) we have $L = (L^+ + L * L^+)/2$. Plugging this into the above formula, we get

$$0 = \gamma + (L^+ + L * L^+) * \gamma/2 + (L^+ + L * L^+) * \tilde{\gamma}/2.$$

Rearranging the terms using associativity and commutativity of convolutions and multiplying by 2 yields

$$0 = 2\gamma + L^+ * \gamma + \tilde{\gamma} * L^+ + (L * \gamma + L * \tilde{\gamma}) * L^+.$$

However, the term in parentheses equals $-\gamma$ by (5.59), so we obtain

$$L^+ * \tilde{\gamma} = -2\gamma, \tag{5.60}$$

which is precisely (5.56).

To prove (5.57), note that (5.42) says

$$0 = \beta + K * \gamma + K * \tilde{\gamma}. \tag{5.61}$$

Substituting $K = L * R$ and $L = (L^+ + L^+ * L)/2$, recalling that $K^+ = L^+ * R/2 + L^+/2$ and rearranging terms gives

$$\begin{aligned} 0 = \beta + K^+ * \gamma + K^+ * \tilde{\gamma} + (L * \gamma + L * \tilde{\gamma}) * K^+ \\ - \tfrac{1}{2} L^+ * (L * \gamma + L * \tilde{\gamma}) - \tfrac{1}{2} L^+ * \gamma - \tfrac{1}{2} L^+ * \tilde{\gamma}. \end{aligned} \tag{5.62}$$

The term in the first set of parentheses is $-\gamma$ by (5.59), and similarly the term in the second set of parentheses is also $-\gamma$. By (5.60), the above identity becomes

$$0 = \beta + \gamma + K^+ * \tilde{\gamma},$$

which is (5.57).

Multiply (5.59) by R in the sense of convolutions. Note that $R * L = K$. Comparing the resulting identity with (5.61) shows that

$$\gamma * R = \beta,$$

which is (5.58). $\square$

Now define $A_0^- \equiv \int_0^1 U^-(t)\,dt$ and $A^- \equiv \int_0^1 B^-(t)\,dt$. It follows by symmetry that $A_0^- =_{\mathscr{D}} A_0^+$ and $A^- =_{\mathscr{D}} A^+$, but in both cases they are dependent. The following calculation of covariances and correlations follows easily from our moment calculations and $A = A^+ + A^-$, $A_0 = A_0^+ + A_0^-$ [or $\int_0^1 B(t)\,dt = A^+ - A^-$, $\int_0^1 U(t)\,dt = A_0^+ - A_0^-$]:

$$\begin{aligned} \operatorname{Cov}(A_0^+, A_0^-) &= \frac{1}{120} - \frac{\pi}{128}, \\ \rho(A_0^+, A_0^-) &= \frac{128 - 120\pi}{6 \cdot 128 - 120\pi} = -0.636791\ldots, \\ \operatorname{Cov}(A^+, A^-) &= \frac{1}{96} - \frac{2}{9\pi}, \\ \rho(A^+, A^-) &= \frac{3\pi - 64}{51\pi - 64} = -0.567185\ldots. \end{aligned}$$

Tables 1 and 2 were computed using Mathematica; see Wolfram (1991).

TABLE 1
First 10 moments of A_0, A, A_0^+ and A^+

k	μ_k	ν_k	μ_k^+	ν_k^+
1	$\frac{1}{4}\sqrt{\frac{\pi}{2}}$	$\frac{4}{3}\frac{1}{\sqrt{2\pi}}$	$\frac{1}{8}\sqrt{\frac{\pi}{2}}$	$\frac{2}{3}\frac{1}{\sqrt{2\pi}}$
2	$\frac{7}{60}$	$\frac{3}{8}$	$\frac{1}{20}$	$\frac{17}{96}$
3	$\frac{21}{512}\sqrt{\frac{\pi}{2}}$	$\frac{263}{315}\frac{1}{\sqrt{2\pi}}$	$\frac{71}{4096}\sqrt{\frac{\pi}{2}}$	$\frac{251}{630}\frac{1}{\sqrt{2\pi}}$
4	$\frac{19}{720}$	$\frac{903}{2560}$	$\frac{211}{18480}$	$\frac{6989}{40960}$
5	$\frac{101}{8192}\sqrt{\frac{\pi}{2}}$	$\frac{2119}{1980}\frac{1}{\sqrt{2\pi}}$	$\frac{15103}{360777252864}\sqrt{\frac{\pi}{2}}$	$\frac{188267}{360360}\frac{1}{\sqrt{2\pi}}$
6	$\frac{70753}{7001290}$	$\frac{37623}{65536}$	$\frac{75233}{16336320}$	$\frac{37235311}{132120576}$
7	$\frac{45493}{7864320}\sqrt{\frac{\pi}{2}}$	$\frac{11074363}{5250960}\frac{1}{\sqrt{2\pi}}$	$\frac{32420011}{1209595520}\sqrt{\frac{\pi}{2}}$	$\frac{1451280043}{1396755360}\frac{1}{\sqrt{2\pi}}$
8	$\frac{206530429}{36714712320}$	$\frac{114752519}{86507520}$	$\frac{75516257}{28555887360}$	$\frac{522258818027}{797253304320}$
9	$\frac{89374187}{23991418880}\sqrt{\frac{\pi}{2}}$	$\frac{3845017725821}{688400856000}\frac{1}{\sqrt{2\pi}}$	$\frac{32582240233}{2145\cdot 2^{33}}\sqrt{\frac{\pi}{2}}$	$\frac{8096107769}{2930256000}\frac{1}{\sqrt{2\pi}}$
10	$\frac{1256447927}{305663155200}$	$\frac{189970427903}{47982837760}$	$\frac{292964136763}{149061732019200}$	$\frac{1011667945773427}{515911471595520}$

6. Expansions of the distributions in Laguerre series. Here we follow Takács (1993a) to express the distribution functions

$$H(x) \equiv P(A_0 \le x),$$
$$H_+(x) \equiv P(A_0^+ \le x),$$
$$K(x) \equiv P(A \le x)$$

and

$$K_+(x) \equiv P(A^+ \le x)$$

and their densities $h \equiv H'$, $h_+ \equiv H'_+$, $k \equiv K'$ and $k_+ \equiv K'_+$ in terms of Laguerre series. The generalized Laguerre polynomials,

$$L_n^{(\alpha)}(x) = \sum_{j=0}^{n} (-1)^j \binom{n+\alpha}{n-j} \frac{x^j}{j!},$$

defined for $n = 0, 1, 2, \ldots$ and $\alpha > -1$, are orthogonal on the interval $0 \le x < \infty$ with respect to the Gamma$(\alpha + 1, 1)$ density

$$g_{\alpha+1}(x) = e^{-x} x^{\alpha} / \Gamma(\alpha + 1).$$

TABLE 2
Numerical values of the first 10 *moments of* A_0, A, A_0^+ *and* A^+

k	μ_k	ν_k	μ_k^+	ν_k^+
1	0.313328534	0.531923041	0.156664267	0.265961520
2	0.116666667	0.375	0.05	0.177083333
3	0.051405463	0.333085142	0.021724928	0.158943670
4	0.026388889	0.352734375	0.011417749	0.170629883
5	0.015452237	0.426948834	0.006876919	0.208423982
6	0.010105723	0.574081421	0.004605260	0.281828252
7	0.007250089	0.841375983	0.003363727	0.414515660
8	0.005625277	1.326503395	0.002644507	0.655072629
9	0.004668917	2.228265881	0.002216275	1.102251713
10	0.004110564	3.959132823	0.001965388	1.960933225

Let

$$G_{\alpha+1}(x) = \int_0^x g_{\alpha+1}(u)\, du$$

be the corresponding Gamma$(\alpha + 1, 1)$ distribution function.

Now we take the approach of Takács (1993a) to expand the distributions H_+ and K_+ as Laguerre series. In the process, we will correct a few minor typographical errors on page 196 of Takács (1993a). By using the results of Uspensky (1927) and Nasarow (1931) [Sansone (1959), Chapter 4], we can show that, for a distribution H of a nonnegative random variable Y which is determined by its moments $\mu_r = EY^r$, $r = 1, 2, \dots$ (such as the areas A, A_0, A^+ and A_0^+),

$$h(x) = g_\alpha(bx) b \sum_{n=0}^{\infty} c_n L_n^{(\alpha-1)}(bx)$$

[this corrects formula (68) in Takács (1993a)] and

$$H(x) = G_\alpha(bx) + a \sum_{n=1}^{\infty} \frac{c_n}{n} g_{\alpha+1}(bx) L_{n-1}^{(\alpha)}(bx) \tag{6.63}$$

TABLE 3
Values of a and b

	A_0	A	A_0^+	A^+
a	$\frac{15\pi}{56-15\pi}$	$\frac{64}{27\pi-64}$	$\frac{5\pi}{32-5\pi}$	$\frac{64}{51\pi-64}$
b	$\frac{60\sqrt{2\pi}}{56-15\pi}$	$\frac{48\sqrt{2\pi}}{27\pi-64}$	$\frac{40\sqrt{2\pi}}{32-5\pi}$	$\frac{96\sqrt{2\pi}}{51\pi-64}$

TABLE 4
Numerical values of a and b

	A_0	A	A_0^+	A^+
a	5.3090699374	3.0735242250	0.9641497577	0.6551339118
b	16.9440997412	5.7781370439	6.1542416479	2.5008652044

for $x \geq 0$, where $a > 0$, $b > 0$ and

$$c_n \binom{n+a-1}{n} = \sum_{r=0}^{n} \frac{(-1)^4}{r!} \binom{n+a-1}{n-r} b^r \mu_r$$

for $n = 0, 1, 2, \ldots$ [(6.63) corrects formula (67) in Takács (1993a)].

As noted in Takács (1993a), if we choose

$$a = \frac{\mu_1^2}{\mu_2 - \mu_1^2} \quad \text{and} \quad b = \frac{a}{\mu_1},$$

then the first and second Laguerre coefficients, c_1 and c_2, are both 0, and the next term in the series to enter is the third term. Tables 3 and 4 give the values of a and b [and hence the leading Gamma(a, b) term] for the four random variables A_0, A, A_0^+ and A^+. We have made numerical comparisons of H and k computed via the Laguerre expansions with H and k computed via the formulas obtained by inversion of the double Laplace transform in Johnson and Killeen (1983) and Takács (1993a), respectively, and have obtained excellent agreement. As seen in Table 5, however, the Laguerre coefficients c_{0n}^+ and c_n^+ decay much more slowly than the corresponding

TABLE 5
Laguerre coefficients c_{0n}, c_n, c_{0n}^+ *and* c_n^+

n	c_{0n}	c_n	c_{0n}^+	c_n^+
3	−0.021454119803	−0.011584328208	0.097883267635	0.157758186834
4	−0.016523337226	−0.027178915962	0.127586565732	0.247981499267
5	−0.007745843462	−0.027729182039	0.109492841572	0.258338193831
6	−0.002656004243	−0.020663009369	0.080609708895	0.226619040144
7	−0.000158072001	−0.013188716669	0.059395115926	0.185964134827
8	0.000961309559	−0.007518446236	0.049017732282	0.153690893619
9	0.001283485814	−0.003542017611	0.046078654907	0.134554052077
10	0.001177477843	−0.000784228414	0.046305580415	0.126580019477
11	0.000899538852	0.001069042587	0.046373029005	0.125557381830
12	0.000596775550	0.002209142203	0.044849197460	0.127540086205
13	0.000335517004	0.002794038532	0.042632790813	0.129818478532
14	0.000137038413	0.002974420142	0.040199307720	0.130995229809
15	0.000001353846	0.002884301321	0.037904739443	0.130667652469
16	−0.000080633690	0.002629846283	0.035932605867	0.129024337933
17	−0.000121149592	0.002287456968	0.034323026910	0.126506297339
18	−0.000132376130	0.001908702360	0.033028868889	0.123580870558

coefficients c_{0n} and c_n. This may be related to the fact that the leading term of the expansions, which is a Gamma(a, b) distribution, has a substantially greater than 1 in the case of both A_0 and A, but a less than 1 (so that the resulting Gamma density is unbounded at the origin) in the case of both A_0^+ and A^+. Thus it seems that the Laguerre series obtained by this method will require a very large number of terms to yield numerically accurate values of H_+ and K_+.

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INSTITUTE FOR MATHEMATICS, PHYSICS
AND MECHANICS
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF LJUBLJANA
JADRANSKA 19
61111 LJUBLJANA
SLOVENIA
E-MAIL: Mihael.Perman@fmf.uni-lj.si

UNIVERSITY OF WASHINGTON
DEPARTMENT OF STATISTICS
BOX 354322
SEATTLE, WASHINGTON 98195-4322
E-MAIL: jaw@stat.washington.edu