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ON THE DISTRIBUTION OF ENERGY IN NOISE-AND SIGNAL-MODULATED WAVES I. AMPLITUDE MODULATION*

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1. Introduction. Because noise is an inevitable and undesirable companion of intelligence transmitted or received by electronic systems, it is essential for any proper theory of communication to provide suitable methods for studying the physical properties of a noise wave and its interaction with a desired signal. On the one hand a successful technique of measurement is required to control or minimize the noise, and on the other an adequate theory is necessary to guide experiment and interpret the data. Accordingly, the purpose of the present paper is to present a number of new results, obtained by the analytical methods developed in recent years, [2-15, 17-20] for the following important problems. (In all cases considered here the noise is assumed to belong to the fluctuation type characteristic of shot and thermal noise, which are described by a normal random process [5, 10]. Impulsive noise, such as atmospheric and solar static, is not treated, although the general methods of analysis remain the same.) Our interest here is confined mainly to amplitude-modulated waves, specifically,

(i) *carrier amplitude-modulated by noise*: this problem considers the amplitude distortion by noise of the carrier wave as the mechanism producing the modulation, and the important case of over-modulation is also examined. This example is of particular interest when normal random noise is used as an approximate model of speech,† or as a form of interference.

(ii) *carrier amplitude-modulated by a signal and noise*: this is a frequent case in practice where a certain amount of noise accompanies the desired signal in the process of modulation. Included also is the problem of speech (normal random noise) accom-

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†Recent experiments of W. B. Davenport, Jr. ("A study of speech probability distributions," Technical Report 148, Research Laboratory of Electronics, (MIT) August 25, 1950) indicate that speech is more satisfactorily described statistically in terms of an impulsive or "static" noise model, where overlapping among individual (and independent) pulses is assumed to be small, of the order of 30-50 per cent of the time. This is different from the usual model of fluctuation or normal random noise, which assumes complete and highly multiple overlapping between the elementary transients. However, because normal random noise has the great advantage of mathematical simplicity, its use as a speech model seems justified on this and physical grounds, at least as a first approximation.

panied by noise; the two noise waves are assumed to be uncorrelated. In a later paper* is considered

(iii) *simultaneous amplitude- and angle-modulation of a carrier by noise*: here the modulating noise waves are correlated, and there may be in general a phase lag of one modulation with respect to the other. The results are of particular interest in connection with the problem of the noise in magnetron generators, in which a simultaneous amplitude- and angle- (i.e., phase- or frequency-) modulation of the oscillations due to the inherent or primary noise of the tube is known to occur.

[A discussion of the problem of carriers angle-modulated by signal and noise has been given elsewhere in a recent report (D. Middleton, 1)].

We remark further that, apart from the specific applications to problems (i)-(iii), the results are needed in the general theory of noise measurement, for here the central objective is to be able to determine by measurements on the output wave, following various linear and nonlinear operations (such as amplification, rectification, clipping, mixing, modulation, discrimination, etc.), the "structure" of the original input disturbance, i.e., whether or not it is an amplitude- or frequency-modulated wave, how the noise and signal occur together, and other qualitative and quantitative data.

The quantities of chief physical interest are (a), the *mean* or steady component of the disturbance, (b), the *mean intensity* of the wave, and (c), the *spectral distribution* $W(f)$ of the mean intensity. This latter quantity is in fact sufficient to give us the other two; the mean intensity is obtained as the area under the spectral density curve $W(f)$, while the (square of the) steady component is given by the constant (or frequency-independent) term in the expression for the spectrum. It is assumed that we are dealing with a stationary (and ergodic) random process, namely, a process for which the underlying mechanism does not change with time. Then time averages and ensemble or statistical averages are equivalent, [20] to within a set of random functions of probability zero, so that if we represent our stochastic, time-dependent disturbance by $y(t)$ we may write the steady component (a) as**

$$\langle y(t_0)\rangle_{\text{av.}} \equiv \lim_{T\to\infty} T^{-1} \int_0^T y(t_0)\, dt_0 = \langle y(t_0)\rangle_{\text{s.av.}} \equiv \int_{-\infty}^{\infty} yW_1(y)\, dy, \tag{1.1}$$

and the mean intensity (b)

$$\langle y(t_0)^2\rangle_{\text{av.}} \equiv \lim_{T\to\infty} T^{-1} \int_0^T y(t_0)^2\, dt_0 = \langle y(t_0)^2\rangle_{\text{s.av.}} \equiv \int_{-\infty}^{\infty} y^2W_1(y)\, dy; \tag{1.2}$$

$W_1(y)\, dy$ is the probability that (at any initial time t_0) y lies in the range y, $y + dy$. The moment of greatest interest, however is given by

$$R(t) \equiv \langle y(t_0)y(t_0 + t)\rangle_{\text{av.}} \equiv \lim_{T\to\infty} T^{-1} \int_0^T y(t_0)y(t_1 + t)\, dt_0\ . \tag{1.3}$$

The quantity $R(t)$ is the *auto-correlation function* of y and may be found statistically when the second-order probability density $W_2(y_1\,, y_2\,; t)$ is known; W_2 has the following interpretation:

*We distinguish here between time and statistical averages by $\langle\ \rangle_{\text{av.}}$ and $\langle\ \rangle_{\text{s.av.}}$ respectively.

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The equivalence of these averages follows from the ergodic theorem [2, 20].

$W_2(y_1, y_2; t)\, dy_1\, dy_2 =$ the joint probability that at some (initial) time t_0, $y(= y_1)$ lies in the range $(y_1, y_1 + dy_1)$ and at a later time $t_0 + t$, $y(=y_2)$ falls in the interval $(y_2, y_2 + dy_2)$. (1.4)

Because time and ensemble averages are equivalent here, Eq. (1.3) becomes

$$R(t) = \langle y_1 y_2 \rangle_{\text{s.av.}} = \iint_{-\infty}^{\infty} y_1 y_2 W_2(y_1, y_2; t)\, dy_1\, dy_2, \tag{1.5}$$

and since the process is stationary, the initial times t_0 do not enter: one is concerned only with the time intervals (t) between observations.

Knowledge of the correlation function $R(t)$ is important, for by the theorem of Wiener [14] and Khintchine [15] the mean intensity spectrum follows at once as the cosine Fourier transform of $R(t)$, namely

$$W(f) = 4 \int_0^{\infty} R(t) \cos \omega t\, dt, \qquad (\omega = 2\pi f) \tag{1.6a}$$

with the inverse relation

$$R(t) = \int_0^{\infty} W(f) \cos \omega t\, df. \tag{1.6b}$$

To determine the desired energy spectrum $W(f)$ the simplest procedure is first to obtain the correlation function and then apply (1.6a). Note from (1.6b) that setting $t = 0$ in $R(t)$ gives the mean total intensity of the random wave, namely

$$R(0) = \int_0^{\infty} W(f)\, df = \lim_{t \to 0} \iint_{-\infty}^{\infty} y_1 y_2 W_2(y_1, y_2; t)\, dy_1\, dy_2 = \langle y^2 \rangle_{\text{s.av.}} \tag{1.7}$$

which is the area under the spectral distribution curve $W(f)$, as expected. On the other hand, allowing t to become infinite in $R(t)$ yields the steady component $\langle y \rangle_{\text{s.av.}}$, since $\lim_{t \to \infty} W_2(y_1, y_2; t) = W_1(y_1) W_1(y_2)$, so that (1.5) becomes

$$\lim_{t \to \infty} R(t) = \iint_{-\infty}^{\infty} y_1 y_2 W_1(y_1) W_1(y_2)\, dy_1\, dy_2 = \langle y \rangle^2_{\text{s.av.}}, \tag{1.8}$$

from (1.1). For a pure noise wave $\langle y \rangle_{\text{s.av.}}$ vanishes, as there are no steady components. However, when y does not represent a purely stochastic variable, but contains steady and periodic terms as well, $\lim_{t \to \infty} R(t)$ will not die down in time, but will oscillate indefinitely. If $R(t)$ is then expanded in a Fourier series, the coefficient of each periodic component represents the mean power (or intensity, as the case may be)* associated with that component; setting $t = 0$ in $R(t)$ still gives the mean total power in the wave.

In a similar way we may find the correlation function for a general function $g(y)$ of the random variable y. By definition (3, 10) we have

$$\begin{aligned} R(t) &\equiv \langle g[y(t_0)] g[y(t_0 + t)] \rangle_{\text{av.}} = \langle g(y_1) g(y_2) \rangle_{\text{s.av.}} \\ &= \iint_{-\infty}^{\infty} g(y_1) g(y_2) W_2(y_1, y_2; t)\, dy_1\, dy_2 . \end{aligned} \tag{1.9}$$

*The mean intensity may be expressed in units of power or mean square amplitude, appropriate to the problem in question.

The spectrum follows from (1.6a). For problems (i)-(iii) the modulated wave $V(t)$ is expressed as a function of a statistical variable y, and the choice of $g(y)$ is based on the pertinent physical model which describes the problem. In general, $g(y)$ is not a linear function of y, and so the evaluation of the auto-correlation function becomes difficult. These remarks are illustrated in the following sections.

The main results of the analysis of a carrier wave amplitude-modulated by normal random noise, or a signal and noise, show that the amplitude-distortion characteristic of over-modulation spreads the spectrum but not significantly; the additional noise components are due to $(n \times n)$ noise modulation products. Furthermore, when a modulating signal accompanies the noise, distortion of the signal also occurs, and $(s \times n)$ as well as $(n \times n)$ noise harmonics are produced. Expressions for the mean total power, the mean continuum power, the mean carrier power, and the mean power in the discrete portions of the spectrum are given, along with a detailed treatment of the spectral distribution of the wave's energy. In the following sections, a discussion of the limiting cases of weak noise, strong signal, etc., is included, and a number of figures illustrate the principal results.

2. Carrier amplitude-modulated by noise. We represent the *IF* (or *RF*) wave by a complex disturbance

$$g(y) = V(t) = A_0(t) \exp(i\omega_0 t), \qquad (\omega_0 = 2\pi f_0), \tag{2.1}$$

where $A_0(t)$ is a real quantity. The amplitude modulation is specifically

$$\left.\begin{aligned} A_0(t) &= A_0(1 + kV_N(t)), \qquad & y = kV_N(t) \geq -1 \\ &= 0, & y = kV_N(t) \leq -1 \end{aligned}\right\}, \tag{2.2}$$

in which $V_N(t)$ is a normal random noise voltage (or current) and k is a modulation index, with dimensions $(\text{volts})^{-1}$. When the instantaneous amplitude $V_N(t)$ is less than $-1/k$, *over-modulation* occurs, and the signal generator does not oscillate until $V_N(t)$ is once more greater than $-1/k$. Since we assume a purely normal random noise, large and even infinite amplitudes are possible, and consequently we may expect over-modulation for a noticeable part of the time, unless the modulating noise is weak. The analysis of this and succeeding sections assumes the common type of modulation in which the instantaneous amplitude of an oscillator's output is modified according to some signal or other low-frequency disturbance applied to a suitable control grid. Frequently, however, a modulated output is produced by applying the *sum* of the separately generated oscillations and the modulation to the input of a (half-wave linear) rectifier. The tube acts now as a mixing device, which yields a suitably modulated output carrier wave *only if the original carrier oscillations are very intense relative to the modulation.* Otherwise one obtains serious distortion due to the significant additional harmonics generated in the nonlinear mixing of the signal (noise), carrier, and background noise. Thus, if a mixer is used, Eqs. (2.1) and (2.2) apply here approximately, provided the modulation is weak, while (2.1) and (2.2) are valid models for all degrees of carrier and modulation strengths when the alternative system of a modulated oscillator is employed.

Let us consider first the simpler and less general situation in which the *r-m-s* noise amplitude $\langle V_N(t)^2\rangle_{s.av}^{1/2}$ is very much less than $1/k\, 2^{1/2}$; this means that for an overwhelmingly large percentage of the time the instantaneous amplitude $kV_N(t)$ is less than unity and therefore over-modulation is for all practical purposes ignorable. The exact expression (2.1) then becomes simply

$$A_0(t) = A_0(1 + kV_N(t)) = A_0(1 + y), \qquad (-\infty < y = kV_N(t) < \infty), \tag{2.3}$$

provided $(0 \leq k(2\psi)^{1/2} \ll 1)$, where by ψ we abbreviate $\langle V_N(t)^2\rangle_{s.av.}$. To see how strict the condition $(0 \leq k(2\psi)^{1/2} \ll 1)$ must be, we ask what fraction of the time α $(0 \leq \alpha \leq 0.5)$, $V(t)$ exceeds an amplitude $V_0 = 1/k$. Since $V(t)$ is normally distributed, we have at once

$$\alpha = \frac{1}{(2\pi\psi)^{1/2}} \int_{V_0=1/k}^{\infty} \exp(-V^2/2\psi)\, dV = \frac{1}{2}[1 - \Theta(1/k(2\psi)^{1/2})], \qquad (\psi \equiv \langle V_N(t)^2\rangle_{s.av.}), \tag{2.4}$$

where Θ is the familiar error function

$$\Theta(x) = 2\pi^{-1/2} \int_0^x \exp(-z^2)\, dz, \qquad \text{and} \qquad \phi^{(n)}(x) \equiv \frac{d^n}{dx^n} \frac{\exp(-x^2/2)}{(2\pi)^{1/2}}. \tag{2.4a}$$

With the help of tables of Θ we readily determine $k(2\psi)^{1/2}$ corresponding to a chosen value of α. We see for example that when $k(2\psi)^{1/2} \leq 0.6$, over-modulation occurs less than 1 per cent of the time, and so for most purposes Eq. (2.3) may replace the more general relation (2.2) when $\langle V_N(t)^2\rangle_{s.av.} \leq 0.18/k$.

The autocorrelation function of the modulated wave is therefore from (1.3) and (2.3)

$$R(t) = (A_0^2/2)\, Re\{\exp(-i\omega_0 t)[1 + k^2\langle V_N(t_0)V_N(t_0 + t)\rangle_{av.}]\} = (A_0^2/2) \cos\omega_0 t\{1 + k^2R_0(t)_N\}, \tag{2.5}$$

in which $R_0(t)_N = \langle V_N(t_0)V_N(t_0 + t)\rangle_{s.av.}$ is the auto-correlation function of the modulating noise wave. The mean intensity spectrum is from (1.6)

$$W(f) = (A_0^2/2)\, \delta(f - f_0) + A_0^2k^2 \int_0^\infty R_0(t)_N \cos(\omega_0 - \omega)t\, dt, \tag{2.6}$$

where we neglect the contribution from $\cos(\omega_0 + \omega)t$ in (2.6), since the spectral width $(\backsim\omega_b)$ of the noise is much less than the carrier frequency $(=f_0)$. The carrier power is unchanged, viz., $W_{f_0} = A_0^2/2$, while the amount of power W_c in the continuum is distributed symmetrically about f_0 , with a total intensity $A_0^2k^2\psi/2$. [Unlike frequency- or phase-modulation (cf. secs. 2, 3 of ref. [1]) there is a larger amount of energy in the wave after modulation than before, and all of the additional power appears in the continuous spectrum.] Now $R_0(t)_N \cos\omega_0 t$ in (2.6) represents the correlation function of the continuous part of the output spectrum, which is the same as the correlation function that one obtains for a narrow-band of noise centered about f_0 , and consequently yields the same *intensity* spectrum. There is, however, an important difference between the two. In the former the component at a frequency $f_0 + f'$ is correlated with the component at the image frequency $f_0 - f'$, while in the latter there is no such coherence between pairs of harmonics symmetrically located about f_0 . Nevertheless, identical forms of correlation function and spectrum occur in either instance because, by definition, these quantities are squares of a modulus, from which all phase factors are excluded. On the other hand, a Fourier analysis of the noise-modulated carrier (2.1), (2.3) and the equivalent (in power) narrow-band noise centered about f_0 show at once coherence in the former and none in the latter. This can be observed directly on a cathode-ray oscilloscope: for a noise-modulated carrier with ignorable over-modulation, the instantaneous envelope will vary randomly but there will be no change in the phase of the carrier, while in the case of the noise band the phase of the carrier will change (relatively slowly) in a random way.

We return now to the general case (2.2), which includes *over-modulation.* To represent the discontinuities in $A_0(t)$, cf. (2.2), we express $A_0(t)$ in terms of its Fourier transform

$$A_0(t) = -A_0(2\pi)^{-1} \int_{\mathbf{C}} z^{-2}\, dz \exp (iz[1\ kV_N(t)]), \tag{2.7}$$

where $\mathbf{C}$ is a contour extending along the real axis from $-\infty$ to $+\infty$ and is indented downward in an infinitesimal semicircle about the singularity at $z = 0$. The correlation function is now

$$R(t) = (A_0^2/2)\, Re\Big\{\exp(-i\omega_0 t)(4\pi^2)^{-1} \int_{\mathbf{C}} z^{-2}\, dz \exp(iz[1 + kV_N(t_0)]) \cdot \int_{\mathbf{C}^*} \xi^{-2}\, d\xi \exp(-i\xi[1 + kV_N(t_0 + t)])\Big\}_{\text{stat.av.}} \tag{2.8}$$

since no coherence between carrier and modulation is assumed. Here $\mathbf{C}^*$ is the contour conjugate to $\mathbf{C}$, extending from $+\infty$ to $-\infty$ and is indented *upward* in an infinitesimal semicircle about the point $\xi = 0$. Furthermore, inasmuch as $V_N(t)$ is real, $A_0(t)$ is also and so $A_0(t)^* = A_0(t)$ and we can replace the integral over $\mathbf{C}^*$ by one in $\mathbf{C}$, setting $i = -i$. The ensemble average in (2.8) may be effected in a straightforward way if we note that since $V_N(t_0)$ $(=V_1)$ and $V_N(t_0 + t)$ $(=V_2)$ are normal random variables, their joint distribution is given by a relation of the form

$$W_2(V_1, V_2; t) = [2\pi\psi(1 - r_0^2)^{1/2}]^{-1} \exp\{-[V_1^2 + V_2^2 - 2V_1V_2r_0]/2(1 - r_0^2)\psi\}, \tag{2.9}$$

and $r_0 = \langle V_1V_2\rangle_{\text{s.av.}}/\langle V^2\rangle_{\text{s.av.}} = \psi(t)/\psi$ is the normalized auto-correlation function of the modulating noise, whose mean intensity spectrum is $W(f)_N$. Substituting (2.9) into (2.8) and observing that the statistical average yields the characteristic function (cf. Eq. (2.16) of ref. [10]) for the noise $F_2(z, \xi; t)_N = \exp\{-\frac{1}{2}k^2\psi(0)(z^2 + \xi^2) - z\xi\psi(t)\}$, we have finally

$$\begin{aligned} R(t) &= (A_0^2/2)\, Re\Big\{\exp(-i\omega_0 t)(4\pi^2)^{-1} \int_{\mathbf{C}} z^{-2}\, dz \exp(iz - k^2\psi z^2/2) \cdot \int_{\mathbf{C}} \xi^{-2}\, d\xi \exp(i\xi - k^2\psi\xi^2/2 - k^2\psi(t)z\xi\Big\} \\ &= (A_0^2h_{0,0}^2/2)\cos\omega_0 t + (A_0^2/2)\sum_{n=1}^{\infty} \frac{[-k^2\psi r_0(t)]^n}{n!}\, h_{0,n}^2 \cos\omega_0 t, \end{aligned} \tag{2.10}$$

where

$$\begin{aligned} h_{0,n} = (2\pi)^{-1}\int_{\mathbf{C}} z^{n-2}\exp(iz - k^2\psi z^2/2)\, dz &= -i^{-n}(k^2\psi)^{(1-n)/2}2^{(n-3)/2} \\ &\cdot \Big\{{}_1F_1\Big(\frac{n-1}{2}; \frac{1}{2}; -\frac{1}{2k^2\psi}\Big)\Big/\Gamma\Big(\frac{3-n}{2}\Big) \\ &+ \frac{2}{(2k^2\psi)^{1/2}}\, {}_1F_1\Big(\frac{n}{2}; \frac{3}{2}; -\frac{1}{2k^2\psi}\Big)\Big/\Gamma\Big(\frac{2-n}{2}\Big)\Big\}, \end{aligned} \tag{2.11}$$

from Eq. (A3.17) of reference (10); ${}_1F_1$ is a confluent hypergeometric function. Specifically (cf. (A.9) of ref. 10), we have for the amplitude functions $h_{0,n}$,

$$h_{0,0} = [1 + \Theta([2k^2\psi]^{-1/2})]/2 + (k^2\psi/2\pi)^{1/2} \exp(-1/2k^2\psi) \tag{2.12a}$$

$$h_{0,1} = i[1 + \Theta([2k^2\psi]^{-1/2})]/2, \tag{2.12b}$$

$$h_{0,n} = (-1)^{n/2}(k^2\psi)^{(1-n)/2}\phi^{(n-2)}([k^2\psi]^{-1/2}), \qquad (n = 2, 3, 4, 5, \cdots) \tag{2.12c}$$

(the definitions of Θ and $\phi^{(n)}$ are given in Eq. (2.4a); see also Appendix III of ref. [10]). By Eq. (1.6a) the mean intensity spectrum is the Fourier transform of (2.10), namely,

$$\begin{aligned} W(f) &= (A_0^2 h_{0,0}^2/2)\,\delta(f - f_0) + A_0^2k^2(-h_{0,1}^2)\psi \int_0^\infty r_0(t) \cos(\omega_0 - \omega)t\,dt \\ &\quad + A_0^2k^2\psi \sum_{n=2}^{\infty} \frac{\phi^{(n-2)}([k^2\psi]^{-1/2})^2}{n!} \int_0^\infty r_0(t)^n \cos(\omega_0 - \omega)t\,dt. \end{aligned} \tag{2.13}$$

When the modulating noise has a gaussian spectrum, $W_N(f) = W_0 \exp(-\omega^2/\omega_b^2)$, this becomes explicitly

$$\begin{aligned} W(f) &= (A_0^2 h_{0,0}^2/2)\,\delta(f - f_0) + \pi^{1/2}A_0^2k^2\psi\omega_b^{-1}\Big\{h_{0,1}^2 \exp[-(\omega_0 - \omega)^2/\omega_b^2] \\ &\quad + \sum_{n=2}^{\infty} \frac{\phi^{(n-2)}([k^2\psi]^{-1/2})^2}{n!\,n^{1/2}} \exp[-(\omega_0 - \omega)^2/n\omega_b^2]\Big\}. \end{aligned} \tag{2.14}$$

Figure (2.1) shows typical intensity spectra for a number of values of $(2k^2\psi)^{1/2}$ between 0 and ∞. We distinguish two limiting cases: $(2k^2\psi)^{1/2} \to \infty$ and $(2k^2\psi)^{1/2} \to 0$; the autocorrelation function (2.10) is accordingly

$$\begin{aligned} (2k^2\psi \to \infty): R(t) &\backsimeq \frac{A_0^2k^2\psi}{4\pi} \cos\omega_0 t\cdot\Big\{1 + \frac{\pi}{2} r_0(t) \\ &\quad + \sum_{n=0}^{\infty} \frac{r_0(t)^{2n+2}(2n)!}{2^{2n}n!^2(2n+2)(2n+1)}\Big\} + 0([2k^2\psi]^{1/2}), \end{aligned} \tag{2.15a}$$

$$\begin{aligned} &= \frac{A_0^2k^2\psi}{4\pi} \cos\omega_0 t\cdot\Big\{r_0(t)\Big[\frac{\pi}{2} + \sin^{-1} r_0\Big] \\ &\quad + (1 - r_0^2)^{1/2}\Big\} + 0([2k^2\psi]^{1/2}) \end{aligned} \tag{2.15b}$$

for the former, while in the latter instance we obtain as expected Eq. (2.5), with a correction term $o(e^{-\frac{1}{2}k^2\psi} n r_0(t)^3)$.

Equations (2.15) apply when the maximum amount of overmodulation, namely 50 per cent of the time, occurs, whereas (2.5) yields the correlation function when essentially no overmodulation takes place. Additional correction terms may be found in straightforward but tedious fashion. We note from Fig. (2.1) that the mean intensity spectrum is here more widely distributed about f_0 than for the case of ignorable over-modulation. The additional harmonics are (carrier $\times$ noise) noise products, stemming from the

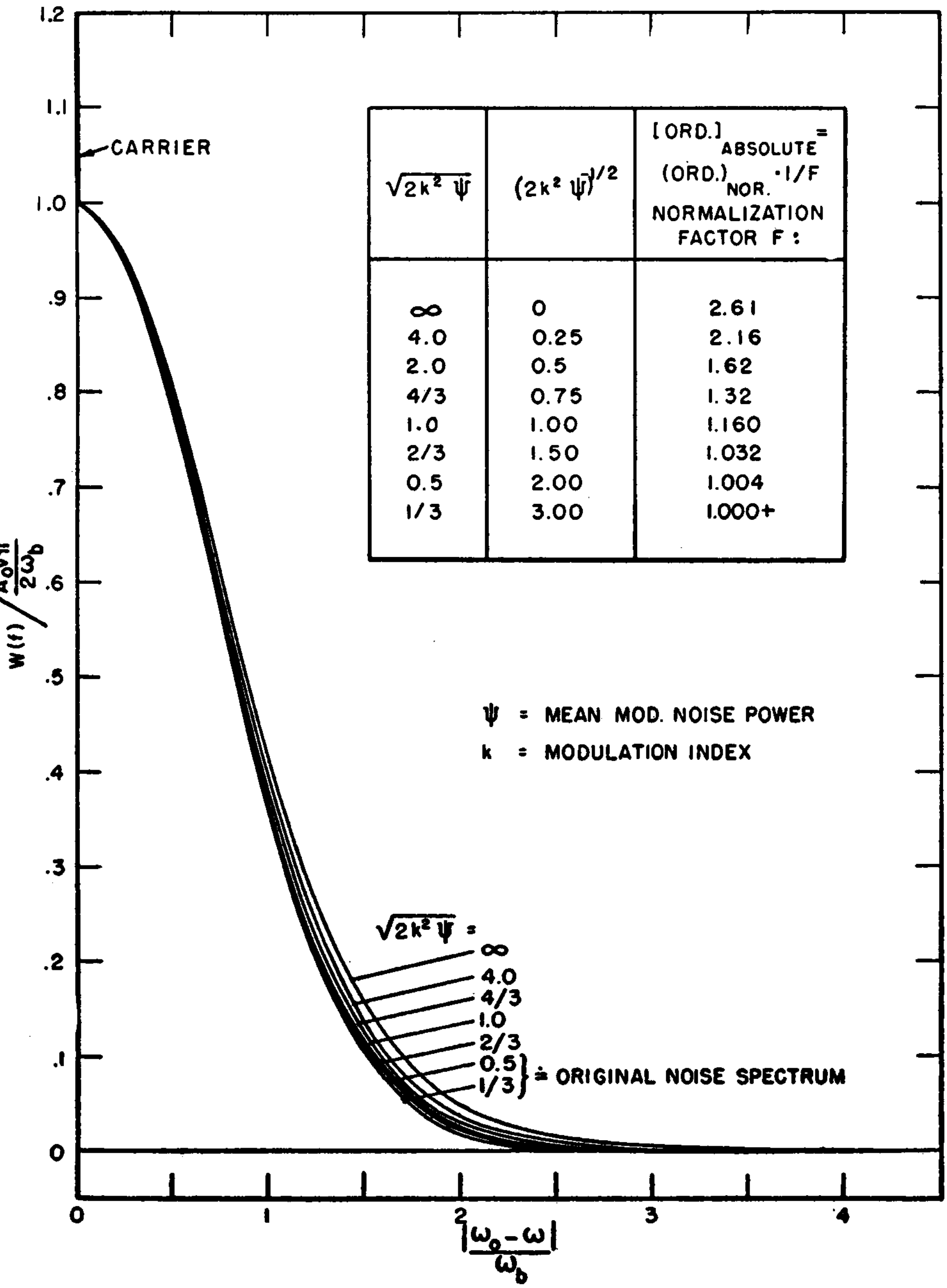

FIG. 2.1. Mean intensity spectrum of a carrier amplitude-modulated by noise.

clipping process inherent in overmodulation. However, [unlike the examples of frequency- or phase-modulation discussed earlier in ref. 1] there is a limit to the spread of the spectrum, determined by the fact that over-modulation can occur at the maximum but 50 per cent of the time. On the other hand, since the amount of clipping inherent in the weak modulation cases is ignorable, no significant spectral spread is obtained, (see Fig. 2.1). In any case the spectral spread is relatively small.

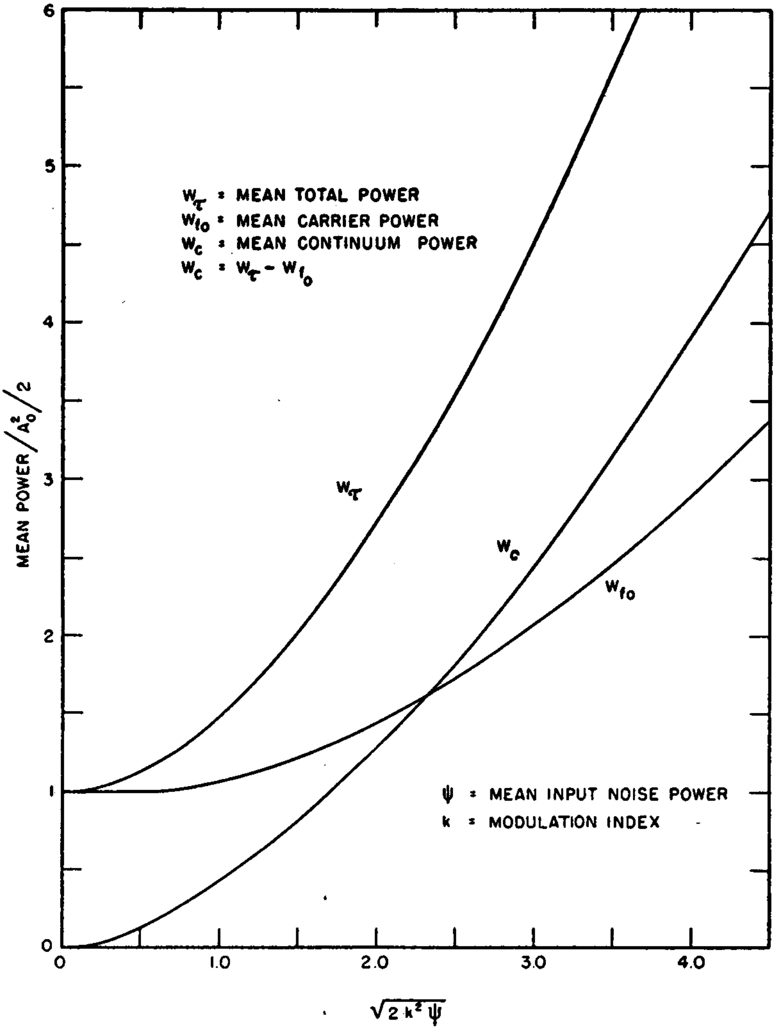

FIG. 2.2. Mean power in a carrier amplitude-modulated by noise.

Whereas the spectrum requires a series development, cf. (2.13), the total mean power W_τ and the total intensity W_c of the continuous part of the disturbance are easily obtained in precise, closed form from Eq. (2.2) if we remember that $kV_N(t) = y$ is normally distributed with the first-order probability density $W_1(y) = (2\pi k^2\psi)^{-1/2} \exp[-y^2/2k^2\psi]$.

The power in the carrier after modulation is (cf. (1.1) and (1.2))

$$\begin{aligned} W_{f_0} &= (A_0^2/2) \left| \int_{-1}^{\infty} (1+y) W_1(y)\, dy \right|^2 = (A_0^2/2) h_{0,0}^2 \\ &= (A_0^2/2)\{[1 + \Theta([2k^2\psi]^{-1/2})]/2 - k^2\psi\phi^{(1)}([k^2\psi]^{-1/2})^2\} \end{aligned} \tag{2.16}$$

and

$$\begin{aligned} W_\tau = R(0) &= (A_0^2/2) \int_{-1}^{\infty} (1+y)^2 W_1(y)\, dy \\ &= (A_0^2/2)\left[\left(\frac{1+k^2\psi}{2}\right)[\Theta([2k^2\psi]^{-1/2}) + 1]\right] - k^2\psi\phi^{(1)}([k^2\psi]^{-1/2})] \end{aligned} \tag{2.17}$$

and so

$$\begin{aligned} \therefore\ W_c = W_\tau - W_{f_0} &= (A_0^2/2)[1/4 + (k^2\psi/2)[\Theta\{(2k^2\psi)^{-1/2}\} + 1] \\ &\quad - \{\Theta[(2k^2\psi)^{-1/2}] - k^2\psi\phi([k^2\psi]^{-1/2})\}^2]. \end{aligned} \tag{2.18}$$

Figure (2.2) illustrates the behavior of the mean intensities W_τ, W_{f_0}, W_c for different degrees of over-modulation. (See also Figs. 3.1 and 3.2 when $\mu = 0$.) Their limiting conditions are instructive: As the intensity of the modulating noise is increased, a correspondingly greater proportion of the modulated wave's power is distributed in the (*noise* $\times$ *noise*) noise sidebands, generated in the process of modulation and the clipping due to possible over-modulation, as shown in Fig. 2.2. We remark that the amount of power in the carrier component and in the continuum is quite independent of the particular spectral distribution of the original (normal random) noise, and depends only on the clipping level at which over-modulation occurs, since power in a given band of frequencies is proportional to the integrated intensity of the band. One can consider also more complicated modulations, such as square-law modulation, viz: $A_0(t) = [1 + kV_N I t)]^2$: the treatment is identical with that of the linear case examined here, following an appropriate modification of the transform relation (2.7).

3. Carrier amplitude-modulated by a signal and noise. The results of the previous section may be generalized to include the important case of modulation of a carrier, $A_0 \exp(i\omega_0 t)$, by a signal with an accompanying noise disturbance. The modulated carrier (2.1) may now be written

$$\left.\begin{aligned} V(t) &= A_0(t) \exp(i\omega_0 t) \\ &= A_0[1 + kV_N(t) + \mu V_S(t)] \exp(i\omega_0 t), \qquad kV_N + \mu V_S \geq -1 \\ &= 0, \qquad\qquad kV_N + \mu V_S \leq -1 \end{aligned}\right\} \tag{3.1}$$

which includes possible over-modulation when the signal and noise are properly phased and sufficiently intense. As before, $A_0(t)$ is represented by a suitable contour integral [cf. (2.7)]. Remembering that $A_0(t)$ is a real quantity, we apply Eq. (3.1) and the results of section 2 to the general relation for the auto-correlation function of the modulated wave and obtain finally

$$R(t) = (1/2)\, Re\{V(t_0)V(t_0 + t)^*\}_{\text{stat.av.}}$$

$$= (A_0^2/2)\, Re\left\{\exp(-i\omega_0 t)(4\pi^2)^{-1} \int_C z^{-2} \exp(iz)\, dz \right. \tag{3.2}$$

$$\left. \cdot \int_C \xi^{-2} \exp(i\xi) F_2(z, \xi; t)_N F_2(z, \xi; t)_S\, d\xi \right\},$$

where $F_2(z, \xi; t)_N$ is the characteristic function for the accompanying noise, and

$$F_2(z, \xi; t)_S = [\exp(i\mu V_S(t_0)z + i\mu V_S(t_0 + t)\xi)]_{\text{stat.av.}}$$

$$= T_0^{-1} \int_0^{T_0} \exp(i\mu V_S(t_0)z + i\mu V_S(t_0 + t)\xi)\, dt_0 \tag{3.3}$$

is the characteristic function for the signal, with T_0 the period of the modulation. Expanding the exponents in (3.3) in a double Fourier series and averaging over the period $T_0(=2\pi/\omega_a)$ we can write the signal's characteristic function in the general form

$$F_2(z, \xi; t)_S = \sum_{m=0}^{\infty} (-1)^m \epsilon_m B_m(z) B_m(\xi) \cos m\omega_a t. \tag{3.4}$$

For signals which are entirely stochastic, we may replace the time-average in (3.3) by its equivalent ensemble average, since we are assuming throughout stationary (ergodic) processes. An example of the latter type is provided when the modulating signal is a (normal) random noise, uncorrelated with the background interference, in which case

$$F_2(z, \xi; t)_S = \exp\{-\mu^2 \psi(0)_S (z^2 + \xi^2) - \mu^2 \psi(t)_S z\xi\},$$

$$\psi(t)_S = \psi(0)_S r_0(t)_S\,, \qquad (\psi(0)_S \equiv \psi_S). \tag{3.5}$$

Here $\psi(t)_S$ is the auto-correlation function of $V(t)_S$. The correlation function of the modulated wave is then given by (2.10), provided we replace $k^2\psi r_0(t)$ therein by $k^2\psi_N r_0(t)_N + \mu^2\psi_S r_0(t)_S$ and $k^2\psi$ in $h_{0,n}$, Eq. (2.11), by $k^2\psi_N + \mu^2\psi_S$. The resulting intensity spectrum is a superposition of $(n \times n)$ noise components, the carrier being the only periodic term.

Let us consider now the more complex situation involving a periodic signal. The auto-correlation function (3.2) becomes with the help of section 2 and (3.4)

$$R(t) = (A_0^2/2) \sum_{m=0}^{\infty} \epsilon_m (-1)^m [\cos(\omega_0 + m\omega_a)t + \cos(\omega_0 - m\omega_a)t]$$

$$\cdot \frac{1}{2}\left\{ h_{m,0}^2 + \sum_{n=2}^{\infty} \frac{(-1)^n (k^2\psi r_0(t))^n}{n!} h_{m,n}^2 \right\} \tag{3.6}$$

where the amplitude functions $h_{m,n}$ are

$$h_{m,n} = (2\pi)^{-1} \int_C z^{n-2} B_m(z) \exp[iz - k^2\psi z^2/2]\, dz. \tag{3.7}$$

The mean intensity spectrum follows at once in the usual way with the aid of the theorem of Wiener and Khintchine, cf. (1.6). Because of the distortion inherent in over-modulation we expect that the original signal will be deformed, and because of the accompanying background noise we may further predict that not only will there be cross-modulation between the components of the noise, but between them and the signal harmonics, modified in the process of over-modulation. Accordingly, we observe from (3.6) that the term in the correlation function for which ($m = 1, n = 0$) corresponds to the carrier component, and the harmonics for which ($m \geq 1, n = 0$) represent the ($s \times s$) signal cross-terms generated in the course of over-modulation. On the other hand, the components for which ($m = 0, n \geq 1$) are attributable to ($n \times n$) noise harmonics, while for ($m \geq 1, n \geq 1$) one has as the noise contribution ($s \times n$) noise products. The mean power associated with the carrier, signal, and continuum are obtained from (3.6) on setting $t = \infty$ for the periodic components and $t = 0$ for the stochastic part of the modulated wave, according to section 1. Note that again, cf. section 2, the power content of the disturbance does not depend on the spectral distribution of the noise and signal modulations. (See also Appendix II of ref. [10].)

We can determine W_τ by a more direct method than expansion in a double series, which involves one less infinite development, unlike (3.6). The procedure is based on the observation that $y = kV_N(t)$ and $z = \mu V_S(t)$ can be treated as independent random variables, whose first-order probability densities $W_1(y)$ and $w_1(z)$ are easily determined. Thus, for the background noise, $W_1(y)$ is a gauss distribution density for which $\langle y^2 \rangle_{\text{s.av.}} = k^2\psi$, $\langle y \rangle_{\text{s.av.}} = 0$, while for a periodic signal

$$w_1(z) = \int_{-\infty}^{\infty} (2\pi)^{-1} \exp(iz\xi)\, d\xi \int_0^{2\pi} (2\pi)^{-1} \exp[i\mu\xi V_S(0, \phi)]\, d\phi. \tag{3.8}$$

The phase ϕ is a purely random quantity, distributed uniformly between 0 and 2π with a probability density $1/2\pi$. Therefore V_S is also a random variable, corresponding physically to the fact that we agree now to observe the periodic wave at (independent) random times. (In this fashion any periodic disturbance can be "randomized" with respect to the observer.) The mean-square value of the modulated carrier (3.1) becomes

$$\begin{aligned} W_\tau &= \langle | V(t_0) |^2 \rangle_{\text{s.av.}}/2 = (A_0^2/2)\langle (1 + y + z)^2 \rangle_{\text{s.av.}}, \text{ (for all } w = y + z \geq -1), \\ &= (A_0^2/2) \int_{-1}^{\infty} W_1^{(1)}(w)(1 + w)^2\, dw = R(0), \end{aligned} \tag{3.9}$$

where the frequency-function $W_1^{(1)}(w)$ for $w = y + z$ is given by

$$\begin{aligned} W_1^{(1)}(w) &= \int_{-\infty}^{\infty} W_1(y) w_1(w - y)\, dy \\ &= \int_{-\infty}^{\infty} (2\pi)^{-1} \exp(-iw\xi - k^2\psi\xi^2/2)\, d\xi \int_0^{2\pi} (2\pi)^{-1} \exp(i\mu\xi V_S(0, \phi))\, d\phi, \end{aligned} \tag{3.10}$$

since the Jacobian $| \partial(y, z)/\partial(y, w) |$ of the transformation is unity. In a similar way one obtains the mean power in the carrier, which is

$$\begin{aligned} W_{f_0} &= (A_0^2/2)h_{0,0}^2 = \langle | V(t) | \rangle_{\text{s.av.}}^2/2 = (A_0^2/2)\, | \langle 1 + y + z \rangle_{\text{av.}} |^2, \\ &\qquad \text{for all } w = y + z \geq -1 \\ &= (A_0^2/2)\left\{ \int_{-1}^{\infty} W_1^{(1)}(w)(1 + w)\, dw \right\}^2, \end{aligned} \tag{3.11}$$

since w is real. Note that this procedure does *not* give the mean power $W_{\text{per.}}$ in the periodic components of the wave, but only the steady or average value of the envelope. To calculate $W_{\text{per.}}$ one must sum the series

$$\sum_{m=0}^{\infty} \epsilon_m(-1)^m h_{m,0}^2 .$$

In the specific case of a sinusoidal signal,

$$V_S = V_0 \cos (\omega_a t + \gamma),$$

we find from (3.3) and (3.4) that now $B_m(z) = J_m(\mu V_0 z)$, and so the amplitude functions (3.7) are explicitly*

$$h_{m,n} = (2\pi)^{-1} \int_C z^{n-2} J_m(\mu V_0 z) \exp (iz - k^2\psi z^2/2)\, dz$$

$$= \frac{-i^{m+n}}{2m!} \cdot \left(\frac{k^2\psi}{2}\right)^{(1-n)/2} \cdot [\mu V_0/(2k^2\psi)^{1/2}]^m \tag{3.12a}$$

$$\cdot \sum_{q=0}^{\infty} \frac{(2)^{q/2}{}_1F_1([q + m + n - 1]/2; m + 1; -\mu^2 V_0^2/2k^2\psi)}{q!(k^2\psi)^{q/2}\Gamma([3 - m - n - q]/2)},$$

or

$$= \frac{-i^{m+n}}{2} \cdot \left(\frac{k^2\psi}{2}\right)^{(1-n)/2} \cdot [\mu V_0/(2k^2\psi)^{1/2}]^m \sum_{q=0}^{\infty} \alpha_{mnq}(\mu^2 V_0^2/2k^2\psi)^q, \tag{3.12b}$$

where

$$\alpha_{mnq} = [1/q!(q + m)!]\left\{\frac{{}_1F_1([2q + m + n - 1]/2; 1/2; - 1/2k^2\psi)}{\Gamma([3 - 2q - m - n]/2)}\right.$$
$$\left. + (2/k^2\psi)^{1/2} \frac{{}_1F_1([2q + m + n]/2; 3/2; - 1/2k^2\psi)}{\Gamma([2 - 2q - m - n]/2)}\right\}. \tag{3.12c}$$

The first result, (3.12a), is convenient when the signal is large relative to the noise $[\mu V_0^2 \gg 2k^2\psi]$, for then we may use the asymptotic form of the confluent hypergeometric function

$${}_1F_1(\alpha; \beta; - x) \backsimeq \frac{\Gamma(\beta)}{\Gamma(\beta - \alpha)} x^{-\alpha}\left\{1 + \frac{\alpha(\alpha - \beta + 1)}{x1!}\right.$$
$$\left. + \frac{\alpha(\alpha + 1)(\alpha - \beta + 1)(\alpha - \beta + 2)}{x^2 2!} + \cdots\right\}, \qquad Re(x) > 0. \tag{3.13}$$

On the other hand, the series (3.12b) is particularly useful for weak signals $[\mu V_0^2 \ll 2k^2\psi]$. From either expression we can easily obtain the important limiting cases of over-modulation and weak noise $(2k^2\psi \to 0)$, the latter by (3.13).

*The details of the integration are given in section 2 and in Appendix III of ref. [10]. See also Appendix IV of ref. [3].

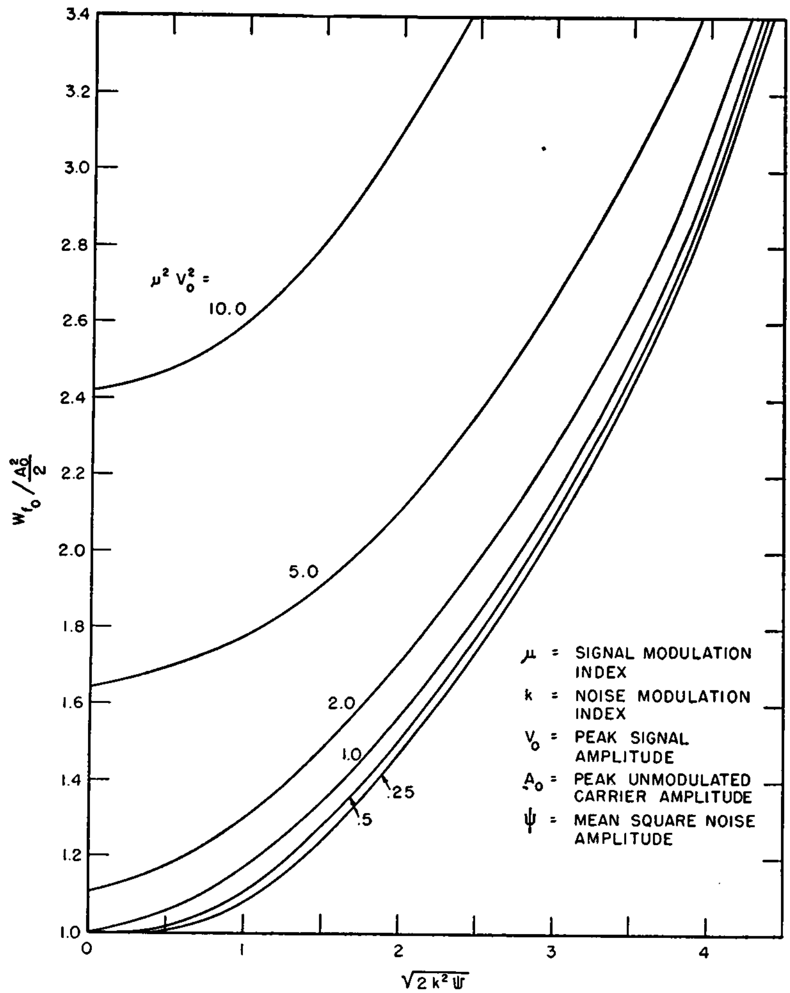

FIG. 3.1. Mean carrier power of a carrier amplitude-modulated by noise and a sinusoidal signal.

The mean power in the carrier is found directly from (3.12) to be in the sinusoidal case $A_0^2 h_{0,0}^2/2$, which is illustrated in Fig. (3.1) for a variety of values of $(2k^2\psi)^{1/2}$ and $\mu^2 V_0^2$. The mean total power W_τ is in the present instance more easily found from (3.8)-(3.10) than from the multiple series expression, where now $V_S(0, \phi) = V_0 \cos(\gamma + \phi)$. We have

$$W_\tau = (A_0^2/2) \int_{-1}^{\infty} (1 + w)^2 \, dw \int_0^\infty \pi^{-1} J_0(\mu V_0 \xi) \cos w\xi \exp(-k \ \psi\xi^2/2) \, d\xi = R(0) \tag{3.14a}$$

$$= (A_0^2/2)\left\{\left(\frac{1}{2} + \frac{k^2\psi}{2} + \frac{\mu^2 V_0^2}{4}\right)[1 + \Theta([2k^2\psi]^{-1/2})] - k^2\psi\phi^{(1)}[(k^2\psi)^{-1/2}] \right. \tag{3.14b}$$

$$\left. + k^2\psi \cdot \sum_{n=2}^{\infty} \left(\frac{\mu^2 V_0^2}{2k^2\psi}\right)^n \frac{1}{n!^2 2^n} \phi^{(2n-3)}[(k^2\psi)^{-1/2}]\right\}.$$

W_T is shown in Fig. (3.2) for representative values of $\mu^2 V_0^2$ and $(2k^2\psi)^{1/2}$. The power associated with the carrier part of the modulated wave is obtained alternatively from (3.11) in a similar manner; we have finally

$$W_{f_0} = (A_0^2/2)\Big\{[1 + \Theta\{(2k^2\psi)^{-1/2}\}]/2 - k^2\psi\phi^{(1)}[(k^2\psi)^{-1/2}] + (k^2\psi)^{1/2} \sum_{n=1}^{\infty} \left(\frac{\mu^2 V_0^2}{2k^2\psi}\right)^n \frac{1}{n!^2 2^n} \phi^{(2n-2)}[(k^2\psi)^{-1/2}]\Big\}^2 \tag{3.15}$$

which is equivalent to our result $A_0^2 h_{0,0}^2/2$. Typical curves for W_{f_0} as a function of $(2k^2\psi)^{1/2}$ are shown in Fig. (3.1). Since $W_{\text{per}} = W_{f_0} + W_{(s\times s)}$ one easily finds the mean power $W_{(s\times s)}$ associated with the signal components once W_{per} and W_{f_0} have been calculated. Furthermore, because $W_c = W_T - W_{\text{per}}$, we obtain also the mean power in the continuum, without having to sum the doubly-infinite series of the direct expansion (3.6).

For a given amount of signal (μV_0 fixed) the power in the periodic and in the carrier components of the modulated wave increases with the amount of modulating noise, as shown (for W_{f_0} only) in Fig. (3.1). The energy available in the signal becomes independent of the noise; however, the noise is then relatively so great that the signal is quite ignorable. This is easily seen from (3.12) in the case of the sinusoidal signal ($\backsim h_{1,0}^2$) when $2k^2\psi \to \infty$. Depending on the strength of the signal and noise relative to the amplitude A_0 of the unmodulated carrier, some of the remaining signal power is distributed in ($s \times s$) "discrete," periodic terms ($m \geq 2$, $n = 0$), which represent a distortion of the original sinusoid, attributable to the clipping inherent in over-modulation. Furthermore, as the noise becomes more intense, over-modulation occurs a significant fraction of the time, up to a maximum of 50 per cent. The amount of signal power in the modulated wave is then one-half that in the original modulating signal, since on the average the comparatively weak signal "rides" on the stronger noise half the time. Additional noise is also generated by the clipping of the wave, and these new noise components appear as ($s \times n$), (m, $n \geq 1$), and ($n \times n$), ($m = 0$, $n \geq 1$) terms, produced by the cross-modulation of the signal and noise harmonics. Only the ($n \times n$) terms are important when the noise is strong compared to the signal. In a similar way one finds that for weak noise and strong signals, the ($s \times n$) noise products are significant provided μV_0 is noticeably greater than unity (over-modulation of the carrier, due essentially to the signal). If μV_0 is less than unity and there is little noise, we expect a negligible amount of over-modulation, and the original signal consequently suffers no appreciable distortion. The same argument applies to describe the variation of the mean-total power W_T and the mean continuum power $W_c = W_T - W_{\text{per}}$ with different amounts of signal and noise modulation. Again, when the noise is the dominant factor, both W_T and W_c become indefinitely large as $2k^2\psi \to \infty$, cf. Fig. (3.2). We note also that W_{per} approaches a fixed limit, independent of the amount of modulating noise, which is one-half the original power in the carrier and modulating signal. Most of the wave's energy goes now into the noise continuum ($s \times n$, $n \times n$), as a result of the very heavy ($\backsim$50 per cent) over-modulation. When the noise becomes weaker, correspondingly less of the modulated carrier's power is distributed in the continuum. In general, different degrees of modulation (i.e., different values of μ and k) will, as expected, critically affect the magnitudes of total, periodic, and continuum powers.

The explicit calculation of spectra, based as it must be on Eq. (3.6), is far more

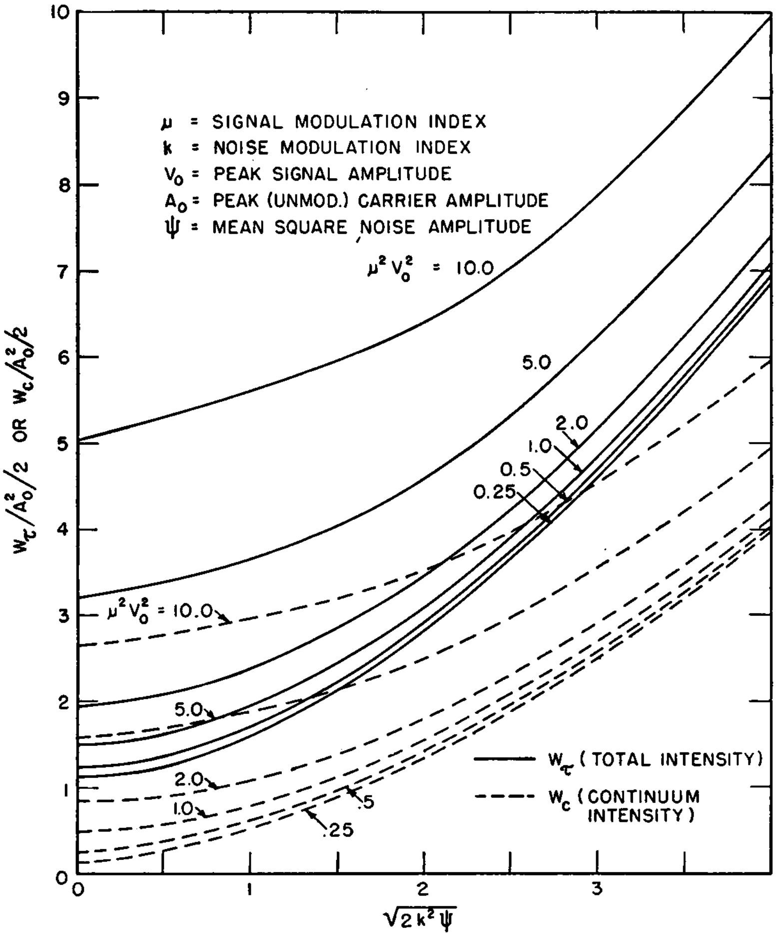

Fig. 3.2. Mean total and continuum powers in a carrier amplitude-modulated by noise and a sinusoidal signal

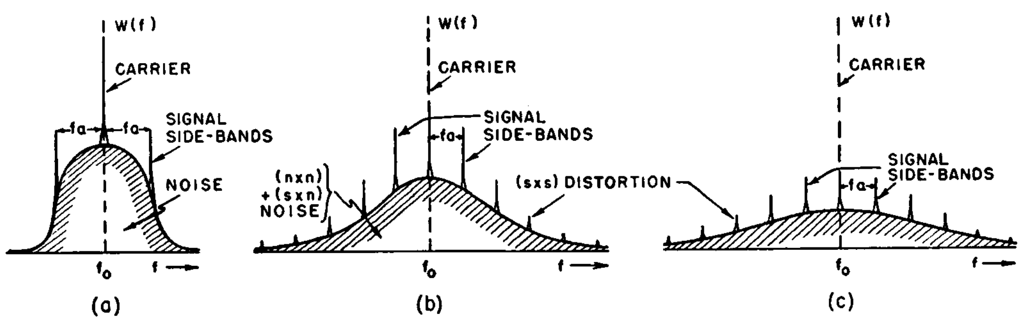

Fig. 3.3. Spectra:
(a) No over-modulation,
(b) Moderate over-modulation,
(c) 50% over-modulation.

tedious than the determination of power. A sketch illustrating typical spectra is given in Fig. (3.3); the precise calculations are reserved for a later paper.

With the help of the properties of the hypergeometric function one can obtain other interesting limiting expressions for the power and the spectrum.* For example, when there is ignorable over-modulation, $(\mu^2\langle V_S^2\rangle_{s.av.} + k^2\langle V_N^2\rangle_{s.av.} \ll 1)$, $h_{m,n}$, $(m \geq 2, n \geq 1)$, approaches zero, and one has as expected

$$R(t) \doteq (A_0^2/2)\{1 + k^2 r_0(t)\psi + \mu^2\langle V_S(t_0)V_S(t_0 + t)\rangle_{av.}\} \cos \omega_0 t, \qquad (\mu^2\langle V_S^2\rangle_{s.av.} + k^2\langle V_N^2\rangle_{s.av.} \ll 1). \tag{3.16}$$

For no signal at all the results of the preceding section can be applied, while at the other extreme of no modulating noise ($\psi = 0$) one finds $R(t)$ from (3.6) and (3.7) on setting $n = 0$' In the specific case of sinusoidal modulation the amplitude functions are** now

$$h_{m,0} = \int_C J_m(\mu V_0 z) \exp(iz)\, dz/2\pi z^2$$

$$= \frac{i^m \mu V_0}{4} \cdot \left\{ \frac{{}_2F_1([m-1]/2, [-m-1]/2; 1/2; 1/\mu^2 V_0^2)}{\Gamma([3+m]/2)\Gamma([3-m]/2)} + \frac{2}{\mu V_0} \frac{{}_2F_1(m/2, -m/2; 3/2; 1/\mu^2 V_0^2)}{\Gamma([2+m]/2)\Gamma([2-m]/2)} \right\}, \qquad (1 \leq \mu V_0) \tag{3.17a}$$

$$= \left.\begin{matrix} -1\;, & m = 0, \\ i\mu V_0/2, & m = 1, \\ 0, & m \geq 2, \end{matrix}\right\}, \qquad (0 \leq \mu V_0 \leq 1). \tag{3.17b}$$

The mean total power is found from (3.14a) to be, $(k^2\psi = 0)$,

$$W_T = R(0) = (A_0^2/2\pi) \int_{-z_0}^{1} \frac{(\mu V_0 z + 1)^2}{(1 - z^2)^{1/2}}\, dz, \qquad z_0 = \begin{cases} 1/\mu V_0\;, & \mu V_0 \geq 1 \\ 1, & 0 \leq \mu V_0 \leq 1 \end{cases}$$

$$= (A_0^2/2)\left\{\left(\frac{1}{2} + \frac{1}{\pi}\sin^{-1} z_0\right)\left(1 + \frac{\mu^2 V_0^2}{2}\right) + \frac{2\mu V_0}{2}\left(1 - \frac{\mu V_0 z_0}{4}\right)(1 - z_0^2)^{1/2}\right\} \tag{3.18}$$

while the mean power in the carrier is

$$W_{f_0} = (A_0^2/2) \left| \int_{-z_0}^{1} \frac{(\mu V_0 z + 1)}{\pi(1 - z^2)^{1/2}}\, dz \right|^2$$

$$= (A_0^2/2)\left\{\frac{1}{2} + \frac{1}{\pi}\sin^{-1} z_0 + \frac{\mu V_0}{\pi}(1 - z_0^2)^{1/2}\right\}^2, \qquad (2k^2\psi \to 0). \tag{3.19}$$

The difference $W_T - W_{f_0}$ now represents the mean power in the continuum, in this case

*See section 4 of ref. [10]; in particular the case of the biased, ν th—law rectifier.

**See Eqs. (4.19), (4.20), and (4.22) of ref. 10.

the discrete spectra consisting of the harmonics of the modulation and its distortion (if any) due to over-modulation. When there is no signal, only terms in (3.6) and (3.7) for which $m = 0$ remain, and our general expressions of the present section reduce to the results of the preceding section, cf. Eq. (3.16) et. seq. Other limiting cases may be treated in the manner of section 4, reference [10].

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