ON THE DISTRIBUTION OF ENERGY IN NOISE- AND SIGNAL-MODULATED WAVES
II. SIMULTANEOUS AMPLITUDE AND ANGLE MODULATION*

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1. Introduction. In Part I[1] we have determined the distribution of energy in a carrier which is amplitude-modulated by a band of normal random noise or by a noise and a signal, and in an earlier work[2] the corresponding problem for frequency- or phase-modulated waves was considered. With the help of these results we now examine the case of a carrier which is simultaneously amplitude- and angle-modulated by a pair of normal random noise waves. Here, either noise wave may be delayed by an amount t_ϕ with respect to the other, but coherence between the two modulations is assumed to exist.

The investigation of this problem is of considerable interest in the study of the noise output of typical magnetron tubes, on the basis of a macroscopic model that treats the magnetron generator as producing a high-frequency oscillation, which in turn is simultaneously amplitude- and phase- or frequency-modulated by a comparatively low-frequency band of normal random noise, itself generated in the oscillator due to the "shot effect" or so-called primary noise which is inherent in the structure of the electron beam moving under the influence of the applied electric and magnetic fields. The present problem also provides us with results useful in communication theory when the interference appears as an amplitude and frequency distortion, or when the modulating signal can be used as a simplified model of a speech signal.† The central feature of this problem, as distinguished from those examined previously in Part I, is the fact that the modulating noise disturbances are now correlated with each other. The result is a "scrambling" of the separate modulations, which in turn produces an *asymmetrical* intensity spectrum, unlike those obtained before. This is a significant phenomenon which is entirely absent in the simpler situations where there is no correlation. Symmetrical distributions can occur for selected values of the relative delay t_ϕ between the amplitude and angle-modulations. Moreover, even in extreme cases, the clipping due to over-modulation causes comparatively little spreading of the spectrum and spectral shape is mainly governed by the degree of angle-modulation. Special attention is given to the example of coherent modulation without over-modulation. Specifically, the modulated carrier is represented by

$$g(y) \equiv V(t) = A_0(t) \exp\,[i(\omega_0 t + \Psi(t))], \qquad (\omega_0 = 2\pi f_0). \tag{2.1}$$

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[1] D. Middleton, Quart. Appl. Math. 9, 337 (1952).

[2] D. Middleton, Technical Report No. 99, Cruft Laboratory, Harvard University (March 1, 1950).

†We assume for the moment that any background noise accompanying the modulated carrier (and hence uncorrelated with it) is quite negligible. It is not, however, difficult to include this additional effect in the analysis.

2. The correlation function. For the amplitude modulation $V_{AN}(t)$ one has from (2.7) of reference 1

$$A_0(t) = (-A_0/2\pi) \int_C z^{-2} \exp(iz[1 + kV_{AN}(t + t_\phi)])\, dz, \tag{2.2a}$$

and for the angle modulation one can write

$$\Psi_{\phi M}(t) = D_0 V_{FN}(t); \qquad \Psi_{FM}(t) = D_0 \int^t V_{FN}(t')\, dt'. \tag{2.2b}$$

As before,[1] k (≥ 0) is an amplitude-modulation index, A_0 is the peak amplitude of the unmodulated carrier, and D_0 is the r-m-s phase or (angular) frequency-deviation per unit r-m-s modulating voltage (or current), viz.*: $D_0^2 = \langle \Psi(t)^2 \rangle_{s\cdot av.} / \langle V_{FN}(t)^2 \rangle_{s\cdot av.} = \omega_d^2 / \langle V_{FN}(t)^2 \rangle_{s\cdot av.}$ or $D_0^2 = \theta_d^2 / \langle V_{FN}(t)^2 \rangle_{s\cdot av.}$ respectively. For the moment we distinguish between the two (real) noise waves $V_{AN}(t + t_\phi)$ and $V_{FN}(t)$, the former of which represents the disturbance producing the amplitude distortion while the latter causes the phase- or frequency-modulation. The effects of intervals of over-modulation, during which the oscillator is cut off, are accounted for in the general expression (2.2a), which gives $A_0[1 + kV_{AN}(t + t_\phi)]$ when kV_{AN} exceeds -1 and vanishes when $kV_{AN} \leq -1$. Here t_ϕ represents a phase lead ($t_\phi > 0$), or lag ($t_\phi < 0$), of the instantaneous amplitude modulation with respect to the phase- or frequency-modulation. Such a phase difference is included because in general the mechanism producing the coherent modulations may not act instantaneously: there may be a definite phase lag or lead of the one over the other. We assume in any case that we are dealing with proper modulations, i.e., those whose highest significant frequency components are much less than the carrier frequency f_0, and that a direct modulation rather than a mixing process is responsible for the perturbed carrier. [See section 2, Part I, for further comments on this point.]

The auto-correlation function of the modulated carrier is written*

$$\begin{aligned} R(t) &= (1/2)\, Re\{\langle V(t_0, t_\phi) V(t_0 + t, t_\phi)^* \rangle_{s.av.}\} \\ &= (A_0^2/2)\, Re\Big\{ \exp(-i\omega_0 t)(4\pi^2)^{-1} \int_C z^{-2} e^{iz}\, dz \int_C \xi^{-2} e^{i\xi}\, d\xi \\ &\qquad \cdot [\exp(ikV_1 z + ikV_2\xi + i\Psi_1 - i\Psi_2)]_{stat.av.} \Big\}. \end{aligned} \tag{2.3}$$

Here V_1 ($\equiv V_{AN}(t_0 + t_\phi)$) is the noise amplitude-modulation at an initial time t_0 and V_2 ($\equiv V_{AN}(t_0 + t_\phi + t)$) is the same disturbance at a time t later; Ψ_1 and Ψ_2 are respectively $\Psi(t_0)$ and $\Psi(t_0 + t)$, the noise angle-modulation. The statistical average is performed over the four random quantities $V_1, \cdots \Psi_2$ as indicated. They are not generally independent random variables, since $V_{AN}(t + t_\phi)$ and $V_{FN}(t)$ are assumed to come from the same source and to contain in common a significant fraction of their total spectral distributions. We note that

*The symbol $\langle\ \rangle_{s.av.}$ denotes the statistical or ensemble average over the appropriate random variables, while $\langle\ \rangle_{av.}$ indicates the infinite time average, cf. ref. 1.

$$[\exp (ikV_1z + ikV_2\xi + i\Psi_1 - i\Psi_2)]_{\text{stat.av.}}$$

$$= F_2(\zeta_1 , \zeta_2 , \zeta_3 , \zeta_4 ; t) = \text{characteristic function of the random variables } kV_1 , kV_2 , \Psi_1 , \text{ and } \Psi_2 , \text{ for which}$$

$$\zeta_1 = z, \qquad \zeta_2 = \xi, \qquad \zeta_3 = 1, \qquad \zeta_4 = -1.$$

Since $V_1 \cdots \Psi_2$ are *normal* random variables we can write at once[3]

$$F_2(\zeta_1 , \zeta_2 , \zeta_3 , \zeta_4 ; t) = \exp\left(-\frac{1}{2}\sum_j^4 \sum_l^4 \zeta_j\zeta_l\mu_{jl}\right) = F_2(z, \xi, 1, -1; t), \qquad \text{when} \quad \zeta_1 = z, \quad \zeta_2 = \xi, \quad \zeta_3 = 1, \quad \zeta_4 = -1, \tag{2.4}$$

where the sixteen variances $u_{jl}(j,l = 1 \cdots 4)$ are

$$\begin{aligned}
&\mu_{11} = \mu_{22} = k^2\langle V_1^2\rangle_{\text{s.av.}} = k^2\langle V_2^2\rangle_{\text{s.av.}} = k^2\Psi_A(0);\\
&\mu_{12} = \mu_{21} = k^2\langle V_1V_2\rangle_{\text{s.av.}} = k^2\Psi_A(t);\\
&\mu_{33} = \mu_{44} = \langle\Psi_1^2\rangle_{\text{s.av.}} = \langle\Psi_2^2\rangle_{\text{s.av.}} = \langle\Psi^2\rangle_{\text{s.av.}} ;\\
&\mu_{34} = \mu_{43} = \langle-\Psi_1\Psi_2\rangle_{\text{s.av.}} ; \qquad \mu_{13} = \mu_{31} = k\langle V_1\Psi_1\rangle_{\text{s.av.}} ;\\
&\mu_{14} = \mu_{41} = -k\langle V_1\Psi_2\rangle_{\text{s.av.}} ; \qquad \mu_{23} = \mu_{32} = k\langle V_2\Psi_1\rangle_{\text{s.av.}} ;\\
&\mu_{24} = \mu_{42} = -k\langle V_2\Psi_2\rangle_{\text{s.av.}} .
\end{aligned} \tag{2.5}$$

The characteristic function (2.4) becomes accordingly

$$F_2(z, \xi; 1, -1; t) = \exp[-\tfrac{1}{2}k^2(z^2 + \xi^2)\psi_A(0) - k^2z\xi\psi_A(t)] \cdot\exp[-D_0^2\Omega(t)]\cdot\exp\{-kD_0[z\Lambda^{(+)}(t) + \xi\Lambda^{(-)}(t)]\}, \tag{2.6}$$

in which

$$\Lambda^{(+)}(t) \equiv \Lambda(t, t_\phi) = \langle V_1\cdot(\Psi_1 - \Psi_2)\rangle_{\text{s.av.}}/D_0 ; \quad \text{and} \quad \langle V_2\cdot(\Psi_1 - \Psi_2)\rangle_{\text{s.av.}}/D_0 \equiv \Lambda^{(-)}(t) = -\Lambda(-t, t_\phi), \tag{2.6a}$$

and $\Omega(t)$ is given by[2]

$$\Omega(t) = \int_0^\infty W_{\phi M}(f)(1 - \cos\omega t)\, df, \qquad \text{or} \tag{2.7a}$$

$$= \int_0^\infty W_{FM}(f)(1 - \cos\omega t)\, df/\omega^2 \tag{2.7b}$$

for phase- or frequency-modulation respectively; $\psi_A(t)$ is the auto-correlation function of the noise wave $V_{AN}(t)$ which produces the amplitude-modulation.

There remains now to evaluate the *cross-correlation function* $\Lambda(t, t_\phi)$, relating the two modulating waves. We consider first phase-modulation. As in previous cases we express the modulating waves in terms of their Fourier transforms $S_T(f)_A$ and $S_T(f)_F$

[3] D. Middleton, Quart. Appl. Math. 5, 445 (1948).

for the amplitude and angle modulations respectively. Replacing statistical averages by their equivalent time averages, in virtue of the assumed stationarity (and ergodicity) of the random processes, we have finally

$$\Lambda(t, t_\phi) = \int_{-\infty}^{\infty} \lim_{T\to\infty} T^{-1}\{S_T(f)_A S_T(-f)_F\} \exp(it_\phi\omega)(1 - e^{-i\omega t})\, df. \tag{2.8}$$

Writing the amplitude-spectral densities $S_T(f)_A$, etc. in terms of a modulus $|S_T(f)_A|$ $(= |S_T(-f)_A|)$ and a phase $\phi(f)_A$ $(= -\phi(-f)_A)$, with similar expressions for $S_T(f)_F$, since we are dealing with a real wave, and noting furthermore that the mean spectral intensities of $V_{AN}(t)$ and $V_{FN}(t)$ are defined by

$$W_{AN}(f) \equiv \lim_{T\to\infty} 2\,|S_T(f)_A|^2 T^{-1}; \qquad W_{FN}(f) \equiv \lim_{T\to\infty} 2\,|S_T(f)_F|^2 T^{-1},$$

we can write the cross-correlation function Λ as

$$\Lambda_{\phi M}(t, t_\phi) = \int_0^\infty [W_{AN}(f)W_{\phi N}(f)]^{1/2} \cdot\{\cos(\omega t + \phi_A - \phi_F) - \cos[\omega(t_\phi - t) + \phi_A - \phi_F]\}\, df, \tag{2.9}$$

since the spectral densities are even functions of frequency, while the phase factors ϕ_F and ϕ_A are odd-functions. In the case of frequency-modulation we replace $S_T(f')_F$ by $S_T(f')_F/i\omega'$ in the above, according to Eq. (2.2b), and obtain finally the cross-correlation function

$$\Lambda_{FM}(t, t_\phi) = -\int_0^\infty [W_{AN}(f)W_{FM}(f)]^{1/2} \cdot\{\sin(\omega t_\phi + \phi_A - \phi_F) - \sin(\omega(t_\phi - t) + \phi_A - \phi_F)\}\, df/\omega. \tag{2.10}$$

The cross-correlation function $\Lambda(t, t_\phi)$ defined above in (2.6a) has a number of important properties:

(*i*) We observe at once from (2.9) or (2.10) that if the two noise waves have no spectral region in common, i.e., $W_{AN}(f)$ and $W_{FN}(f)$ do not overlap, the cross-correlation function vanishes, even though the two disturbances arise from a common source. Thus the amplitude- and frequency-modulations are no longer in any way correlated. In general, however, $W_{AN}(f)$ and $W_{FN}(f)$ may have a significant proportion of their power distributed in the same spectral region; the precise amount is measured by a quantity which we shall call the *overlap- or coherence-function* $\chi(t_\phi)$, viz.:

$$\chi(t_\phi) \equiv \langle V_1\Psi_1\rangle_{\text{s.av.}}/D_0 (\equiv \langle V_2\Psi_2\rangle_{\text{s.av.}}/D_0)$$

$$= \int_0^\infty [W_{AN}(f)W_{\phi N}(f)]^{1/2} \cos(\omega t_\phi + \phi_A - \phi_F)\, df$$

$$= \Lambda_{\phi M}(\infty, t_\phi) \tag{2.11a}$$

$$= -\int_0^\infty [W_{AN}(f)W_{FN}(f)]^{1/2} \sin(\omega t_\phi + \phi_A - \phi_F)\, df/\omega$$

$$= \Lambda_{FM}(\infty, t_\phi) + A \tag{2.11b}$$

respectively for phase- or frequency-modulation; A, as we shall see presently, cf. (2.15), is a constant depending on the behavior of both spectral densities $W_{AN}(f)$ and $W_{FN}(f)$ as $f \to 0$.

(*ii*) When t approaches zero, $\Lambda(t, t_\phi)$ vanishes identically; this is a direct consequence of the definition (2.6a) of the cross-correlation function.

(*iii*) When the time lag (or lead) t_ϕ of the amplitude-modulation with respect to the angle-modulation becomes indefinitely great (i.e., when $t_\phi \to \pm\infty$) we observe in the case of phase-modulation that since

$$\lim_{t_\phi \to \pm\infty} [\cos(\omega t_\phi + \phi_A - \phi_F) - \cos(\omega t_\phi + \phi_A - \phi_F - \omega t)] \\ = \mp 2\pi\omega \sin\frac{\omega t}{2}\,\delta(\omega - 0) = \mp\omega\left(\sin\frac{\omega t}{2}\right)\cdot\delta(f - 0), \tag{2.12}$$

we have*

$$\lim_{t_\phi \to \pm\infty} \Lambda_{\phi M}(t, t_\phi) = \mp\lim_{f \to 0}\left\{\frac{1}{2}\,[W_{AN}(f)W_{\phi N}(f)]^{1/2}\omega \sin\frac{\omega t}{2}\right\}, \tag{2.13}$$

which usually vanishes, unless the spectral densities together approach zero as $0(\omega^{-n})$, $(n \geq 2)$. For a band of noise whose zero-frequency density is finite or zero, (2.13) always vanishes. In a similar fashion one finds that in frequency-modulation

$$\lim_{t_\phi \to \pm\infty} \Lambda_{FM}(t, t_\phi) = \pm\frac{1}{2}\int_0^\infty [W_{AN}(f)W_{FN}(f)]^{1/2}[\delta(f - 0) - \delta(f - 0)]\,df = 0 \tag{2.14}$$

identically and quite independently of the spectral distributions of the modulating noise waves in the neighborhood of $f = 0$. The vanishing of the cross-correlation function in the limit of infinite delay times is explained if we note that the original wave-forms are now so separated in time that even a partial coherence between the two waves is completely lost. This is not true however when the major portion of the noise waves' energy lies in the almost zero-frequency region, cf. (2.13), as then the waveform changes so slowly in the course of time that even an infinite delay does not destroy coherence entirely. This can also be seen directly from the overlap function $\chi(t_\phi)$ when $t_\phi \to \pm\infty$.

(*iv*) A similar argument may be applied when $|t| \to \infty$ to show that

$$\lim_{t \to \pm\infty} \Lambda_{\phi M}(t, t_\phi) = \chi_{\phi M}(t_\phi); \qquad \lim_{t \to \pm\infty} \Lambda_{FM}(t, t_\phi) = \chi_{FM}(t_\phi) \mp \frac{1}{4}\,W_{AN}(0)W_{FN}(0); \tag{2.15}$$

here the time-dependent part of the cross-correlation function may or may not vanish, depending again on how much of the wave's energy is located in the spectral region $(0, \epsilon)$, as $\epsilon \to 0$.

(*v*) If the amplitude- and angle-modulations have the same spectra, i.e., $W_{AN}(f) = W_{FN}(f)$; $\phi_A = \phi_F$, a simplifying assumption that is ordinarily not at all critical, the calculation of the auto-correlation function (2.3) and the corresponding mean intensity spectrum $W(f)$ is greatly reduced, since the cross-correaltion function Λ takes a comparatively simple form. Note also that when the delay t_ϕ vanishes, one has

$$\Lambda_{\phi M}(t, 0) = \Omega_{\phi M}(t), \qquad \text{cf.} \qquad (2.7\text{a}); \qquad \psi_A(t) = \Omega_{\phi M}(\infty) - \Omega_{\phi M}(t) \tag{2.16a}$$

*A rigorous treatment recognizes these results as a form of the Dirichlet conditions.[4]

[4]H. S. Carslaw, *Theory of Fourier's Series and Integrals*, 3rd rev. ed. (Dover, N. Y., 1949), pp. 92-94.

$$\Lambda_{FM}(t, 0) = \Omega_{FM}(t), \qquad \text{cf.} \qquad (2.7\text{b}); \qquad \psi_A(t) = \ddot{\Omega}_{FM}(t). \tag{2.16b}$$

For example, consider the case of identical gaussian noise spectra, $W(f) = W_0 \exp [-\omega^2/\omega_b^2]$; one readily finds from (2.9)-(2.11) that

$$\Lambda_{\phi M}(t, t_\phi) = (W_0\omega_b/4\pi^{1/2})[\exp(-\omega_b^2 t_\phi^2/4) - \exp(-\omega_b^2(t - t_\phi)^2/4)]; \tag{2.17a}$$

$$\chi_{\phi M}(t_\phi) = (W_0\omega_b/4\pi^{1/2}) \exp(-\omega_b^2 t_\phi^2/4),$$

$$\left.\begin{aligned} \Lambda_{FM}(t, t_\phi) &= (W_0/4)[\Theta(\omega_b t_\phi/2) - \Theta(\omega_b[t_\phi - t]/2)]; \\ \chi_{FM}(t_\phi) &= (W_0/4)\Theta(\omega_b t_\phi/2), \end{aligned}\right\} \tag{2.17b}$$

where $\Theta(x) = 2\pi^{-1/2} \int_0^x \exp(-y^2)\, dy$.

The complete auto-correlation function (2.3) for the noise-modulated carrier can now be written

$$R(t) = (A_0^2/2) \exp\{-D_0^2\Omega(t)\}\, Re\Big\{\exp(-i\omega_0 t)(4\pi^2)^{-1}$$

$$\cdot \int_C z^{-2}\, dz \exp[iz - k^2\psi_A z^2/2 - kD_0\Lambda^{(+)}(t)z] \tag{2.18a}$$

$$\cdot \int_C \xi^{-2}\, d\xi \exp[i\xi - kD_0\Lambda^{(-)}(t)\xi - k^2\psi_A\xi^2/2]\cdot \exp[-k^2\psi_A(t)\xi z]\Big\}$$

$$= (A_0^2/2) \exp\{-D_0^2\Omega(t)\} \sum_{n=0}^{\infty} [(-k^2\psi_A)^n/n!]r_0(t)_A^n$$

$$\cdot \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} ([-kD_0(\psi_A\psi_F)^{1/2}]^{j+l}/j!l!)\lambda^{(+)}(t)^j\lambda^{(-)}(t)^l \tag{2.18b}$$

$$\cdot\, Re\{\exp(-i\omega_0 t)h_{0,n+j}h_{0,n+l}\},$$

where

$$h_{0,q+n} = (2\pi)^{-1} \int_C \exp(iz - k^2\psi_A z^2/2)z^{q+n-2}\, dz,$$

and (2.18c)

$$\Lambda^{(\pm)}(t)/(\psi_A\psi_F)^{1/2} \equiv \lambda^{(\pm)}(t); \qquad \psi_{A,F}(t)/\psi_{A,F} = r_0(t)_{A,F}\,.$$

The amplitude functions $h_{0,q+n}$ are given explicitly by Eqs. (2.11-2.12) of reference 1. The spectrum is found in the usual way by taking the Fourier transform of $R(t)$.

3. Remarks on power and spectra. As before, the mean total power W_T is obtained if one sets $t = 0$ in (2.18b) above, namely in the auto-correlation function $R(t)$, and the mean carrier power W_{f_0} follows from (2.18b) also, as t is allowed to become infinite. In the former case one gets a single infinite series ($n \geq 0$, $l = j = 0$), while in the latter a double series development ($n = 0$, j, $l \geq 0$), is the result. W_T and W_{f_0} may, however, be obtained in closed form, if we follow the procedure of section 2, ref. 1. We need now the joint first-order probability density $W_1(y, x)$ for the two correlated, normal random variables $y = kV_{AN}(t + t_\phi)$ and $x = D_0 V_{\phi N}(t)$ or $D_0 \int^t V_{FN}(t')\,dt'$. This may be obtained directly from the characteristic function (2.4) and (2.5) for the *second-order* distribution on setting $\zeta_2 = \zeta_4 = 0$, which gives the characteristic function $F_1(\zeta_1, \zeta_3)$.

By inversion one finds[3] that

$$W_1(y, x) = \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} F_1(\zeta_1, \zeta_3) \exp(-iy\zeta_1 - ix\zeta_3)\, d\zeta_1\, d\zeta_3$$

$$= (2\pi\Delta^{1/2})^{-1} \exp\left\{-\frac{1}{2\Delta}[D_0^2\Omega(\infty)y^2 + k^2\psi_A x^2 - 2kD_0\chi(t_\phi)xy]\right\}, \tag{3.1}$$

$$\Delta = k^2\psi_A D_0^2\Omega(\infty) - (kD_0\chi)^2.$$

Expressing $V(t)$, (2.1), now in terms of the random variables y and x, namely,

$$V(t) = A_0(1 + y)\exp(ix + i\omega_0 t), \qquad (y \geq -1);\ = 0, \qquad (y \leq -1),$$

we have

$$W_T = R(0) = \frac{1}{2}\langle | A_0(t)e^{i\Psi(t)} |^2\rangle_{s.av.}$$

$$= (A_0^2/2)\cdot\int_{y=-1}^{\infty} dy\,(1 + y)^2 \int_{-\infty}^{\infty} e^{ix-ix}W_1(y, x)\,dx \tag{3.2}$$

$$= (A_0^2/2)\int_{-1}^{\infty}(1 + y)^2 W_1(y)\,dy,$$

where $W_1(y)$ is a normal probability density for which $\bar{y} = 0$, $\overline{y^2} = \psi_A k^2$, showing that the mean total power is the same as if there were amplitude-modulation only. This is what we would expect, since angle-modulation merely redistributes the original wave's energy among the carrier and sidebands without changing the total power (see sec. 2 of ref. 2). On the other hand, the mean power W_{f_0} remaining in the carrier is influenced by the phase- or frequency-modulation, because of the coherence between modulations. We have

$$W_{f_0} = \frac{1}{2}|\langle A_0(t)e^{i\Psi(t)}\rangle_{s.av.}|^2 = (A_0^2/2)\left|\int_{-1}^{\infty}(1 + y)\,dy\int_{-\infty}^{\infty} e^{ix}W_1(y, x)\,dx\right|^2$$

$$= (A_0^2/2)\left|\int_{-1}^{\infty}(1 + y)\,dy\int_{-\infty}^{\infty}(2\pi)^{-1}F_1(\zeta_1, 1)\,d\zeta_1\right|^2 \tag{3.3a}$$

$$= (A_0^2/2)\exp[-D_0^2\Omega(\infty)]\left|\frac{1}{\pi^{1/2}}\int_{-(2k^2\psi_A)^{-1/2}}^{\infty} dz\right.$$

$$\left.\cdot(1 + z[2k^2\psi_A]^{1/2})\exp[-(z - iD_0\chi/(2\psi_A)^{1/2})^2]\right|^2, \tag{3.3b}$$

which may be integrated in straightforward fashion with the help of the MacLaurin series for $\phi^{(0)}(x \pm a)$. We find the mean carrier power to be finally

$$W_{f_0} = (A_0^2/2)\exp[-D_0^2\Omega(\infty)]\left| h_{0,0} + ikD_0\chi \mid h_{0,1} \mid + k\psi_A^{1/2}\sum_{n=2}^{\infty}\frac{1}{n!}[iD_0\chi/\psi_A^{1/2}]^n\phi^{(n-2)}([k^2\psi_A]^{-1/2})\right|^2, \tag{3.4}$$

representative values of which are shown in Fig. 3.1, along with curves for the mean continuum power $W_c = W_T - W_{f_0}$. (Typical curves for W_T are given in Fig. 2.2 of ref. 1.)

It is immediately apparent from (3.3b) that if there is no angle- or amplitude-modulation (i.e., $D_0 \to 0$, or $k \to 0$) one obtains the previous result for W_{f_0}, namely Eqs. (2.21), ref. 2 or (2.16), ref. 1, with corresponding expressions for the mean continuum power W_c. Furthermore, if there is no coherence between the modulations (so that $\chi \to 0$), the resulting power is proportional to the product of the individual carrier powers for angle- and amplitude-modulation alone, viz.:

$$W_{f0} = (A_0^2/2)h_{0,0}^2\cdot\exp[-D_0^2\Omega(\infty)], \qquad (\chi \to 0). \tag{3.5}$$

We distinguish now between phase- and frequency-modulation, according to the remarks of sec. 2, ref. 2. Figure 3.1 shows only the case of phase-modulation by noise with a gaussian spectrum for which $W_{\phi M}(0)$ is finite,* cf. Eq. (2.35), ref. 2. Increasing θ_d $(=D_0^2\langle V(t)_N^2\rangle_{\text{s-av.}})$, either by lessening the rate of sweep or by increasing the r-m-s phase deviation D_0, decreases the amount of power remaining in the carrier and makes noticeably greater the amount of energy available in the continuum. However, because of coupling with the amplitude-modulation, the scale of this effect is determined by (a), the extent of the amplitude variation $(\backsim\psi_A k^2)$, and (b), the amount of coherence $(\backsim\chi)$, which depends jointly on the time-delay t_ϕ and on the magnitude of the angle-modulation $(\backsim D_0^2\psi_F)$. For very heavy over-modulation $(k \to \infty)$ Eq. (3.4) reduces to

$$W_{f_0} \simeq (A_0^2/2)\left| [k^2\psi_A/2\pi]^{1/2}[1 + \exp(D_0^2\chi^2/2\psi_A) - (2^{1/2}D_0\chi/\psi_A^{1/2})\cdot\int_0^{D_0\chi/(2\psi_A)^{1/2}}\exp(y^2)\,dy + \frac{iD_0\chi\pi^{1/2}}{(2\psi_A)^{1/2}}\right|^2 \cdot\exp[-D_0^2\Omega(\infty)], \qquad (k\to\infty); \tag{3.6}$$

The integral in (3.6) is tabulated on page 32 of Jahnke and Emde's Tables (Dover, 1943).[5] Integrating (3.3b) by parts gives the appropriate asymptotic development when $D_0\chi/(2\psi_A)^{1/2}$ is large, for any value of $k(\geq 0)$. We have finally†

$$W_{f_0} \simeq (A_0^2/2)(k^2\psi_A^3/2\pi D_0\chi^4)\cdot\mid(1 + 2i/kD_0\chi + \cdots)\mid^2 \cdot\exp\{-1/k^2\psi_A + D_0^2[\chi^2/\psi_A - \Omega(\infty)]\}, \qquad (D_0\chi/(2\psi_A)^{1/2} \gg 1). \tag{3.7}$$

We return to the expression (2.18b) for the auto-correlation function of the modulated carrier. Applying the theorem of Wiener and Khintchine [(1.6), reference 1], we

*Since $W_{FM}(0) > 0$ for the same type of spectrum, $\Omega_{FM}(\infty)$ becomes infinite in such a way that W_{f_0} above vanishes. Consequently, all the wave's energy is distributed in the continuum; therefore $W_T = W_c$ and W_T are independent of χ and D_0.

[5]Jahnke and Emde, *Tables of Functions*, 4th rev. ed. (Dover, N. Y., 1945).

†k never appears to any lower power than k^{-1} in the series in Eq. (3.7).

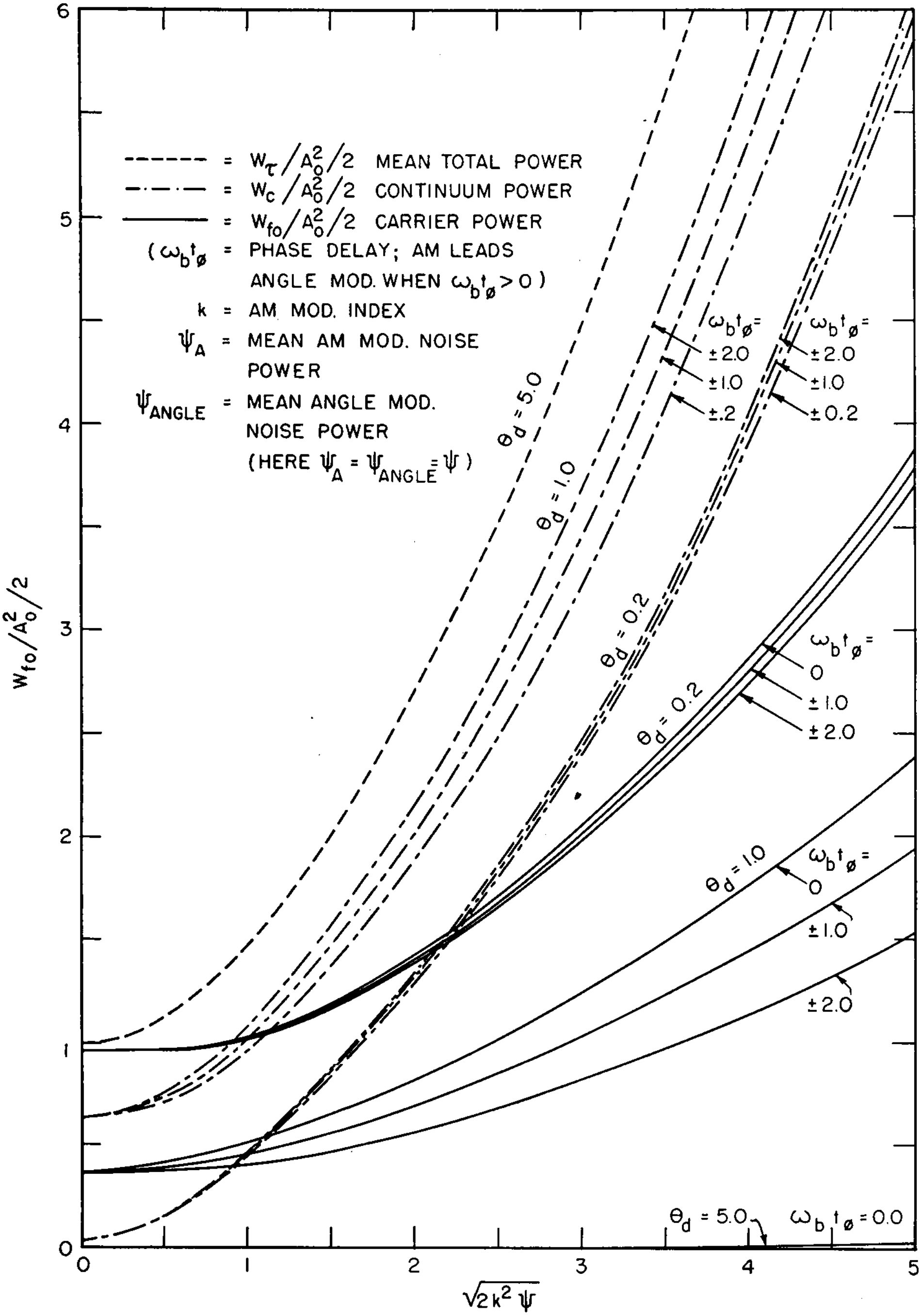


FIG. 3.1. Mean power in a carrier simultaneously amplitude and phase modulated by correlated random noise.

distinguish three general types of spectral contribution: (i) "pure" angle-modulation of the carrier by noise, i.e., (*noise* $\times$ *carrier*), or $(n \times c)$ terms, for which $(n = j = l = 0)$; (ii) "pure" angle-modulation times "pure" amplitude-modulation harmonics, namely $(n \times c)_{\text{angle}} \cdot (n \times c)_{AM}$ components, for which in turn $(n \geq 1, j = l = 0)$; and finally (iii), the cross-modulation products $(n \times c)_{\text{angle}} \times (n \times c)_{AM}$, $(n \geq 0, j + l \geq 1)$, which arise from the coherence of the simultaneous angle- and amplitude-modulations. (We use the term "pure" above to indicate the independent or incoherent character of the modulation vis-à-vis the carrier and the other modulation.) Components of types (i) and (ii) are readily seen to yield only the usual symmetrical power spectra (about f_0). Modulation products belonging to group (iii), however, are more complex: these yield both symmetrical and *anti-symmetrical* distributions. As we shall see presently, this anti-symmetry is a direct consequence of the coherence between the two noise modulations and depends critically on the amount of delay t_ϕ between them.

We observe that the terms in (2.18b) for which $j + l = \text{odd}$ $(n \geq 1)$ are responsible for the spectral asymmetry. To see this we note first that terms involving $\lambda^{(+)}(t)$ and $\lambda^{(-)}(t)$ in the auto-correlation function $R(t)$, that are odd in t, yield spectral contributions which are odd (i.e., antisymmetrical) in $(f_0 - f)$. Furthermore, such terms have as a factor $\sin \omega_0 t$, so that $R(t)$ itself remains an even function, as required from the nature of $R(t)$. Now $\lambda^{(+)}$ and $\lambda^{(-)}$ may be split into two parts, one of which is even and the other odd, in such a way that if we let $\mu^{(+)}(t)^j \mu^{(-)}(t)^l / j!l!$ represent the summand of the double series in (2.18b), including the amplitude factors $h_{0,q}$, we may write symbolically $(E_j + 0_j)^j(-E_l + 0_l)^l/j!l!$, where E and 0 are respectively the even and odd parts of $\mu^{(\pm)}(t)$. Then by direct expansion one gets for $(j + l = \text{odd})$ only odd terms, and for $(j + l = \text{even})$, only even terms. Let us illustrate with the simplest, but most important case, for which $(j + l = 1, n = 0)$, and to facilitate the discussion without in any way changing its essential features, let us further require that $D_0\psi_F^{1/2}$ be so small that we can set $\exp[-D_0^2\Omega(t)]$ equal to unity. We have as before to distinguish between the generally dissimilar situations of phase- or frequency-modulation. From (2.18b), (2.9) and (2.10), we have for the part of the auto-correlation function for which $(j + l = 1, n = 0)$

$$R(t)_{n=0;j+l=1} \doteq (A_0^2/2)kD_0(\psi_A\psi_F)^{1/2}[\lambda^{(+)}(t) + \lambda^{(-)}(t)]\, Re\{\exp(-i\omega_0 t)h_{0,0}\cdot h_{0,1}\} \tag{3.8a}$$

$$\begin{Bmatrix}\phi M\\ FM\end{Bmatrix} \qquad = A_0^2 k D_0 \sin \omega_0 t\{-ih_{0,0}\cdot h_{0,1}\}\left[\int_0^\infty [W_{AN}(f')W_{(\phi N,FN)}(f')]^{1/2}\right.$$

$$\left.\sin \omega' t \begin{Bmatrix} \sin(\omega' t_\phi + \phi_A' - \phi_F') \\ \cos(\omega' t_\phi + \phi_A' - \phi_F')/\omega' \end{Bmatrix} df'\right], \tag{3.8b}$$

respectively for phase- or frequency-modulation. The power spectrum for these cross-products becomes

$$W(f)_{(\phi M,FM)} = A_0^2 k D_0(-ih_{0,0}\cdot h_{0,1})$$

$$\cdot \int_0^\infty Q_{(\phi M,FM)}(f')[\delta(f' - [f_0 + f]) - \delta(f' - [-f_0 - f]) \tag{3.9}$$

$$+ \delta(f' - [f_0 - f]) - \delta(f' - [-f_0 + f])]\, df',$$

where

$$Q_{\phi M}(f') = [W_{AN}(f')W_{\phi N}(f')]^{1/2} \sin [\omega' t_\phi + \phi_A' - \phi_F'],$$

and (3.9a)

$$Q_{FM}(f') = [W_{AN}(f')W_{FN}(f')]^{1/2} \cos (\omega' t_\phi + \phi_A' - \phi_F')/\omega'.$$

Since both f_0 and f are positive, for obvious physical reasons, it follows immediately that the spectral density (3.9) reduces to

$$W(f) = A_0^2 k D_0(-ih_{0,0}\cdot h_{0,1}) \begin{Bmatrix} Q(f_0 - f), & f_0 \geq f \geq 0 \\ -Q(f - f_0), & f \geq f_0 > 0 \end{Bmatrix}, \tag{3.10}$$

inasmuch as the integrals over Q in (3.9) are zero for all negative values of f_0 and f, and because $Q(f_0 + f)$ is vanishingly small, Q being a *low-frequency* spectral distribution of the type $W_0 \exp(-\omega^2/\omega_b^2)$. The asymmetrical nature of the spectrum is at once evident from (3.10). Actually, the discontinuity at $f = f_0$ does not exist, but is due here to our simplifying approximation that $\exp[-D_0^2\Omega(t)] \doteq 1$. Including the exponential factor removes the discontinuity but does not alter the asymmetry; a specific example is considered in section 4(b) following. Similar remarks apply to the terms in $R(t)$, (2.18b), for which $n \geq 1$ and $l + j = \text{odd} \geq 1$.

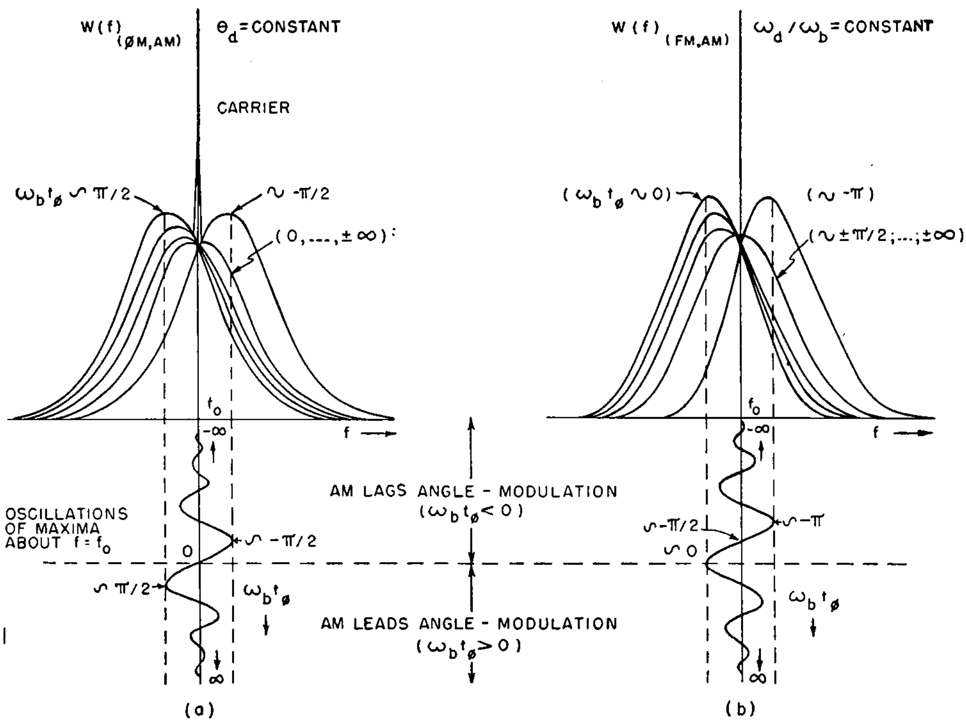


FIG. 3.2. Symmetrical and asymmetrical noise spectra for correlated noise modulation.

Figure (3.2) shows typical spectra for a fixed amount of over-modulation and for various values of delay; (a) illustrates the case of phase-modulation by noise, while (b) gives the corresponding case of frequency-modulation by the same noise wave. The noise causing the angle-modulation is here identical with that producing the amplitude-distortion of the carrier, but is delayed by t_ϕ seconds. We note at once that phase-modulation *leads* the frequency-modulation by approximately a quarter period. This is understandable if we remember that the frequency-modulation is the *integral* of the phase-modulation (for the same modulating wave), cf. (3.8), and consequently lags behind the latter in time by a delay $t_\phi \backsim \pi/2\omega_b$ on the average. In the case of simultaneous amplitude- and angle-modulation by a sine-wave alone this is exactly the delay. Now, as the delay or lead becomes progressively greater, the maxima of the spectral distributions, and therefore the distributions themselves, oscillate about f_0 in ever-decreasing swings until for infinite lags or leads all coherence between the modulations is lost, and only a symmetrical spectrum remains. The decay of the swings' maxima is quite rapid for phase-modulation and somewhat less so for frequency-modulation, cf. (2.17). The oscillations of the spectral maxima are also shown in Fig. (3.2) above; we remark, however, that values (of $\omega_b t_\phi$) at the points of symmetry ($f = f_0$) and maximum asymmetry are only approximate and do not represent a purely periodic phenomenon. The precise values depend in a complicated way on the spectral shape of the modulating noise, the amount of over-modulation, and the intensity of the angle-modulation. From this we see also that a symmetrical spectrum alone is not sufficient evidence upon which to postulate a lack of coherence between modulations. For a given level of clipping, the power (i.e. the area under the spectral curve) in the continuum is a constant, independent of the phase-differences between modulations, (but not necessarily independent of the type of angle-modulation).

As noted above, the energy spectrum may be resolved into three principal orders of modulation products (i)-(iii), viz.:

$$W(f) = W(f)_{(angle, n=0=j=l)} + W(f)_{(angle\cdot AM; n\geq 1, j=l=0)} + W(f)_{(angle\times AM; n\geq 0, j+l\geq 1)} = W(f)_{\mathrm{I}} + W(f)_{\mathrm{II}} + W(f)_{\mathrm{III}}, \tag{3.11}$$

respectively. When the deviation of the angle-modulation is very large compared to the rate of deviation, so that asymptotic developments are possible in the manner of section 2 (reference 2), we find from (2.18) that

$$W(f)_{\mathrm{I+II}} \backsimeq A_0^2 \sum_{n=0}^{\infty} \{(-k^2\psi_A)^n/n!\} h_{0,n}^2 \cdot \sum_{q=0}^{\infty} \int_0^\infty c^{(2q)}(t) r_0(t)_A^n \exp(-D_0^2\Omega^{(2)} t^2/2) \cos(\omega_0 - \omega)t \, dt, \tag{3.12}$$

where the polynomials $c^{(2q)}(t)$ yield the asymptotic development in descending powers of ω_d^2/ω_b^2 or θ_d^2; thus $c^{(2q)}(t)$ yields $(\omega_d/\omega_b)^{-2q}$ or θ_d^{-2q} multiplied by appropriate constants. Specifically we have

$$c^{(0)}(t) = 1; \qquad c^{(2)}(t) = \frac{-D_0^2\Omega^{(4)}}{4!} t^4; \qquad c^{(4)}(t) = \left(\frac{-D_0^2\Omega^{(6)}}{6!} t^6 + \frac{D_0^4\Omega^{(4)2}}{2!4!^2} t^8\right);$$

$$c^{(6)}(t) = \left(\frac{-D_0^2\Omega^{(8)}}{8!} t^8 + \frac{2}{2!6!4!} D_0^4\Omega^{(4)}\Omega^{(6)} t^{10} - \frac{D_0^6\Omega^{(4)3}}{3!4!^3} t^{12}\right);$$

$$\left.\begin{aligned} c^{(8)}(t) = \Big(\frac{-D_0^2\Omega^{(10)}}{10!}\, t^{10} + \frac{D_0^4}{2!}\left[\frac{\Omega^{(6)2}}{6!^2} + \frac{2\Omega^{(4)}\Omega^{(8)}}{8!4!}\right]t^{12} \\ - \frac{3D_0^6\Omega^{(4)2}\Omega^{(6)}}{3!4!^2 6!}\, t^{14} + \frac{D_0^8\Omega^{(4)4}}{4!\cdot 4!^4}\, t^{16}\Big); \qquad \text{etc.} \end{aligned}\right\} \tag{3.13}$$

where

$$\Omega_{FM}^{(2n)} = (-1)^{n-1} b_{2n-2}\,; \qquad n \geq 1; \qquad \Omega_{\phi M}^{(2n)} = b_{2n}(-1)^{n-1}; \qquad n \geq 0;$$

$$b_n = \int_0^\infty \omega^2 W(f)_{(FM,\phi M)}\, df.$$

The spectrum of the cross-modulation is more complicated, but can be handled in a similar fashion. One has then to consider terms of the following type:

$$(-1)\, Re\Big\{h_{0,n+j}\cdot h_{0,n+l} \int_0^\infty r_0(t)^n c^{(2q)}(t)\Lambda(t,\, t_\phi)^j\,\Lambda(-t,\, t_\phi)^l \cdot \exp\,(-D_0^2\Omega^{(2)}t^2/2 - i\omega_0 t)\,\cos\omega t\, dt\Big\}.$$

On the other hand, if the amplitude-modulation is excessive, so that over-modulation occurs approximately half the time, the asymmetrical components are suppressed, and one has essentially a symmetrical spectrum, which nevertheless is influenced by the coherence between the angle- and amplitude-modulations. The analytical explanation is immediate: only $h_{0,n+q}$ ($n + q =$ even) does not vanish when $(2k^2\psi_A)^{1/2} \to \infty$, except $h_{0,1}$, which becomes independent of $(2k^2\psi_A)^{1/2}$, cf. Eqs. (2.12a-c), ref. 1. Consequently, $j + l$ is always even, and only symmetrical terms remain. Since these components are all of order $(k^2\psi_A)$, the contribution of the single remaining antisymmetrical pair ($j + l = 1$, $n = 0$), which are $0(k\psi_A^{1/2})$, becomes ignorable in the limit $(2k^2\psi_A)^{1/2} \to \infty$. Physically, it is the amplitude distortion which effectively determines the "cut-off" and "cut-in" of the oscillator. The angle-modulation alone does not alter the amplitude, changing only the spacing between zero-crossings, and therefore one expects that the strong amplitude-modulation controls the duration of the "on"- and "off"-periods of the wave.

4. Coherent and incoherent modulations. Two important cases remain to be considered in some detail: *one*, the situation of no coherence ($\Lambda = 0$) between modulations, and *two*, the case of coherence, but no over-modulation. A discussion of *one* and *two* follows below in parts (a) and (b).

(a) *No coherence*:

The general problem simplifies greatly when there is no coherence or coupling between the modulating waves, a condition represented analytically by the fact that $\Lambda(t,\, t_\phi)$ vanishes, and physically, by the complete independence of the noise waves. The auto-correlation function (2.18) reduces now to

$$\begin{aligned} R(t) &= r_0(t)_{angle}\cdot R_0(t)_{AM}\, \cos\omega_0 t \\ &= (A_0^2/2)\, \exp\,[-D_0^2\Omega(t)] \sum_{n=0}^\infty \{(-k^2\psi_A)^n/n!\}\, r_0(t)_A^n h_{0,n}^2\, \cos\omega_0 t, \end{aligned} \tag{4.1}$$

and the intensity spectrum $W(f)$ $(=W(f)_{\mathrm{I}} + W(f)_{\mathrm{II}})$, cf. Eq. (3.11), contains no cross terms of the type $W(f)_{\mathrm{III}}$. The mean total power in the wave is given by Eq. (2.17), reference 1, as before; the power in the carrier and continuum is also easily found from (3.4).

Let us consider the specific example of phase-modulation, and let us assume a gaussian spectral distribution for the modulating noise; (in general, the two noise waves need not have identical spectra or powers.) The spectrum follows at once from the cosine Fourier transform of Eq. (4.1) and we obtain finally with the help of (c), section 2, reference 2,

$$W(f) = \frac{A_0^2\pi^{1/2}\exp(-\theta_d^2)}{\omega_b}\left\{h_{0,0}^2\,\delta(f - f_0) + h_{0,0}^2\sum_{q=1}^{\infty}\frac{\theta_d^{2q}\exp(-\beta^2/q)}{q!q^{1/2}}\right.$$
$$\left.+\sum_{q=0}^{\infty}\sum_{n=1}^{\infty}\frac{\theta_d^{2q}}{q!}\frac{(-k^2\psi_A)^n}{n!}h_{0,n}^2\frac{\exp(-\beta^2/(q + n\omega_A^2/\omega_b^2))}{(q + n\omega_A^2/\omega_b^2)^{1/2}}\right\},\qquad \beta = \frac{\omega_0 - \omega}{\omega_b}. \tag{4.2}$$

The first term represents the carrier's contribution, the second, the spectrum due to phase-modulation alone, and the final term, the distribution $W(f)_{\mathrm{II}}$, attributable to the noncoherent modulation products between the angle- and amplitude-modulations. When excessive over-modulation occurs, we find that

$$W(f) \backsimeq A_0^2\frac{k^2\psi_A\exp(-\theta_d^2)}{2\pi^{1/2}\omega_b}\left\{\delta(f - f_0) + \sum_{q=1}^{\infty}\frac{\theta_d^{2q}}{q!}\left(\left(\frac{\exp(-\beta^2/q)}{q^{1/2}}\right.\right.\right.$$
$$\left.+\frac{\pi\exp[-\beta^2/(q + \omega_A^2/\omega_b^2)]}{2(q + \omega_A^2/\omega_b^2)^{1/2}}\right)$$
$$+\sum_{n=0}^{\infty}\left(\frac{(2n)!\exp[-\beta^2/(q + (n + 2)\omega_A^2/\omega_b^2)]}{n!^22^{2n}(2n + 2)(2n + 1)[q + (n + 2)\omega_A^2/\omega_b^2]^{1/2}}\right.$$
$$\left.\left.\left.+\frac{(2n)!\exp(-\beta^2/(n + 2)\omega_A^2/\omega_b^2)}{n!^2 2^{2n}(2n + 2)(2n + 1)[(n + 2)\omega_A^2/\omega_b^2]^{1/2}}\right)\right)\right\},\qquad k^2\to\infty, \tag{4.3}$$

and when there is ignorable over-modulation we have simply

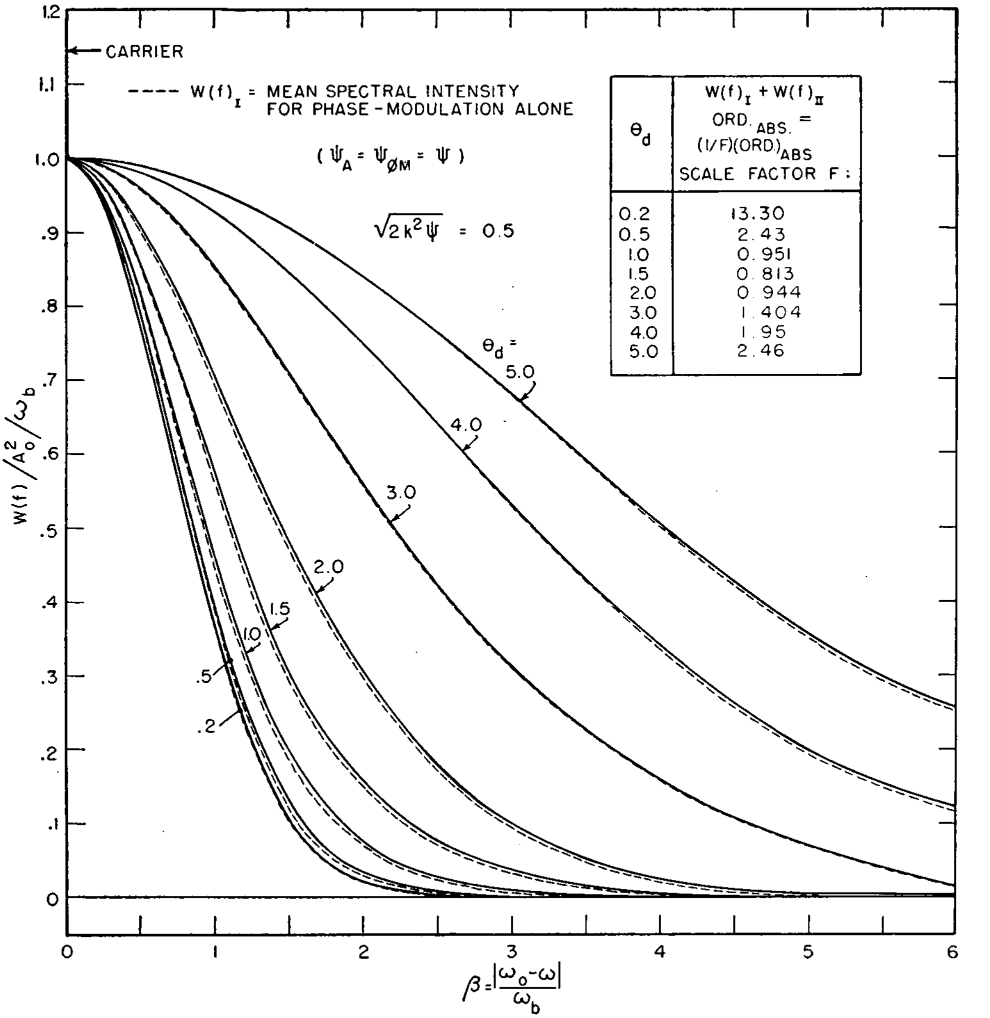
$$W(f) \doteq A_0^2\frac{\pi^{1/2}\exp(-\theta_d^2)}{\omega_b}\left\{\delta(f - f_0) + \sum_{q=1}^{\infty}\frac{\theta_d^{2q}}{q!q^{1/2}}\exp(-\beta^2/q)\right.$$
$$\left.+k^2\psi_A\sum_{q=0}^{\infty}\frac{\theta_d^{2q}\exp[-\beta^2/(q + \omega_A^2/\omega_b^2)]}{q!(q + \omega_A^2/\omega_b^2)^{1/2}}\right\},\qquad (2k^2\psi_A)^{1/2} \le 0.6. \tag{4.4}$$

Figure (4.1) shows the energy spectrum for $(2k^2\psi)^{1/2} = 0.5$ and for various values of the deviation ratio θ_d. A general, but very slight broadening of the spectrum results from the additional modulation. Similar remarks apply for frequency-modulation (detailed calculations for which may be carried out along the lines of section 2(b), reference 2). In general, because the spread in the spectrum due to clipping is small, even in the extreme case of 50 per cent over-modulation, as can be seen from Fig. (2.1), reference 1, we expect little additional broadening of the spectrum due to this effect. For large values of θ_d^2 or ω_d^2/ω_b^2 the spectral intensity $W(f)$ is obtained in asymptotic form from

(3.12) and (3.13); in the instance of a gaussian spectrum for the amplitude-modulation ($\backsim W_A \exp(-\omega^2/\omega_A^2)$) we have

$$W(f) \backsimeq \sum_{n=0}^{\infty} \frac{(-k^2\psi_A)^n}{n!} h_{0,n}^2 \; \{\text{Eq. (2.33), ref. 2, modified}\}, \tag{4.5}$$

where now Eq. (2.33), reference 2 is easily modified by first multiplying numerator and denominator of each term which contains a factor $\phi^{(2q)}$ by a suitable power of D_0^2, so



θ_d	$W(f)_I + W(f)_{II}$ ORD. ABS. = (1/F)(ORD.)ABS SCALE FACTOR F:
0.2	13.30
0.5	2.43
1.0	0.951
1.5	0.813
2.0	0.944
3.0	1.404
4.0	1.95
5.0	2.46

Fig. 4.1. Mean intensity spectrum for simultaneous uncorrelated amplitude and phase-modulation of a carrier by noise (no over-modulation).

that $[D_0^2\Omega^{(2)}]^q$ results in the denominator, and then replacing $[D_0^2\Omega^{(2)}]^q$ so obtained by $[D_0^2\Omega^{(2)} + \omega_A^2 n/2]^q$; the argument of the $\phi^{(2q)}$ is similarly changed from β'' to $\beta''/(1 + \omega_A^2 n/2D_0^2\Omega^{(2)})^{1/2}$.

(b) *Coherence, but no over-modulation*

This is a particularly important case because it is the simplest one which incorporates all the new features of coherence between modulations, and because it offers a promising macroscopic model for the behavior of noise in a magnetron, as described at the beginning of this section.

When over-modulation effects are ignorable (<1 per cent of the time), only the amplitude functions $h_{0,0}$ ($\doteq 1$) and $h_{0,1}$ ($\doteq i$) yield significant contributions, and we obtain from Eq. (2.18b) the following spectral components: (noise $\times$ carrier)$_{\text{angle mod.}}$, for which ($n = j = l = 0$); $(n \times c)_{\text{angle}} \cdot (n \times c)_{AM}$, for which ($n = 1, j = l = 0$); and finally $(n \times c)_{\text{angle}} \times (n \times c)_{AM}$, where ($n = 0, l + j = 1$), all of which are represented in (4.6) below in the order listed:

$$\begin{aligned} R(t) \doteq (A_0^2/2) \exp\,[-D_0^2\Omega(t)]\Big\{&\cos\omega_0 t + k^2\psi_A r_0(t)_A \cos\omega_0 t \\ &+ 2kD_0\Big(\int_0^\infty Q(f')\sin\omega' t\,df'\Big)\sin\omega_0 t\Big\}, \qquad (2k^2\psi_A)^{1/2} \le 0.6. \end{aligned} \tag{4.6}$$

Here $Q(f')$ is given by Eq. (3.9a). We observe at once that the mean total power is the same as if there were no angle-modulation at all, which is not surprising, as angle-modulation alone does not change the energy content of a modulated carrier. Note also that the cross-correlation term never contributes to the total power (by definition, cf. (2.6a)), but represents a redistribution of the existing energy, determined by the carrier strength and degree of amplitude-modulation. The mean power in the carrier is the coefficient of $\cos\omega_0 t$ or $\sin\omega_0 t$ as t becomes infinite:

$$W_{f_0} = (A_0^2/2)\exp\,[-D_0^2\Omega(\infty)]\{1 + (kD_0/2)\lim_{f'\to 0}\,[\omega' Q(f')]\}, \tag{4.7}$$

showing that W_{f_0} is determined essentially by the type of angle-modulation. The second term of (4.7) vanishes for the usual types of phase-modulation, while for frequency-modulation the exponential factor vanishes, as explained earlier in section 2, reference 2, with the result that W_{f_0} is here described by the previous expression (2.21), reference 2. The continuum power is

$$W_c = (A_0^2/2)(1 + k^2\psi_A - \exp\,[-D_0^2\Omega(\infty)]). \tag{4.8}$$

Let us consider specifically the example of phase-modulation when the modulating noise waves are identical, so that

$$\begin{aligned} W_{AN}(f) = W_{\phi N}(f) = W_0\exp(-\omega^2/\omega_b^2); \qquad &\psi_A = \psi_F = W_0\omega_b/4\pi^{1/2} = \psi; \\ \therefore\ \phi_A = \phi_F\,; \qquad &r_0(t) = \exp\,[-\omega_b^2 t^2/4]. \end{aligned} \tag{4.9}$$

To obtain the desired spectrum we need the following integrals:

$$\begin{aligned} &\int_{-\infty}^{\infty}\begin{Bmatrix}\sin at\\ \cos at\end{Bmatrix}\exp(-b^2t^2 \pm ct)\,dt \\ &\qquad = (\pi^{1/2}/b)\exp\,[(c^2 - a^2)/4b^2]\begin{Bmatrix}\sin(\pm ac/2b^2)\\ \cos(\pm ac/2b^2)\end{Bmatrix}, \qquad Re\,(|\,b\,|) > 0. \end{aligned} \tag{4.10}$$

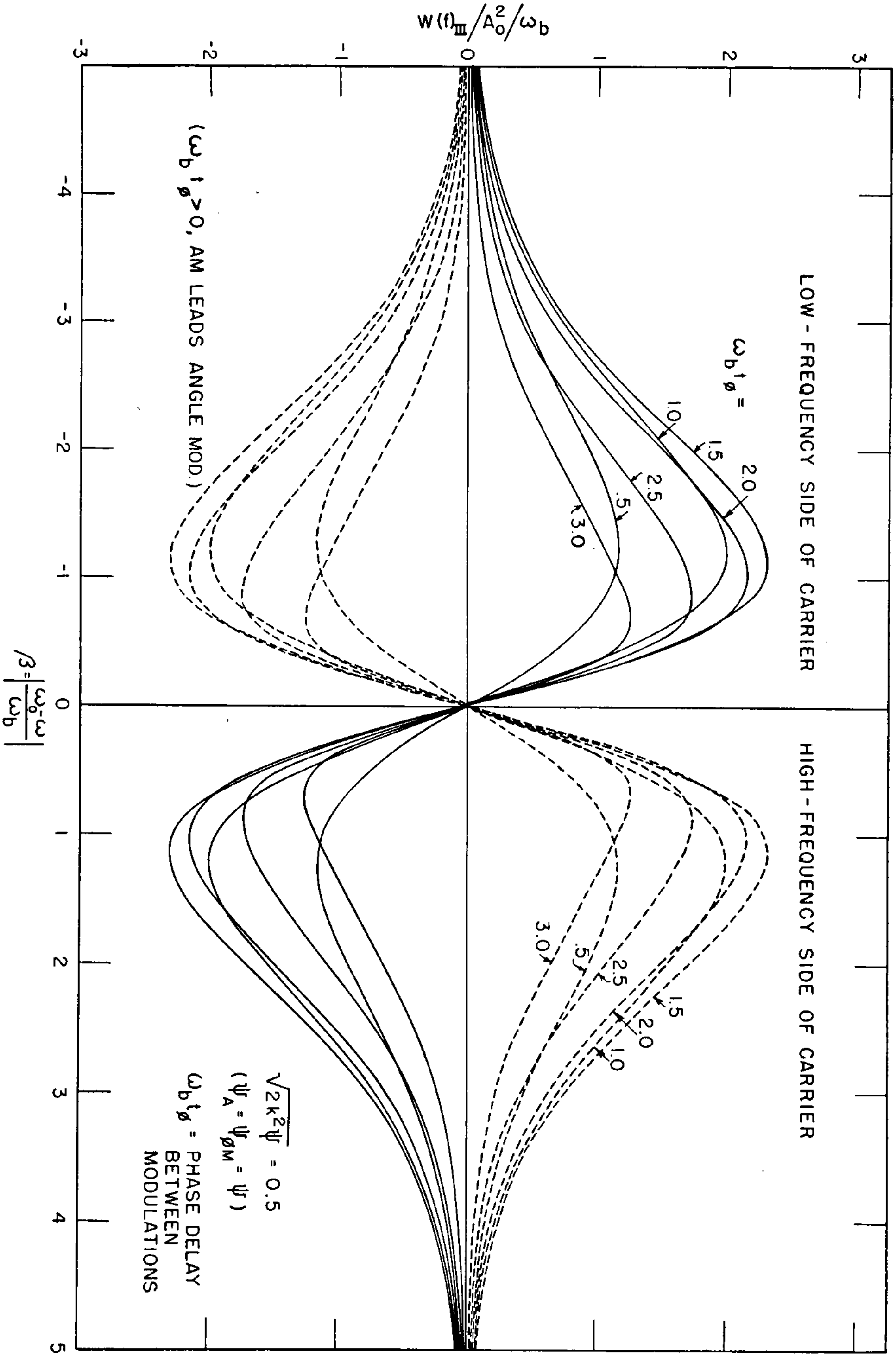


FIG. 4.2. Cross-spectral intensity of a carrier simultaneously amplitude and phase modulated by noise (no over-modulation).

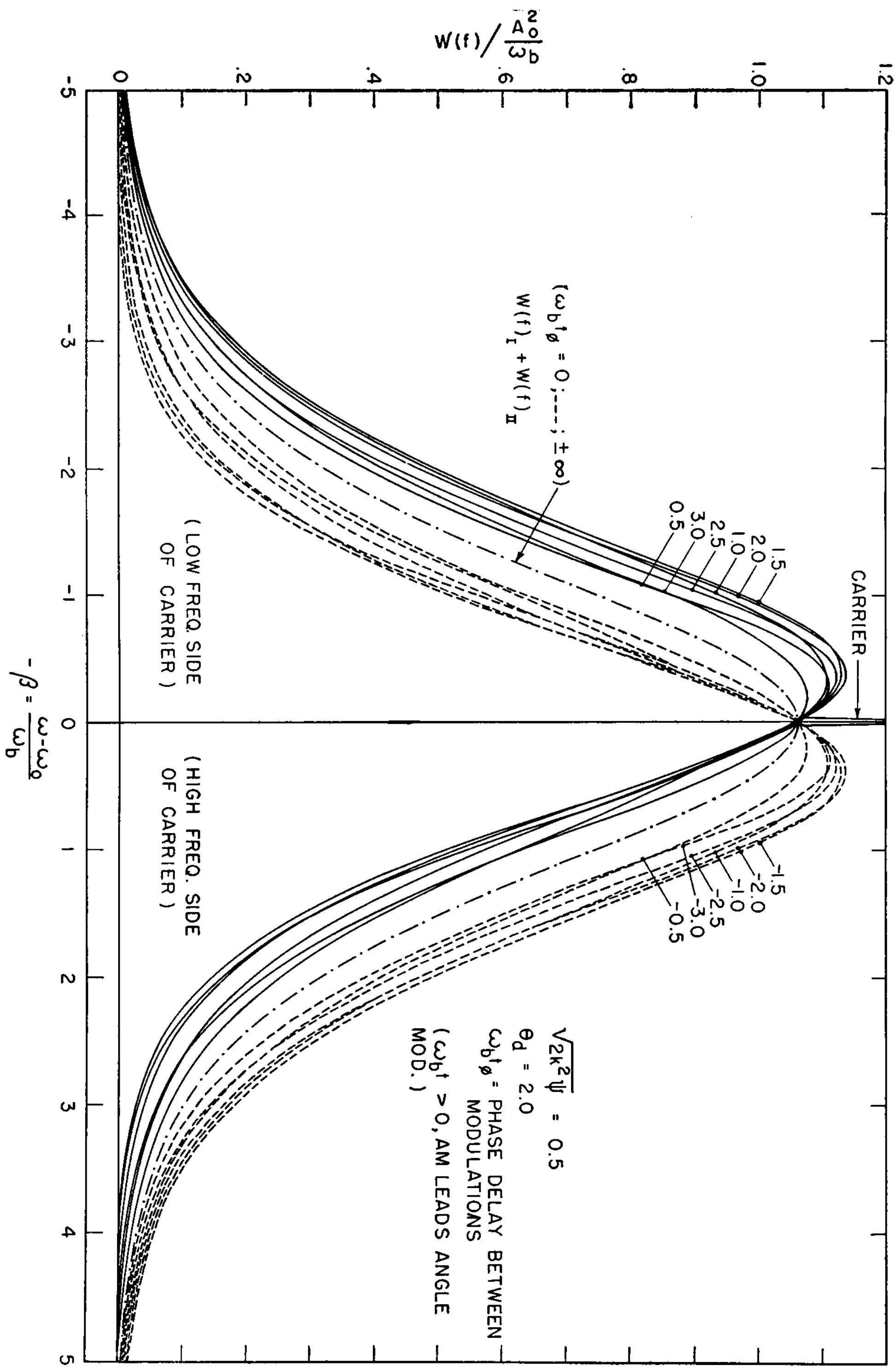


FIG. 4.3. Mean spectral intensity of a carrier simultaneously amplitude and phase modulated by correlated random noise (no over-modulation, θ_d constant, various delays).

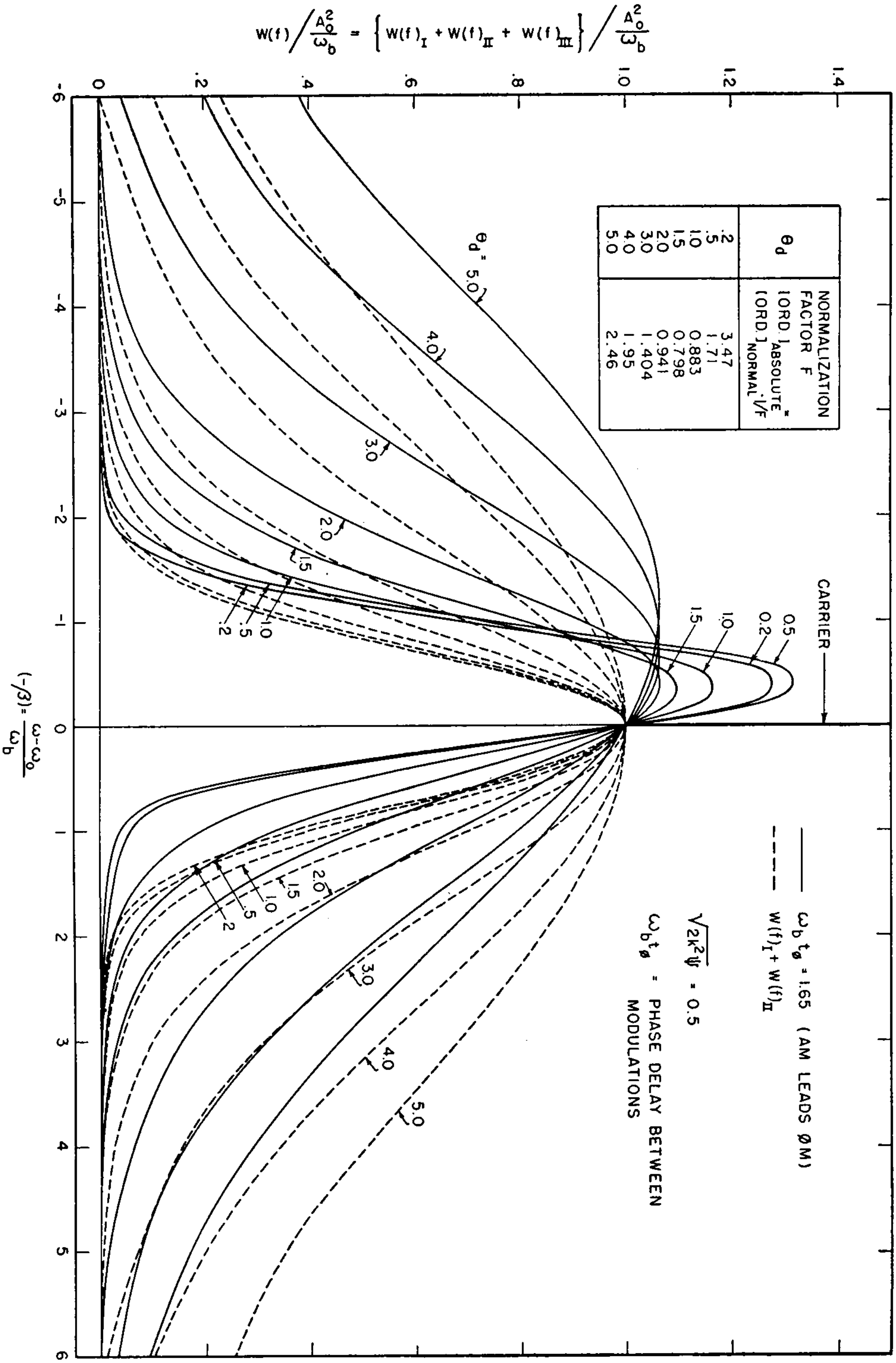


Fig. 4.4. Mean spectral intensity of a carrier simultaneously amplitude and phase modulated by correlated random noise (no over-modulation, t_ϕ constant, various degrees of modulation).

Application of the Wiener-Khintchine theorem yields

$$W(f) = W(f)_{\phi M} + W(f)_{\phi M \cdot AM} + W(f)_{\phi M \times AM}\,, \tag{4.11}$$

where $W(f)_{\phi M} = W(f)_{\text{I}}$ is given by Eq. (2.48), reference 2, $W(f)_{\phi M \cdot AM} = W(f)_{\text{II}}$ is obtained directly from the last term of (4.4) when $\omega_A = \omega_b$, and

$$\begin{aligned} W(f)_{\phi M \times AM} &= W(f)_{\text{III}} = \theta_d (k^2 \psi)^{1/2} \exp(-\theta_d^2)(A_0^2 \pi^{1/2}/\omega_b) \\ &\cdot \sum_{n=0}^{\infty} \frac{\theta_d^{2n}}{n!(n+1)^{1/2}} \exp[-\beta^2/(n+1)] \left\{ 2 \sin\left(\frac{\omega_b t_\phi \beta}{n+1}\right) \exp[-n(\omega_b t_\phi)^2/4(n+1)] \right\}, \end{aligned} \tag{4.12}$$

$$\beta = (\omega_0 - \omega)/\omega_b\,.$$

Typical spectra are shown in Figs. (4.2)-(4.4) for a variety of time-delays and for various values of θ_d. The spectra are in general asymmetrical, in the manner of Fig. (3.2); the spectral spread increases with the larger values of θ_d. We remark that, unlike the case of no coherence, it is not possible here to speak of a phase-modulating noise equivalent to a frequency-modulating wave (in the sense of Eq. (2.19), reference 2) which produces the same final spectrum, even if we assume the same disturbance causes both amplitude- and angle-modulations. The explanation lies in the entirely different types of modulation, which are coupled together in such a way that altering one does not produce a compensating change in the other. When the angle-modulation is sufficiently intense (i.e., high modulation indices), we obtain from (3.12) and (3.13) the needed asymptotic representation of the spectrum, namely

$$\begin{aligned} W(f) \simeq A_0^2 \sum_{q=0}^{\infty} \int_0^\infty dt \Big\{ &(1 + k^2 \psi_A r_0(t)_A) \cos(\omega_0 - \omega)t + 2kD_0 \sin(\omega_0 - \omega)t \\ &\cdot \int_0^\infty Q(f', t_\phi) \sin \omega' t \, df' \Big\} c^{(2q)}(t) \exp(-D_0^2 \Omega^{(2q)} t^2/2). \end{aligned} \tag{4.13}$$

The precise forms of $\Omega^{(2)}$ etc. that appear in the exponent and in the polynomials $c^{(2)}(t)$, Eq. (3.13), are described in Eq. (2.34), reference 2. $W(f)_{\text{I}}$ and $W(f)_{\text{II}}$ follow from sec. 2, reference 2, and the above $W(f)_{\text{III}}$ has the following general form

$$W(f)_{\text{III}} \simeq -\pi k D_0 A_0^2 \sum_{m=0}^{\infty} \frac{Q_{2m+1}(t_\phi) \cdot G^{(2m+1)}(\beta)}{(2m+1)![D_0^2 \Omega^{(2)}]^{m+1}}, \tag{4.14}$$

where

$$Q_{2m+1}(t_\phi) \equiv \int_{-\infty}^{\infty} Q(f', t_\phi) \omega'^{2m+1} \, df',$$

and $G^{(2m+1)}(\beta)$ is the $(2m + 1)$st derivative with respect to β of the expression in the parentheses of Eq. (2.33), reference 2. For phase- or frequency-modulation by noise with a gaussian spectrum one easily shows that when the amplitude- and angle-modulations are identical

$$\begin{aligned} W(f)_{\text{III}} \Big|_{\phi M} &\simeq (2\pi)^{3/2} (2k^2 \psi)^{1/2} \frac{A_0^2}{\omega_b \theta_d} \sum_{m=0}^{\infty} \frac{(-1)^m \phi^{(2m+1)}(\omega_b t_\phi / 2^{1/2})}{(2m+1)! \theta_d^{2m}} G^{(2m+1)}(\beta'') \\ &\theta_d^2 \gg 1, \qquad \beta'' = \left(\frac{\omega_0 - \omega}{\omega_b}\right) \frac{2^{1/2}}{\theta_d} \end{aligned} \tag{4.15}$$

or

$$W(f)_{\mathrm{III}}\Big|_{FM} \backsimeq -\pi(2\pi)^{1/2}\frac{A_0^2}{\omega_d}[2(2k^2\psi)]^{1/2}\sum_{m=0}^{\infty}\frac{(-1)^m}{2^m}\left(\frac{\omega_b}{\omega_d}\right)^{2m}\frac{\phi^{(2m)}(\omega_b t_\phi/2^{1/2})}{(2m+1)!}$$
$$\cdot\, G^{(2m+1)}(\beta'); \qquad (\omega_b/\omega_d)^2 \ll 1, \qquad \beta' = \left(\frac{\omega_0-\omega}{\omega_b}\right)\left(\frac{\omega_b}{\omega_d}\right). \tag{4.16}$$

We observe that if the amplitude-modulation is particularly weak, the (angle $\times$ AM), or $W(f)_{\mathrm{III}}$, components are strong compared to the incoherent terms $W(f)_{\mathrm{II}}$, since the latter are $0(k^2\psi_A)$, whereas the former appear as $0(k^2\psi_A)^{1/2}$. The "pure" angle-modulation terms $W(f)_{\mathrm{I}}$, however, remains unaffected.

5. Remarks on signal and noise modulations. When a signal accompanies the noise there are a number of interesting possibilities: (i) amplitude-modulation by signal and noise and angle-modulation by noise, (ii), amplitude-modulation by noise and angle-modulation by a signal and noise, and (iii), angle- and amplitude-modulations by signal and noise; in each case the noise waves and the signal waves are respectively coherent, but there is no coupling between signal and noise. We shall outline the principal steps in the process of obtaining the spectrum, but without giving a detailed treatment, inasmuch as the more complex problems of signal and noise in this case introduce no new concepts or techniques not already considered in the preceding sections.

Since (iii) contains (i) and (ii) as special cases, let us examine the more general case first. The modulated carrier is now

$$g(y) = V(t) = A_0(t)_{S+N}\exp\left[i(\omega_0 t + \Psi(t)_N + \Psi(t)_S)\right] \tag{5.1}$$

where $A_0(t)_{S+N}$ is an obvious generalization of (2.2a), viz.:

$$A_0(t)_{S+N} = (-A_0/2\pi)\int_C z^{-2}\,dz\,\exp\{iz[1 + kV_{AN}(t+t_\phi) + \mu V_S(t+t_\phi)]\} \tag{5.2}$$

and $\Psi(t)_N$, $\Psi(t)_S$ are specified by [(2.1), (2.2), (3.1b), ref. 2], or by (2.2). The autocorrelation function (2.3) becomes

$$R(t) = (A_0^2/2)\,Re\Big\{e^{-i\omega_0 t}(4\pi^2)^{-1}\int_C z^{-2}e^{iz}\,dz$$
$$\cdot\int_C \xi^{-2}e^{i\xi}\,d\xi\,[\exp\{ikV_{N1}z + ikV_{N2}\xi + i\Psi_{N1} - i\Psi_{N2}\}]_{\mathrm{stat.av.}} \tag{5.3}$$
$$\cdot\,[\exp\{i\mu V_{S1}z + i\mu V_{S2}\xi + i\Psi_{S1} - i\Psi_{S2}\}]_{\mathrm{stat.av.}}\Big\}$$

where the subscripts 1 and 2 refer to the wave at an initial and final time, t seconds apart. The first ensemble average has already been found, namely Eq. (2.6), and the second, which is the characteristic function for the joint signal modulations may be expressed as*

*It is assumed that the fundamental frequencies of the two signals are here commensurable; otherwise there will be no coherence between signals, and the characteristic function then factors into the product of two characteristic functions $F_2(z, \xi; t)_{S,AM}\cdot F_2(z, \xi, t)_{S,\mathrm{angle}\ M}$, representing the (now) independent amplitude- and angle-modulations.

$$F_2(z, \xi; 1, -1; t)_S$$

$$= T_0^{-1} \int_0^{T_0} \exp\,[i\mu V_S(t_0 + t_\phi)z + i\mu V_S(t_0 + t_\phi + t)\xi + i\Psi(t_0)_S - i\Psi(t_0 + t)_S]\, dt_0 \quad (5.4)$$

$$= \sum_{m=0}^{\infty} \epsilon_m(-1)^m C_m(z) C_m(\xi) \cos m\omega_a t,$$

since F_2 is an even function of t. The auto-correlation function (2.18b) may still be used, provided that we replace the amplitude functions $h_{0,n+q}$ by $h_{m,n+q}$ according to (3.7) of ref. 1, with $C_m(z)$ in place of $B_m(z)$ therein, insert $(-1)^m\epsilon_m \cos m\omega_a t$ as an additional trigonometric factor, and perform the summation over all values of m. The precise form of $C_m(z)$ depends, of course, on the wave shape of the modulating signals, and except for the simple sinusoidal cases, are very complicated functions. In general, the spectrum will be asymmetrical in the discrete or signal portions of the distribution, as well as in the continuum.

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