# ON THE DISTRIBUTION OF ORBITS OF GEOMETRICALLY FINITE HYPERBOLIC GROUPS ON THE BOUNDARY (WITH APPENDIX BY FRANÇOIS MAUCOURANT) 

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#### Abstract

We investigate the distribution of orbits of a non-elementary discrete hyperbolic subgroup $\Gamma$ acting on $\mathbb{H}^{n}$ and its geometric boundary $\partial_{\infty}\left(\mathbb{H}^{n}\right)$. In particular, we show that if $\Gamma$ admits a finite Bowen-Margulis-Sullivan measure (for instance, if $\Gamma$ is geometrically finite), then every $\Gamma$-orbit in $\partial_{\infty}\left(\mathbb{H}^{n}\right)$ is equidistributed with respect to the Patterson-Sullivan measure supported on the limit set $\Lambda(\Gamma)$. The appendix by Maucourant is the extension of a part of his thesis where he obtains the same result as a simple application of Roblin's theorem.

Our approach is via establishing the equidistribution of solvable flows on the unit tangent bundle of $\Gamma \backslash \mathbb{H}^{n}$, which is of independent interest.


## 1. Introduction

Let $G$ be the group of orientation preserving isometries of the hyperbolic space $\mathbb{H}^{n}$ and $\Gamma<G$ a torsion-free non-elementary ( $=$ not virtually abelian) discrete subgroup. The action of $\Gamma$ extends to $\overline{\mathbb{H}^{n}}:=\mathbb{H}^{n} \cup \partial_{\infty}\left(\mathbb{H}^{n}\right)$ where $\partial_{\infty}\left(\mathbb{H}^{n}\right)$ denotes the geometric boundary of $\mathbb{H}^{n}$, and we define the limit set $\Lambda(\Gamma)$ as the set of accumulation points of a $\Gamma$-orbit in $\overline{\mathbb{H}^{n}}$.

If we denote by $\delta_{\Gamma}$ the critical exponent of $\Gamma$, then there exists a $\Gamma$-invariant conformal density $\left\{\nu_{x}: x \in \mathbb{H}^{n}\right\}$ of dimension $\delta_{\Gamma}$ on $\Lambda(\Gamma)$ by Patterson [14] for $n=2$ and Sullivan [19] for $n$ general. We consider the Bowen-Margulis-Sullivan measure $m_{\Gamma}^{\mathrm{BMS}}$ on the unit tangent bundle $\mathrm{T}^{1}\left(\Gamma \backslash \mathbb{H}^{n}\right)$ associated to the density $\left\{\nu_{x}\right\}$ (Def. 2.1). When the total mass $\left|m_{\Gamma}^{\mathrm{BMS}}\right|$ finite, the geodesic flow is ergodic on $\mathrm{T}^{1}\left(\Gamma \backslash \mathbb{H}^{n}\right) \quad 19$.

For a subset $\Omega \subset \partial_{\infty}\left(\mathbb{H}^{n}\right)$ and $x \in \mathbb{H}^{n}$, we denote by $S_{x}(\Omega) \subset \mathbb{H}^{n}$ the set of all points lying in geodesics emanating from $x$ toward $\Omega$, and by $B_{T}(x) \subset \mathbb{H}^{n}$ the hyperbolic ball of radius $T$ centered at $x$.

Our main theorem is the following:
Theorem 1.1. Suppose that the total mass $\left|m_{\Gamma}^{\mathrm{BMS}}\right|$ is finite. Let $\Omega_{1}$ and $\Omega_{2}$ be Borel subsets of $\partial_{\infty}\left(\mathbb{H}^{n}\right)$ whose boundaries are of zero Patterson-Sullivan measure. Then, for any $x, y \in \mathbb{H}^{n}$ and $\xi \in \partial_{\infty}\left(\mathbb{H}^{n}\right)$, as $T \rightarrow \infty$,

$$
\#\left\{\gamma^{-1}(y) \in S_{x}\left(\Omega_{1}\right) \cap B_{T}(x): \quad \gamma(\xi) \in \Omega_{2}\right\} \sim \frac{\nu_{x}\left(\Omega_{1}\right) \nu_{y}\left(\Omega_{2}\right)}{\delta_{\Gamma} \cdot\left|m_{\Gamma}^{\mathrm{BMS}}\right|} \cdot e^{\delta_{\Gamma} T}
$$

[^0]

Figure 1. Orbits of $\Gamma$ on $\mathbb{H}^{n} \times \partial_{\infty}\left(\mathbb{H}^{n}\right)$

If $\Gamma$ is geometrically finite, that is, if the unit neighborhood of the convex cor $\rrbracket^{1} \mathcal{C}_{\Gamma}$ has finite volume, then $\left|m_{\Gamma}^{\mathrm{BMS}}\right|<\infty$ [20]. However the above theorem is not restricted only to those groups as there are geometrically infinite groups with $\left|m_{\Gamma}^{\mathrm{BMS}}\right|<\infty($ see [15]).

We remark that the assumption of $\left|m_{\Gamma}^{\mathrm{BMS}}\right|<\infty$ implies that the conformal density $\left\{\nu_{x}\right\}$ is determined uniquely up to homothety (see [17, Coro.1.8]).

When $\Omega_{1}=\Omega_{2}=\partial_{\infty}\left(\mathbb{H}^{n}\right)$, the above counting problem is simply the nonEuclidean lattice point counting problem, and was solved by Lax and Phillips 10 for geometrically finite groups with $\delta_{\Gamma}>(n-1) / 2$. Theorem 1.1 for $\Omega_{2}=\partial_{\infty}\left(\mathbb{H}^{n}\right)$ is due to Roblin [17. When $\Gamma$ is a lattice, the same type of orbital counting result for $\Omega_{2}=\partial_{\infty}\left(\mathbb{H}^{n}\right)$ was obtained in a much more general setting of Riemannian symmetric spaces (see [11], [2], [5], 6], etc.). Theorem 1.1 for general $\Omega_{1}, \Omega_{2}$ was proved in [7] for all lattices in semisimple Lie groups (see also [9] for the case when $\left.\Omega_{1}=\partial_{\infty}\left(\mathbb{H}^{n}\right)\right)$.

We highlight Theorem 1.1 for the Möbius transformation action of $\mathrm{PSL}_{2}(\mathbb{C})$, that is, the action on the extended complex plane $\widehat{\mathbb{C}}=\mathbb{C} \cup\{\infty\}$ by

$$
\left(\begin{array}{ll}
a & b \\
c & d
\end{array}\right)(z)=\frac{a z+b}{c z+d}
$$

where $a, b, c, d \in \mathbb{C}$ with $a d-b c=1$ and $z \in \widehat{\mathbb{C}}$. In the upper half-space model $\mathbb{H}^{3}=\{(x, y, r): r>0\}$ of the hyperbolic 3 -space with the metric $d=\frac{\sqrt{d x^{2}+d y^{2}+d r^{2}}}{r}$, the Möbius transformations by elements of $\mathrm{PSL}_{2}(\mathbb{C})$ give rise to all orientation preserving isometries of $\mathbb{H}^{3}$.

[^1]For $g=\left(\begin{array}{ll}a & b \\ c & d\end{array}\right) \in \operatorname{PSL}_{2}(\mathbb{C})$, we have

$$
\cosh (d(g(j), j))=\frac{|a|^{2}+|b|^{2}+|c|^{2}+|d|^{2}}{2}
$$

where $j=(0,0,1)$. Hence the following follows from Theorem 1.1.
Corollary 1.2. Let $\Gamma<\mathrm{PSL}_{2}(\mathbb{C})$ be a non-elementary geometrically finite discrete subgroup. For any Borel subset $\Omega$ of $\widehat{\mathbb{C}}$ with $\nu_{j}(\partial(\Omega))=0$, we have, as $T \rightarrow \infty$,
$\#\left\{\left(\begin{array}{ll}a & b \\ c & d\end{array}\right) \in \Gamma:|a|^{2}+|b|^{2}+|c|^{2}+|d|^{2}<2 \cosh T, \quad \frac{a z+b}{c z+d} \in \Omega\right\} \sim \frac{\left|\nu_{j}\right| \cdot \nu_{j}(\Omega)}{\delta_{\Gamma} \cdot\left|m_{\Gamma}^{\mathrm{BMS}}\right|} \cdot e^{\delta_{\Gamma} T}$.
A similar result holds for the linear fractional transformation action of nonvirtually cyclic and finitely generated subgroups of $\mathrm{PSL}_{2}(\mathbb{R})$ on $\widehat{\mathbb{R}}$.

After the submission, we were pointed out by the referee that in F. Maucourant's thesis [12, Theorem 1.1 was already proved in the case when the sector is taken to be the whole ball (i.e., $\Omega_{1}=\partial_{\infty}\left(\mathbb{H}^{n}\right)$ ) and that his approach which elegantly uses a theorem of Roblin [17, Theorem 4.11] can be extended to obtain Theorem 2 of the Appendix. As Maucourant's result is not published, Maucourant agreed to write an appendix on his result.

Our approach is different from his, as we do not rely on the aforementioned theorem of Roblin but on a recent result of Oh and Shah (see Theorem 2.3). In section 2 , we obtain the main ergodic theorem which is the equidistribution of solvable flows (Theorem 2.7) which is of independent interest. In section 3, we relate the counting function in Theorem 1.1 with an average over a solvable flow of a certain function on $\mathrm{T}^{1}\left(\Gamma \backslash \mathbb{H}^{n}\right)$ (Lemma 3.1) and then apply the results in section 2 to conclude Theorem 1.1. Some computations such as Lemma 3.3 are a bit tricky due to the fact that the Burger-Roblin measure $m_{\Gamma}^{\mathrm{BR}}$ is not an invariant measure in general.

This approach of establishing the equidistribution of $\Gamma$-orbits on the boundary via the study of solvable flows on $\mathrm{T}^{1}\left(\Gamma \backslash \mathbb{H}^{n}\right)$ was first used in [7].

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## 2. EQUidistribution of solvable flows

For $x, y \in \mathbb{H}^{n}$ and $\xi \in \partial_{\infty}\left(\mathbb{H}^{n}\right)$, the Busemann function $\beta$ is defined as follows:

$$
\beta_{\xi}(x, y)=\lim _{t \rightarrow \infty}\left\{d\left(x, \xi_{t}\right)-d\left(y, \xi_{t}\right)\right\} .
$$

where $\xi_{t}$ is a geodesic ray toward $\xi$.
For a unit tangent vector $u \in \mathrm{~T}^{1}\left(\mathbb{H}^{n}\right)$, we denote by $\pi(u)$ the base point of $u$ and by $u^{+}$(resp. $u^{-}$) the forward (resp. backward) endpoint of the geodesic determined by $u$.

Let $\Gamma$ be a non-elementary discrete subgroup of $G=\operatorname{Isom}^{+}\left(\mathbb{H}^{n}\right)$. Let $\left\{\nu_{x}\right.$ : $\left.x \in \mathbb{H}^{n}\right\}$ denote a Patterson-Sullivan density for $\Gamma$, i.e., each $\nu_{x}$ is a finite measure supported on $\partial_{\infty}\left(\mathbb{H}^{n}\right)$ satisfying: for any $x, y \in \mathbb{H}^{n}, \xi \in \partial_{\infty}\left(\mathbb{H}^{n}\right)$ and $\gamma \in \Gamma$,

$$
\gamma_{*} \nu_{x}=\nu_{\gamma x} ; \quad \text { and } \quad \frac{d \nu_{y}}{d \nu_{x}}(\xi)=e^{-\delta_{\Gamma} \beta_{\xi}(y, x)}
$$

where $\gamma_{*} \nu_{x}(R)=\nu_{x}\left(\gamma^{-1}(R)\right)$.

Definition 2.1. The Bowen-Margulis-Sullivan measure $m_{\Gamma}^{\mathrm{BMS}}$ associated to $\left\{\nu_{x}\right\}$ ([3], [11], [20]) is defined as the measure induced on $\mathrm{T}^{1}\left(\Gamma \backslash \mathbb{H}^{n}\right)$ of the following $\Gamma$-invariant measure $\widetilde{m}^{\mathrm{BMS}}$ on $\mathrm{T}^{1}\left(\mathbb{H}^{n}\right)$ :

$$
d \widetilde{m}^{\mathrm{BMS}}(u)=e^{\delta_{\Gamma} \beta_{u^{+}}(x, \pi(u))} e^{\delta_{\Gamma} \beta_{u^{-}}(x, \pi(u))} d \nu_{x}\left(u^{+}\right) d \nu_{x}\left(u^{-}\right) d t .
$$

We denote by $\left\{m_{x}: x \in \mathbb{H}^{n}\right\}$ a $G$-invariant conformal density of dimension $n-1$, which is unique up to homothety.

Definition 2.2. The Burger-Roblin measure $m_{\Gamma}^{\mathrm{BR}}$ associated to $\left\{\nu_{x}\right\}$ and $\left\{m_{x}\right\}$ ( 4 , [17]) is defined as the measure induced on $\mathrm{T}^{1}\left(\Gamma \backslash \mathbb{H}^{n}\right)$ of the following $\Gamma$-invariant measure $\widetilde{m}^{\mathrm{BR}}$ on $\mathrm{T}^{1}\left(\mathbb{H}^{n}\right)$ :

$$
d \widetilde{m}^{\mathrm{BR}}(u)=e^{(n-1) \beta_{u}+(x, \pi(u))} e^{\delta_{\Gamma} \beta_{u^{-}}(x, \pi(u))} d m_{x}\left(u^{+}\right) d \nu_{x}\left(u^{-}\right) d t
$$

The measure $\widetilde{m}^{\mathrm{BR}}$ is supported on the set of unit tangent vectors $u$ such that $u^{-}$belongs to the limit set $\Lambda_{\Gamma}$.

We fix $x \in \mathbb{H}^{n}$ and $\xi \in \partial_{\infty}\left(\mathbb{H}^{n}\right)$ in the rest of this section. Let $K$ be the stabilizer of $x$ in $G$ and $P$ denote the stabilizer of $\xi \in \partial_{\infty}\left(\mathbb{H}^{n}\right)$. The subgroup $P$ is a minimal parabolic subgroup of $G$ and is the normalizer of its unipotent radical $N$. Without loss of generality, we may assume that $m_{x}$ is the probability measure.

Denote by $X_{0} \in \mathrm{~T}^{1}\left(\mathbb{H}^{n}\right)$ the unit vector based at $x$ such that $X_{0}^{-}=\xi$. We set

$$
\xi_{x}:=X_{0}^{+}
$$

Setting $A=\left\{a_{t}:=\exp \left(t X_{0}\right): t \in \mathbb{R}\right\}$, we have (cf. [7, Lem. 4.1])

- $G=K A^{+} K$ where $A^{+}:=\left\{a_{t}: t \geq 0\right\}$;
- $P=M A N$ where $M$ is the centralizer of $A$ in $K$ and $M=K \cap P$;
- $N$ is the expanding horospherical subgroup of $G$ with respect to $A^{+}$, i.e., $N=\left\{g \in G: a_{t} g a_{-t} \rightarrow e \quad\right.$ as $\left.t \rightarrow \infty\right\}$.
The above Cartan decomposition $G=K A^{+} K$ says that for any $g \in G$, there exists a unique element $a \in A^{+}$such that $g=k_{1} a k_{2}$, for $k_{1}, k_{2} \in K$. Moreover, $k_{1} a k_{2}=k_{1}^{\prime} a k_{2}^{\prime}$ implies that $k_{1}=k_{1}^{\prime} m$ and $k_{2}=m^{-1} k_{2}^{\prime}$ for some $m \in M$.

We may identify $G / K$ with $\mathbb{H}^{n}$ where $g K$ corresponds to $g(x)$ and $G / M$ with $\mathrm{T}^{1}\left(\mathbb{H}^{n}\right)$ where $g M$ corresponds to $g\left(X_{0}\right)$.

Let $B_{0}$ be the maximal split solvable subgroup of $G$ given by

$$
B_{0}=A N
$$

For $T>0$ and a subset $\Omega \subset K$ with $\Omega M=\Omega$, set

$$
B_{0}(T, \Omega):=B_{0} \cap \Omega A_{T}^{+} K
$$

where $A_{T}^{+}:=\left\{a_{t}: 0 \leq t \leq T\right\}$. Our aim in this section is to prove an equidistribution of $B_{0}(T, \Omega)$ on $\mathrm{T}^{1}\left(\Gamma \backslash \mathbb{H}^{n}\right)$ : Theorem 2.7

The following is the main ergodic ingredient we use.
Theorem 2.3. [13] Suppose that $\left|m_{\Gamma}^{\mathrm{BMS}}\right|<\infty$. Let $\Omega$ be a Borel subset of $K$ with $\Omega M=\Omega$ and with $\nu_{x}\left(\partial\left(\Omega\left(\xi_{x}\right)\right)\right)=0$. For any $\varphi \in C_{c}(\Gamma \backslash G)^{M}$,

$$
e^{\left(n-1-\delta_{\Gamma}\right) t} \int_{s \in \Omega / M} \varphi\left(s a_{t}\right) d m_{x}(s) \sim \frac{\nu_{x}\left(\Omega\left(\xi_{x}\right)\right)}{\left|m_{\Gamma}^{\mathrm{BMS}}\right|} \cdot m_{\Gamma}^{\mathrm{BR}}(\varphi) \quad \text { as } t \rightarrow+\infty
$$

By the Iwasawa decomposition $G=A N K$, the map

$$
K \longrightarrow B_{0} \backslash G: k \mapsto B_{0} k
$$

is a diffeomorphism, say, $\iota$. Let $N^{-}$be the contracting horospherical subgroup of $G$ with respect to $A^{+}: N^{-}=\left\{g \in G: a_{-t} g a_{t} \rightarrow e \quad\right.$ as $\left.t \rightarrow \infty\right\}$.

The map $M \times N^{-} \rightarrow B_{0} \backslash G, m n \mapsto B_{0} m n$, composed with $\iota^{-1}$, is an $M$ equivariant map $M \times N^{-} \rightarrow K$ which is a diffeomorphism onto its image, which is a Zariski open subset. Let $S$ be the image of $\{e\} \times N^{-}$under this map. We note that the complement of $M \backslash M S$ in $M \backslash K$ is a point.
Lemma 2.4. Let $s \in S$. If $V \subset S$ is a neighborhood of $s$ and $S_{0}$ is a compact subset of $S$, there exists $C=C\left(S_{0}\right)>1$ such that for any $m \in M$,

$$
M S_{0} m \subset M V s^{-1} m a_{-t} \quad \text { for all } t>C
$$

Proof. Since $e \in V s^{-1}$, the conjugation by $a_{t}$ expands $V s^{-1} \subset S$ by the factor of $e^{t}$, and hence we can find $C>1$ such that

$$
S_{0} \subset a_{t} V s^{-1} a_{-t}
$$

for all $t>C$. Hence

$$
B_{0} M S_{0} m \subset B_{0} M a_{t} V s^{-1} a_{-t} m=B_{0} M V s^{-1} m a_{-t}
$$

as $a_{t} \in B_{0}$.
By the uniqueness of the decomposition $G=B_{0} K$, we have the desired inclusion.

We denote by $d h$ the Haar measure on $G$ such that for $h=k_{1} a_{t} k_{2} \in K A^{+} K$,

$$
d h=2^{n-1}(\sinh t \cosh t)^{(n-1) / 2} d k_{1} d t d k_{2}
$$

where $d k$ denotes the probability Haar measure on $K$.
We denote by $\rho_{\ell}$ the left-invariant Haar measure on $B_{0}$ given by the relation:

$$
d h=d \rho_{\ell}(b) d k
$$

where $h=b k \in B_{0} K$.
In the rest of this section, we assume that $\left|m_{\Gamma}^{\mathrm{BMS}}\right|<\infty$.
The following lemma is a special case of [16, Prop. 3.1]:
Lemma 2.5. Any sphere centered at $\xi_{0} \in \Lambda(\Gamma)$ has measure zero with respect to $\nu_{x}$.
Proposition 2.6. Let $V$ be an open neighborhood of $e$ in $K$ such that $M V=V$. Let $\Omega$ be a Borel subset of $K$ with $\Omega M=\Omega$ and with $\nu_{x}\left(\partial\left(\Omega\left(\xi_{x}\right)\right)\right)=0$. Then for any $\psi \in C_{c}(\Gamma \backslash G)$,

$$
\int_{V} \int_{B_{0}(T, \Omega)} \psi(b k) d \rho_{\ell}(b) d k \sim \frac{e^{\delta_{\Gamma} T} \nu_{x}\left(\Omega\left(\xi_{x}\right)\right)}{\delta_{\Gamma} \cdot\left|m_{\Gamma}^{\mathrm{BMS}}\right|} \cdot m_{\Gamma}^{\mathrm{BR}}\left(\psi * \chi_{V}\right)
$$

as $T \rightarrow \infty$, where $\psi * \chi_{V}(h)=\int_{k \in V} \psi(h k) d k$ and $B_{0}(T, \Omega):=B_{0} \cap \Omega A_{T}^{+} K$.
Proof. Note that

$$
\begin{aligned}
B_{0}(T, \Omega) V & =B_{0} V \cap \Omega A_{T}^{+} K \\
& =\left\{k_{1} a_{t} k_{2}: k_{1} \in \Omega_{k_{2}}(t), 0<t<T, k_{2} \in K\right\}
\end{aligned}
$$

where $\Omega_{k_{2}}(t)=\Omega \cap B_{0} V k_{2}^{-1} a_{-t}$.

Setting $\Xi(t)=2^{n-1}(\sinh t \cosh t)^{(n-1) / 2}$, we have

$$
\begin{aligned}
& \int_{B_{0}(T, \Omega)} \psi(b k) d \rho_{\ell}(b) d k \\
& =\int_{h \in B_{0}(T, \Omega)} \psi(h) d h \\
& =\int_{k_{1} a_{t} k_{2} \in B_{0}(T, \Omega)} \psi\left(k_{1} a_{t} k_{2}\right) \Xi(t) d k_{2} d t d k_{1} \\
& =\int_{k_{2} \in K} \int_{0<t<T} \int_{k_{1} \in \Omega_{k_{2}}(t)} \psi\left(k_{1} a_{t} k_{2}\right) \Xi(t) d k_{2} d t d k_{1} .
\end{aligned}
$$

Set for $m \in M$,

$$
\Omega_{m}:=\Omega \cap M S m^{-1}
$$

where $S$ is the image of $N^{-}$in $K$. We note that since $S \subset M \backslash K$ is an open Zariski dense subset whose complement is a point and $\nu_{x}$ is atom free, $\nu_{x}(\Omega)=\nu_{x}\left(\Omega_{m}\right)$ and $\nu_{x}(\partial(\Omega))=\nu_{x}\left(\partial\left(\Omega_{m}\right)\right)$. Write $V=M V_{0}$ for $V_{0} \subset S$. Let $k_{2}=m s \in M S$ with $s \in V_{0}$.

By Lemma 2.5, for any fixed $\epsilon>0$, we can take a compact subset $S_{\epsilon} \subset S$ such that $\nu_{x}\left(\Omega\left(\xi_{x}\right)-S_{\epsilon}\left(\xi_{x}\right)\right)<\epsilon$ and $\nu_{x}\left(\partial\left(S_{\epsilon}\left(\xi_{x}\right)\right)\right)=0$. If we set $\Omega_{m}\left(S_{\epsilon}\right):=$ $\Omega \cap M S_{\epsilon} m^{-1}$, then $\nu_{x}\left(\Omega_{m}\left(\xi_{x}\right)-\Omega_{m}\left(S_{\epsilon}\right)\left(\xi_{x}\right)\right)<\epsilon$ and $\nu_{x}\left(\partial\left(\Omega_{m}\left(S_{\epsilon}\right)\left(\xi_{x}\right)\right)\right)=0$ since $\left.\partial\left(\Omega_{m}\left(S_{\epsilon}\right)\left(\xi_{x}\right)\right) \subset \partial\left(\Omega\left(\xi_{x}\right)\right) \cup \partial\left(S_{\epsilon}\left(\xi_{x}\right)\right)\right)$.

By Lemma 2.4 there exists $C_{\epsilon}>1$ such that for all $t>C_{\epsilon}$,

$$
\Omega_{m}\left(S_{\epsilon}\right) \subset \Omega_{m s}(t)
$$

On the other hand, as $a_{t} \in B_{0}$,

$$
\Omega_{m s}(t) \subset \Omega \cap B_{0} M V s^{-1} m^{-1} a_{-t}=\Omega \cap B_{0} M\left(a_{t} V s^{-1} a_{-t}\right) m^{-1} \subset \Omega \cap M S m^{-1}=\Omega_{m}
$$

Without loss of generality we assume below that $\psi$ is non-negative. Hence for all $t>C_{\epsilon}$,

$$
\int_{k_{1} \in \Omega_{m}\left(S_{\epsilon}\right)} \psi\left(k_{1} a_{t} s m\right) d k_{1} \leq \int_{k \in \Omega_{m s}(t)} \psi\left(k_{1} a_{t} m s\right) d k_{1} \leq \int_{k_{1} \in \Omega_{m}} \psi\left(k_{1} a_{t} m s\right) d k_{1}
$$

Note that by applying Theorem 2.3

$$
\begin{aligned}
& \int_{k_{1} \in \Omega_{m}\left(S_{\epsilon}\right)} \psi\left(k_{1} a_{t} m s\right) d k_{1} \\
& =\int_{s \in \Omega_{m}\left(S_{\epsilon}\right) / M} \int_{m_{1} \in M} \psi\left(s a_{t} m_{1} m s\right) d m_{1} d m_{x}(s) \\
& =\int_{s \in \Omega_{m}\left(S_{\epsilon}\right) / M} \psi_{m s}\left(s a_{t}\right) d m_{x}(s) \\
& \sim e^{-\left(n-1-\delta_{\Gamma}\right) t} \frac{1}{\left|m_{\Gamma}^{\mathrm{BMS}}\right|} m_{\Gamma}^{\mathrm{BR}}\left(\psi_{m s}\right) \nu_{x}\left(\Omega_{m}\left(S_{\epsilon}\right)\left(\xi_{x}\right)\right)
\end{aligned}
$$

where $\psi_{m s}(h):=\int_{m_{1} \in M} \psi\left(h m_{1} m s\right) d m_{1}$.

Hence

$$
\begin{aligned}
& \lim _{t} \inf e^{\left(n-1-\delta_{\Gamma}\right) t} \int_{k_{1} \in \Omega_{m s}(t)} \psi\left(k_{1} a_{t} m s\right) d k_{1} \\
& \geq \liminf e^{\left(n-1-\delta_{\Gamma}\right) t} \int_{k_{1} \in \Omega_{m}\left(S_{\epsilon}\right)} \psi\left(k_{1} a_{t} m s\right) d k_{1} \\
& =\frac{1}{\left|m_{\Gamma}^{\mathrm{BMS}}\right|} m_{\Gamma}^{\mathrm{BR}}\left(\psi_{m s}\right) \nu_{x}\left(\Omega_{m}\left(S_{\epsilon}\right)\left(\xi_{x}\right)\right) \\
& \geq \frac{1}{\left|m_{\Gamma}^{\mathrm{BMS}}\right|} m_{\Gamma}^{\mathrm{BR}}\left(\psi_{m s}\right)\left(\nu_{x}\left(\Omega_{m}\left(\xi_{x}\right)\right)-\epsilon\right)
\end{aligned}
$$

and similarly
$\limsup _{t} e^{\left(n-1-\delta_{\Gamma}\right) t} \int_{k_{1} \in \Omega_{m s}(t)} \psi\left(k_{1} a_{t} m s\right) d k_{1} \leq \frac{1}{\left|m_{\Gamma}^{\mathrm{BMS}}\right|} m_{\Gamma}^{\mathrm{BR}}\left(\psi_{m s}\right) \nu_{x}\left(\left(\Omega_{m}\left(\xi_{x}\right)\right)+\epsilon\right)$.
As $\epsilon>0$ is arbitrary and $\Omega_{m}\left(\xi_{x}\right)=\Omega\left(\xi_{x}\right)$, we deduce

$$
\lim _{t \rightarrow \infty} e^{\left(n-1-\delta_{\Gamma}\right) t} \int_{k \in \Omega_{m s}(t)} \psi\left(k_{1} a_{t} m s\right) d k_{1}=\frac{1}{\left|m_{\Gamma}^{\mathrm{BMS}}\right|} m_{\Gamma}^{\mathrm{BR}}\left(\psi_{m s}\right) \nu_{x}\left(\Omega\left(\xi_{x}\right)\right)
$$

Using that $\Xi(t) \sim_{t} e^{(n-1) t}$, we obtain that for any $m s \in M V_{0}$, as $T \rightarrow \infty$,

$$
\begin{aligned}
& \int_{A_{T}^{+}} \int_{\Omega_{m s}(t)} \psi\left(k_{1} a_{t} m s\right) \Xi(t) d k_{1} d t \\
& \sim \frac{e^{\delta_{\Gamma} T}}{\delta_{\Gamma} \cdot\left|m_{\Gamma}^{\mathrm{BMS}}\right|} m_{\Gamma}^{\mathrm{BR}}\left(\psi_{m s}\right) \nu_{x}\left(\Omega\left(\xi_{x}\right)\right) .
\end{aligned}
$$

Now for $m s \notin M V_{0}$, we claim that

$$
\limsup _{T} e^{-\delta_{\Gamma} T} \int_{A_{T}^{+}} \int_{\Omega_{m s, t}} \psi\left(k_{1} a_{t} m s\right) \Xi(t) d k_{1} d t=0
$$

Consider the set

$$
\Omega_{m s, t}^{c}:=\Omega \cap B_{0}\left(M S-M V_{0}\right) s^{-1} m^{-1} a_{-t}
$$

As $s \in M S-M V_{0}$, we have by the previous case that

$$
\lim _{T} e^{-\delta_{\Gamma} T} \int_{A_{T}^{+}(C)} \int_{\Omega_{m s, t}^{c}} \psi\left(k_{1} a_{t} m s\right) \Xi(t) d k_{1} d t=\frac{1}{\delta_{\Gamma} \cdot\left|m_{\Gamma}^{\mathrm{BMS}}\right|} m_{\Gamma}^{\mathrm{BR}}\left(\psi_{m s}\right) \nu_{x}\left(\Omega_{m}\left(\xi_{x}\right)\right)
$$

Since $\Omega_{m s, t}^{c} \subset \Omega_{m}$ and

$$
\lim _{T} e^{-\delta_{\Gamma} T} \int_{A_{T}^{+}(C)} \int_{\Omega_{m}} \psi\left(k_{1} a_{t} m s\right) \Xi(t) d k_{1} d t=\frac{1}{\delta_{\Gamma} \cdot\left|m_{\Gamma}^{\mathrm{BMS}}\right|} m_{\Gamma}^{\mathrm{BR}}\left(\psi_{m s}\right) \nu_{x}\left(\Omega_{m}\left(\xi_{x}\right)\right),
$$

the claim follows.
Since the image of $S$ is an open Zariski dense subset of $M \backslash K$, we may replace $K$ by $M S$ in the integration over $K$ and hence

$$
\begin{aligned}
& \int_{k_{2} \in K} \int_{A_{T}^{+}} \int_{k_{1} \in \Omega_{k_{2}}(t)} \psi\left(k_{1} a_{t} m s\right) \Xi(t) d k_{1} d t d k_{2} \\
& \sim \int_{m s \in M V_{0}} \int_{A_{T}^{+}} \int_{k_{1} \in \Omega_{k_{2}}(t)} \psi\left(k_{1} a_{t} m s\right) \Xi(t) d k_{1} d t d k_{2} \\
& \sim \frac{e^{\delta_{\Gamma} T}}{\delta_{\Gamma} \cdot\left|m_{\Gamma}^{\mathrm{BMS}}\right|} m_{\Gamma}^{\mathrm{BR}}\left(\psi * \chi_{V}\right) \nu_{x}\left(\Omega\left(\xi_{x}\right)\right) .
\end{aligned}
$$

This completes the proof of Proposition 2.6
Theorem 2.7. Let $\Omega$ be a Borel subset of $K / M$ with $\nu_{x}\left(\partial\left(\Omega\left(\xi_{x}\right)\right)\right)=0$. Then for any $\psi \in C_{c}(\Gamma \backslash G)$, as $T \rightarrow \infty$,

$$
\int_{B_{0}(T, \Omega)} \psi(b) d \rho_{\ell}(b) \sim \frac{e^{\delta_{\Gamma} T}}{\delta_{\Gamma}} \cdot \frac{\nu_{x}\left(\Omega\left(\xi_{x}\right)\right)}{\left|m_{\Gamma}^{\mathrm{BMS}}\right|} \cdot m_{\Gamma}^{\mathrm{BR}}(\psi)
$$

Proof. Let $V_{\epsilon}$ be an $\epsilon$-neighborhood of $e$ in $K$ such that $V_{\epsilon} M=V_{\epsilon}$. For any $\psi \in C_{c}(\Gamma \backslash G)^{M}$ and $\epsilon>0$, define functions $\psi_{\varepsilon}^{ \pm} \in C_{c}(\Gamma \backslash G)^{M}$ as follows:

$$
\psi_{\varepsilon}^{+}(h):=\sup _{k \in V_{\varepsilon}} \psi(h k), \text { and } \psi_{\varepsilon}^{-}(h):=\inf _{k \in V_{\varepsilon}} \psi(h k) .
$$

Let $\eta>0$. By the uniform continuity of $\psi$ and the $M$-invariance, there exists $\epsilon=\epsilon(\eta)$ such that $\left|\psi_{\varepsilon}^{+}(h)-\psi_{\epsilon}^{-}(h)\right|<\eta$ for all $h \in G$.

Without loss of generality we may assume $\psi \geq 0$. Note that, by applying Proposition 2.6

$$
\begin{aligned}
& \limsup _{T} e^{-\delta_{\Gamma} T} \int_{B_{0}(T, \Omega)} \psi(b) d \rho_{\ell}(b) \\
& \leq \limsup e^{-\delta_{\Gamma} T} \operatorname{Vol}\left(V_{\epsilon}\right)^{-1} \int_{k \in V_{\epsilon}} \int_{B_{0}(T, \Omega)} \psi_{\epsilon}^{+}(b k) d \rho_{\ell}(b) d k \\
& =\operatorname{Vol}\left(V_{\epsilon}\right)^{-1} \frac{1}{\delta_{\Gamma}\left|m_{\Gamma}^{\mathrm{BMS}}\right|} m_{\Gamma}^{\mathrm{BR}}\left(\psi_{\epsilon}^{+} * \chi_{V_{\epsilon}}\right) \nu_{x}\left(\Omega\left(\xi_{x}\right)\right)
\end{aligned}
$$

Similarly

$$
\begin{aligned}
& \liminf _{T} e^{-\delta_{\Gamma} T} \int_{B_{0}(T, \Omega)} \psi(b) d \rho_{\ell}(b) \\
& \geq \liminf _{T} e^{-\delta_{\Gamma} T} \operatorname{Vol}\left(V_{\epsilon}\right)^{-1} \int_{k \in V_{\epsilon}} \int_{B_{0}(T, \Omega)} \psi_{\epsilon}^{-}(b k) d \rho_{\ell}(b) d k \\
& =\operatorname{Vol}\left(V_{\epsilon}\right)^{-1} \frac{1}{\delta_{\Gamma}\left|m_{\Gamma}^{\mathrm{BMS}}\right|} m_{\Gamma}^{\mathrm{BR}}\left(\psi_{\epsilon}^{-} * \chi_{V_{\epsilon}}\right) \nu_{x}\left(\Omega\left(\xi_{x}\right)\right) .
\end{aligned}
$$

Since

$$
m_{\Gamma}^{\mathrm{BR}}\left(\psi_{\epsilon}^{ \pm} * \chi_{V_{\epsilon}}\right)=m_{\Gamma}^{\mathrm{BR}}(\psi) \operatorname{Vol}\left(V_{\epsilon}\right)+O(\eta)
$$

we deduce that

$$
\underset{T}{\limsup } e^{-\delta_{\Gamma} T} \int_{B_{0}(T, \Omega)} \psi(b) d \rho_{\ell}(b)=\frac{1}{\delta_{\Gamma}\left|m_{\Gamma}^{\mathrm{BMS}}\right|} m_{\Gamma}^{\mathrm{BR}}(\psi) \nu_{x}\left(\Omega\left(\xi_{x}\right)\right)+O(\eta)
$$

and

$$
\liminf _{T} e^{-\delta_{\Gamma} T} \int_{B_{0}(T, \Omega)} \psi(b) d \rho_{\ell}(b)=\frac{1}{\delta_{\Gamma}\left|m_{\Gamma}^{\mathrm{BMS}}\right|} m_{\Gamma}^{\mathrm{BR}}(\psi) \nu_{x}\left(\Omega\left(\xi_{x}\right)\right)+O(\eta)
$$

As $\eta>0$ is arbitrary, this proves the claim.

## 3. Proof of Theorem 1.1

Fix $x \in \mathbb{H}^{n}$ and $\xi \in \partial_{\infty}\left(\mathbb{H}^{n}\right)$. We keep the same notation from the previous section. Let $y \in \mathbb{H}^{n}$ and choose $g \in G$ such that $g(x)=y$.

For a subset $W$ of $G$, we denote by $W^{g}$ the conjugate $g W g^{-1}$. Note that $K^{g}$ is the stabilizer of $y$ and that $B:=B_{0}^{g}$ stabilizes $g(\xi)=g\left(X_{0}^{-}\right)$.

For $\hat{\Omega}_{1}, \hat{\Omega}_{2} \subset \partial_{\infty}\left(\mathbb{H}^{n}\right)$, we set

$$
\Omega_{1}:=\left\{k \in K / M: k \xi_{x} \in \hat{\Omega}_{1}\right\}, \quad \Omega_{2}:=\left\{k \in K^{g} / M^{g}: k(g(\xi)) \in \hat{\Omega}_{2}\right\}
$$

so that $\hat{\Omega}_{1}=\Omega_{1}\left(\xi_{x}\right)$ and $\hat{\Omega}_{2}=\Omega_{2}(g(\xi))$. We assume that the boundaries of $\hat{\Omega}_{i}$ have measure zero with respect to the Patterson-Sullivan density.

In this notation, we have

$$
\left\{z \in S_{x}\left(\hat{\Omega}_{1}\right), d(z, x)<T\right\}=\Omega_{1} A_{T}^{+}(x)
$$

and hence the condition $\gamma^{-1} y \in S_{x, T}\left(\hat{\Omega}_{1}\right)$ becomes $\gamma \in g K A_{T}^{-} \Omega_{1}^{-1}$. And $\gamma(\xi) \in \hat{\Omega}_{2}$ is equivalent to $\gamma g^{-1}(g(\xi)) \in \Omega_{2}(g(\xi))$ and hence to $\gamma g^{-1} \in \Omega_{2} B$.

For $h \in G$, we write

$$
h=h_{K^{g}} h_{B} g
$$

where $h_{K^{g}} \in K^{g}$ and $h_{B} \in B$ are uniquely determined.
Hence setting

$$
B\left(T, \Omega_{1}\right)=B \cap g \Omega_{1} A_{T}^{+} K g^{-1}
$$

the number we want to count is the following:

$$
\begin{aligned}
N_{T}\left(\hat{\Omega}_{1}, \hat{\Omega}_{2}\right) & :=\#\left\{\gamma \in \Gamma: \gamma(g(\xi)) \in \hat{\Omega}_{2}, \gamma^{-1}(y) \in S_{x}\left(\hat{\Omega}_{1}\right)\right\} \\
& =\# \Gamma \cap g K A_{T}^{-} \Omega_{1}^{-1} \cap \Omega_{2} B g \\
& =\left\{\gamma \in \Gamma: \gamma_{K^{g}} \in \Omega_{2}, \gamma_{B}^{-1} \in B\left(T, \Omega_{1}\right)\right\} .
\end{aligned}
$$

Let $V_{\epsilon}$ be an $\epsilon$-neighborhood of $e$ in $K$ such that $M V_{\epsilon} M=V_{\epsilon}$.
For the $\epsilon$-neighborhood $A_{\varepsilon}=\left\{a_{t}:|t|<\epsilon\right\}$ of $e$ in $A$, by the strong wavefront Lemma (see [8] or [7]) there exists a symmetric neighborhood $\mathcal{O}_{\varepsilon}^{\prime}$ of $e$ in $G$ and $C>1$ such that for all $k \in K$ and all $t>C$,

$$
\begin{equation*}
g k a_{t} K g^{-1} \mathcal{O}_{\varepsilon}^{\prime} \subset g k V_{\epsilon} a_{t} A_{\epsilon} K g^{-1} \tag{3.1}
\end{equation*}
$$

Choose a symmetric neighborhood $\tilde{V}_{\epsilon} \subset V_{\epsilon}$ so that

$$
\begin{equation*}
\operatorname{Vol}\left(V_{\epsilon}^{+}-V_{\epsilon}^{-}\right)<\eta \operatorname{Vol}\left(V_{\epsilon}\right) \tag{3.2}
\end{equation*}
$$

where $V_{\epsilon}^{+}:=V_{\epsilon} \tilde{V}_{\epsilon}$ and $V_{\epsilon}^{-}:=\cap_{u \in \tilde{V}_{\epsilon}} V_{\epsilon} u$.
We may assume without loss of generality that $\mathcal{O}_{\epsilon}^{\prime}$ satisfies

$$
a_{t} n k\left(g^{-1} \mathcal{O}_{\epsilon}^{\prime} g\right) \subset a_{t} A_{\epsilon} N k \tilde{V}_{\epsilon}
$$

for all $a_{t} n k \in A N K$.
We set

$$
\mathcal{O}_{\epsilon}:=\mathcal{O}_{\epsilon}^{\prime} \cap B
$$

and note that $\mathcal{O}_{\epsilon}^{-1}=\mathcal{O}_{\epsilon}$.
Fix $\eta>0$. Then there exists $0<\epsilon(\eta)<\eta$ such that for all $0<\epsilon<\epsilon(\eta)$,

$$
\begin{equation*}
\nu_{x}\left(\Omega_{1, \epsilon}^{+}\left(\xi_{x}\right)-\Omega_{1, \epsilon}^{-}\left(\xi_{x}\right)\right)<\eta \tag{3.3}
\end{equation*}
$$

where $\Omega_{1, \epsilon}^{+}=\Omega_{1} V_{\epsilon}^{+}$and $\Omega_{1, \epsilon}^{-}=\cap_{k \in V_{\epsilon}^{-}} \Omega_{1} k$. This is possible since the boundary of $\hat{\Omega}_{1}$ has measure zero with respect to $\nu_{x}$.

Similarly, we may assume that

$$
\begin{equation*}
\nu_{y}\left(\Omega_{2, \epsilon}^{+}(g(\xi))-\Omega_{2, \epsilon}^{-}(g(\xi))\right)<\eta \tag{3.4}
\end{equation*}
$$

where $\Omega_{2, \epsilon}^{+}=\Omega_{2} U_{\epsilon}^{+}$and $\Omega_{2, \epsilon}^{-}=\cap_{k \in U_{\epsilon}^{-}} \Omega_{2} k$ where $U_{\epsilon}^{ \pm}:=g V_{\epsilon}^{ \pm} g^{-1}$. We also set $U_{\epsilon}=g V_{\epsilon} g^{-1}$.

We choose $\phi_{\epsilon}^{ \pm}=\phi_{\Omega_{2}, \epsilon}^{ \pm} \in C_{c}\left(K^{g}\right)^{M^{g}}$ such that $0 \leq \phi_{\Omega_{2}, \epsilon}^{-} \leq \phi_{\Omega_{2}, \epsilon}^{+} \leq 1, \phi_{\epsilon}^{+}(k)=1$ for $k \in \Omega_{2}, \phi_{\epsilon}^{+}(k)=0$ for $k \notin \Omega_{2} U_{\epsilon}, \phi_{\epsilon}^{-}(k)=1$ for $k \in \cap_{u \in U_{\epsilon}} \Omega_{2} u$, and $\phi_{\epsilon}^{-}(k)=0$ for $k \notin \Omega_{2}$.

We denote by $\rho$ the left invariant Haar measure on $B$ given by: for $\psi \in C_{c}(B)$,

$$
\int_{B} \psi(b) d \rho(b):=\int_{b_{0} \in B_{0}} \psi\left(g^{-1} b_{0} g\right) d \rho_{\ell}\left(b_{0}\right)
$$

Choosing a non-negative function $\psi_{\epsilon} \in C_{c}(B)$ supported on $\mathcal{O}_{\epsilon}$ and with $\int_{B} \psi_{\epsilon}(b) d \rho(b)=$ 1, we define a function $f_{\Omega_{2}, \eta}^{ \pm}$on $G=K^{g} B g$ by

$$
f_{\Omega_{2}, \eta}^{ \pm}(h)=\phi_{\Omega_{2}, \epsilon(\eta)}^{ \pm}\left(h_{K^{g}}\right) \psi_{\epsilon(\eta)}\left(h_{B}\right)
$$

where $h=h_{K^{g}} h_{B} g \in G$ with $h_{K} \in K^{g}$ and $h_{B} \in B$ uniquely determined and $\epsilon=\epsilon(\eta)$. Define

$$
F_{\Omega_{2}, \eta}^{ \pm}(h)=\sum_{\gamma \in \Gamma} f_{\Omega_{2}, \eta}^{ \pm}(\gamma h)
$$

which is an integrable function defined on $\Gamma \backslash G$.
We set

$$
\begin{gathered}
B_{0}^{C}\left(T, \Omega_{1}\right):=B_{0} \cap \Omega_{1} A_{T}^{+}(C) K \\
B^{C}\left(T, \Omega_{1}\right):=B \cap g \Omega_{1} A_{T}^{+}(C) K g^{-1} \\
N_{T}^{C}\left(\hat{\Omega}_{1}, \hat{\Omega}_{2}\right):=\# \Gamma \cap g K A_{T}^{-}(C) \Omega_{1}^{-1} \cap \Omega_{2} B g
\end{gathered}
$$

where $A_{T}^{-}(C)=\left\{a_{-t}: C<t<T\right\}$ and $A_{T}^{+}(C)=\left\{a_{t}: C<t<T\right\}$. When $C=0$, we simply omit the superscript 0 from the above notation.

Note that

$$
N_{T}^{C}\left(\hat{\Omega}_{1}, \hat{\Omega}_{2}\right)=\left\{\gamma \in \Gamma: \gamma_{K^{g}} \in \Omega_{2}, \gamma_{B}^{-1} \in B^{C}\left(T, \Omega_{1}\right)\right\}
$$

Lemma 3.1. Let $C>1$ be taken so that (3.1) holds. For any $T>1$ and small $\eta>0$, we have

$$
\begin{align*}
& N_{T}^{C}\left(\hat{\Omega}_{1}, \hat{\Omega}_{2}\right) \leq \int_{B_{0}\left(T+\varepsilon, \Omega_{1, \epsilon}^{+}\right)} F_{\Omega_{2}, \eta}^{+}\left(b_{0}\right) d \rho_{\ell}\left(b_{0}\right)  \tag{1}\\
& \int_{B_{0}^{C}\left(T-\varepsilon, \Omega_{1, \epsilon}^{-}\right)} F_{\Omega_{2}, \eta}^{-}\left(b_{0}\right) d \rho_{\ell}\left(b_{0}\right) \leq N_{T}\left(\hat{\Omega}_{1}, \hat{\Omega}_{2}\right) \tag{2}
\end{align*}
$$

where $\Omega_{1, \epsilon}^{+}=\Omega_{1} V_{\epsilon}$ and $\Omega_{1, \epsilon}^{-}=\cap_{k \in V_{\epsilon}} \Omega_{1} k$ and $\epsilon=\epsilon(\eta)$.
Proof. For simplicity, we set $F^{ \pm}:=F_{\Omega_{2}, \eta}^{ \pm}$and $\Omega_{1}^{ \pm}:=\Omega_{1, \epsilon}^{ \pm}$. We have

$$
\begin{aligned}
& \int_{B_{0}\left(T+\varepsilon, \Omega_{1, \epsilon}^{+}\right)} F^{+}\left(b_{0}\right) d \rho_{\ell}\left(b_{0}\right) \\
& =\int_{B\left(T+\epsilon, \Omega_{1}^{+}\right)} F^{+}\left(g^{-1} b g\right) d \rho(b) \\
& \geq \int_{B\left(T+\epsilon, \Omega_{1}^{+}\right)} \sum_{\gamma \in \Gamma} \chi_{\Omega_{2}}\left(\gamma_{K^{g}}\right) \psi_{\epsilon}\left(\gamma_{B} b\right) d \rho(b) \\
& =\sum_{\gamma \in \Gamma, \gamma_{K^{g}} \in \Omega_{2}} \int_{\gamma_{B} B\left(T+\epsilon, \Omega_{1}^{+}\right) \cap \mathcal{O}_{\epsilon}} \psi_{\epsilon}(b) d \rho(b)
\end{aligned}
$$

since $\rho$ is left-invariant. Since we have chosen $\mathcal{O}_{\varepsilon}$ so that $B^{C}\left(T, \Omega_{1}\right) \mathcal{O}_{\epsilon} \subset B(T+$ $\left.\epsilon, \Omega_{1}^{+}\right)$, for any $\gamma \in \Gamma$ such that $\gamma_{B}^{-1} \in B^{C}\left(T, \Omega_{1}\right)$,

$$
B\left(T+\epsilon, \Omega_{1}^{+}\right) \cap \gamma_{B}^{-1} \mathcal{O}_{\epsilon}=\gamma_{B}^{-1} \mathcal{O}_{\varepsilon}
$$

and hence

$$
\int_{\gamma_{B} B^{C}\left(T+\epsilon, \Omega_{1}^{+}\right) \cap \mathcal{O}_{\epsilon}} \psi_{\epsilon}(b) d \rho(b)=\int_{\mathcal{O}_{\epsilon}} \psi_{\epsilon}(b) d \rho(b)=1
$$

It follows that

$$
\begin{aligned}
\int_{B\left(T+\epsilon, \Omega_{1}^{+}\right)} F^{+}(b) d \rho_{\ell}(b) & \geq \#\left\{\gamma \in \Gamma: \gamma_{K^{g}} \in \Omega_{2}, \gamma_{B}^{-1} \in B^{C}\left(T, \Omega_{1}\right)\right\} \\
& =N_{T}^{C}\left(\hat{\Omega}_{1}, \hat{\Omega}_{2}\right)
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
& \int_{B_{0}^{C}\left(T-\epsilon, \Omega_{1}^{-}\right)} F^{-}\left(b_{0}\right) d \rho_{\ell}\left(b_{0}\right) \\
& =\int_{B^{C}\left(T-\epsilon, \Omega_{1}^{-}\right)} F^{-}\left(g^{-1} b g\right) d \rho(b) \\
& \leq \int_{B^{C}\left(T-\epsilon, \Omega_{1}^{-}\right)} \sum_{\gamma \in \Gamma} \chi_{\Omega_{2}}\left(\gamma_{K^{g}}\right) \psi_{\epsilon}\left(\gamma_{B} b\right) d \rho(b) \\
& =\sum_{\gamma \in \Gamma, \gamma_{K} \in \in \Omega_{2}} \int_{\gamma_{B} B^{C}\left(T-\epsilon, \Omega_{1}^{-}\right) \cap \mathcal{O}_{\epsilon}} \psi_{\epsilon}(b) d \rho(b)
\end{aligned}
$$

Since $\Omega_{1}^{-} V_{\epsilon} \subset \Omega_{1}$, we have

$$
B^{C}\left(T-\epsilon, \Omega_{1}^{-}\right) \mathcal{O}_{\epsilon} \subset B\left(T, \Omega_{1}\right)
$$

Therefore for $\gamma \in \Gamma$ such that $\gamma_{B}^{-1} \notin B\left(T, \Omega_{1}\right)$, we have $\rho_{\ell}\left(B^{C}\left(T-\epsilon, \Omega_{1}^{-}\right) \cap \gamma_{B}^{-1} \mathcal{O}_{\varepsilon}\right)=$ 0.

Hence it follows that

$$
\int_{B^{C}\left(T-\epsilon, \Omega_{1}^{-}\right)} F(b) d \rho(b) \leq \#\left\{\gamma \in \Gamma: \gamma_{K^{g}} \in \Omega_{2}, \gamma_{B}^{-1} \in B\left(T, \Omega_{1}\right)\right\}=N_{T}\left(\hat{\Omega}_{1}, \hat{\Omega}_{2}\right)
$$

Lemma 3.2. Let $k \in K$ and $k_{2} \in K^{g}$. Writing $k^{-1} k_{2} g=a_{r} n k_{0} \in A N K$, we have

$$
r=\beta_{k \xi}(y, x)
$$

Proof. Since $\xi=\lim _{t \rightarrow \infty} a_{-t} x$, we compute that

$$
\begin{aligned}
& \beta_{k \xi}(y, x)=\beta_{k \xi}\left(k_{2} y, x\right) \\
& =\beta_{\xi}\left(k^{-1} k_{2} y, x\right) \\
& =\lim _{t \rightarrow \infty} d\left(a_{r} n k_{0} x, a_{-t} x\right)-t \\
& =\lim _{t \rightarrow \infty} d\left(a_{r}\left(a_{t} n a_{-t}\right) a_{t} k_{0} x, x\right)-t=r
\end{aligned}
$$

For simplicity, we set $F_{\eta}^{ \pm}:=F_{\Omega_{2}, \eta}^{ \pm}$and $f_{\eta}^{ \pm}:=f_{\Omega_{2}, \eta}^{ \pm}$.

Lemma 3.3. We have

$$
\limsup _{\eta} m_{\Gamma}^{\mathrm{BR}}\left(F_{\eta}^{+}\right)=\liminf _{\eta} m_{\Gamma}^{\mathrm{BR}}\left(F_{\eta}^{-}\right)=\nu_{y}\left(\hat{\Omega}_{2}\right) .
$$

Proof. We use the formula for $\tilde{m}^{\mathrm{BR}}$ : for any $\Psi \in C_{c}(G)^{M}$,

$$
\widetilde{m}^{\mathrm{BR}}(\Psi)=\int_{K A N} \Psi\left(k a_{r} n\right) e^{-\delta r} d n d r d \nu_{x}(k(\xi))
$$

Define functions $\mathfrak{R}_{\epsilon}, \mathfrak{R}_{\epsilon}^{+}, \mathfrak{R}_{\epsilon}^{-}$on $G$ : for $h=a_{r} n k \in A N K$,

$$
\mathfrak{R}_{\epsilon}(h)=e^{-\delta_{\Gamma} r} \chi_{V_{\epsilon}}(k), \mathfrak{R}_{\epsilon}^{+}(h)=e^{-\delta_{\Gamma} r} \chi_{V_{\epsilon}^{+}}(k), \Re_{\epsilon}(h)=e^{-\delta_{\Gamma} r} \chi_{V_{\epsilon}^{-}}(k) .
$$

Note that

$$
\int_{B} \psi_{\epsilon}\left(b^{-1}\right) d \rho(b)=\int_{A N} \psi_{\epsilon}\left(g a_{t} n g^{-1}\right) e^{-(n-1) t} d t d n
$$

and hence

$$
e^{-(n-1) \epsilon} \leq \int_{B} \psi_{\epsilon}\left(b^{-1}\right) d \rho(b) \leq e^{(n-1) \epsilon}
$$

We then have

$$
\begin{aligned}
& m_{\Gamma}^{\mathrm{BR}}\left(F_{\eta}^{+} * \chi_{V_{\epsilon}}\right)=\widetilde{m}^{\mathrm{BR}}\left(f_{\eta}^{+} * \chi_{V_{\epsilon}}\right) \\
& =\int_{K A N} \int_{k_{1} \in V_{\epsilon}} f_{\eta}^{+}\left(k\left(a_{r} n k_{1}\right)\right) \chi_{V_{\epsilon}}\left(k_{1}\right) e^{-\delta r} d k_{1} d n d r d \nu_{x}(k(\xi)) \\
& =\int_{k \in K} \int_{h \in G} f_{\eta}^{+}(k h) \Re_{\epsilon}(h) d h d \nu_{x}(k(\xi)) \\
& =\int_{k \in K} \int_{h \in G} f_{\eta}^{+}(h) \Re_{\epsilon}\left(k^{-1} h\right) d h d \nu_{x}(k(\xi)) \\
& =\int_{k \in K} \int_{k_{2} \in K^{g}} \int_{b \in B} f_{\eta}^{ \pm}\left(k_{2} b^{-1} g\right) \Re_{\epsilon}\left(k^{-1} k_{2} b^{-1} g\right) d \rho(b) d k_{2} d \nu_{x}(k(\xi)) \\
& =\int_{k \in K} \int_{k_{2} \in K^{g}} \int_{b \in B} \phi_{\Omega_{2}, \epsilon(\eta)}^{+}\left(k_{2}\right) \psi_{\epsilon(\eta)}\left(b^{-1}\right) \Re_{\epsilon}\left(k^{-1} k_{2} b^{-1} g\right) d \rho(b) d k_{2} d \nu_{x}(k(\xi)) \\
& =(1+O(\eta)) \int_{k_{2} \in K^{g}} \int_{k \in K} \phi_{\Omega_{2}, \epsilon(\eta)}^{+}\left(k_{2}\right) \Re_{\epsilon}^{+}\left(k^{-1} k_{2} g\right) d k_{2} d \nu_{x}(k(\xi)) .
\end{aligned}
$$

For $h \in G$, define $\hat{k}_{h} \in K$ to be the unique element such that

$$
h \in B_{0} \hat{k}_{h}
$$

We note that

$$
\hat{k}_{k^{-1} k_{2} g}=\hat{k}_{k^{-1} g}\left(g^{-1} k_{2} g\right)
$$

Hence together with Lemma 3.2 ,

$$
\mathfrak{R}_{\epsilon}^{ \pm}\left(k^{-1} k_{2} g\right)=\chi_{V_{\epsilon}^{ \pm}}\left(\hat{k}_{k^{-1} g}\left(g^{-1} k_{2} g\right)\right) \cdot \epsilon^{-\delta_{\Gamma} \beta_{k \xi}(y, x)}
$$

Define functions $\tilde{\phi}_{\Omega_{2}, \epsilon}^{ \pm} \in C\left(K^{g}\right)^{M^{g}}$ by

$$
\tilde{\phi}_{\Omega_{2}, \epsilon}^{+}\left(k_{2}\right):=\sup _{k \in U_{\epsilon}^{+}} \phi_{\Omega_{2}, \epsilon}^{+}\left(k_{2} k\right) \quad \text { and } \quad \tilde{\phi}_{\Omega_{2}, \epsilon}^{-}\left(k_{2}\right):=\inf _{k \in U_{\epsilon}^{-}} \phi_{\Omega_{2}, \epsilon}^{-}\left(k_{2} k\right)
$$

Note that $0 \leq \tilde{\phi}_{\Omega_{2}, \epsilon}^{+} \leq 1$ vanishes outside $\Omega_{2} U_{\epsilon}^{+}$and is 1 on $\Omega_{2}$.
Therefore, using the conformal property of $\left\{\nu_{x}: x \in \mathbb{H}^{n}\right\}$ :

$$
e^{-\delta_{\Gamma} \beta_{k \xi}(y, x)} d \nu_{x}(k \xi)=d \nu_{y}(k \xi)
$$

we have

$$
\begin{aligned}
& \int_{k_{2} \in K^{g}} \int_{k \in K} \phi_{\Omega_{2}, \epsilon(\eta)}^{+}\left(k_{2}\right) \chi_{V_{\epsilon}^{+}}\left(\hat{k}_{k^{-1} g}\left(g^{-1} k_{2} g\right)\right) \cdot e^{-\delta_{\Gamma} \beta_{k \xi}(y, x)} d k_{2} d \nu_{x}(k(\xi)) \\
& =\int_{k_{2} \in K^{g}} \int_{k \in K} \phi_{\Omega_{2}, \epsilon(\eta)}^{+}\left(g \hat{k}_{k^{-1} g}^{-1} g^{-1} k_{2}\right) \chi_{V_{\epsilon}^{+}}\left(g^{-1} k_{2} g\right) d k_{2} d \nu_{y}(k(\xi)) \\
& \leq \int_{k_{2} \in K^{g}} \int_{k \in K} \tilde{\phi}_{\Omega_{2}, \epsilon(\eta)}^{+}\left(g \hat{k}_{k^{-1} g}^{-1} g^{-1}\right) \chi_{V_{\epsilon}^{+}}\left(g^{-1} k_{2} g\right) d k_{2} d \nu_{y}(k(\xi)) \\
& =(1+O(\eta)) \operatorname{Vol}\left(V_{\epsilon}\right) \int_{k \in K} \tilde{\phi}_{\Omega_{2}, \epsilon(\eta)}^{+}\left(g \hat{k}_{k^{-1} g}^{-1} g^{-1}\right) d \nu_{y}(k(\xi))
\end{aligned}
$$

Since $k B_{0}=\left(g \hat{k}_{k^{-1} g}^{-1} g^{-1}\right)\left(g B_{0}\right)$ and $B_{0}$ stabilizes $\xi$, we have

$$
\begin{equation*}
k(\xi)=\left(g \hat{k}_{k^{-1} g}^{-1} g^{-1}\right)(g \xi) \tag{3.5}
\end{equation*}
$$

Therefore we have

$$
\begin{aligned}
& m_{\Gamma}^{\mathrm{BR}}\left(F_{\eta}^{+} * \chi V_{\epsilon}\right) \\
& =(1+O(\eta)) \operatorname{Vol}\left(V_{\epsilon}\right) \int_{k^{\prime} \in K^{g}} \tilde{\phi}_{\Omega_{2}, \epsilon(\eta)}^{+}\left(k^{\prime}\right) d \nu_{y}\left(k^{\prime}(g(\xi))\right) \\
& \left.=(1+O(\eta)) \operatorname{Vol}\left(V_{\epsilon}\right) \nu_{y}\left(\Omega_{2}(g(\xi))\right) \quad \text { by } 3.4\right) .
\end{aligned}
$$

Hence we conclude

$$
\limsup _{\epsilon} \frac{m_{\Gamma}^{\mathrm{BR}}\left(F_{\eta}^{+} * \chi_{V_{\epsilon}}\right)}{\operatorname{Vol}\left(V_{\epsilon}\right)}=(1+O(\eta)) \nu_{y}\left(\Omega_{2}(g(\xi))\right)
$$

Similarly we can deduce

$$
\liminf _{\epsilon} \frac{m_{\Gamma}^{\mathrm{BR}}\left(F_{\eta}^{-} * \chi_{V_{\epsilon}}\right)}{\operatorname{Vol}\left(V_{\epsilon}\right)}=(1+O(\eta)) \nu_{y}\left(\Omega_{2}(g(\xi))\right)
$$

On the other hand, it is not hard to deduce from the continuity of $F_{\eta}^{ \pm}$that

$$
m_{\Gamma}^{\mathrm{BR}}\left(F_{\eta}^{ \pm}\right)=\lim _{\epsilon} \frac{m_{\Gamma}^{\mathrm{BR}}\left(F_{\eta}^{ \pm} * \chi_{V_{\epsilon}}\right)}{\operatorname{Vol}\left(V_{\epsilon}\right)}
$$

Hence $\lim \sup _{\eta} m_{\Gamma}^{\mathrm{BR}}\left(F_{\eta}^{+}\right)=\nu_{y}\left(\Omega_{2}(g(\xi))\right)=\liminf _{\eta} m_{\Gamma}^{\mathrm{BR}}\left(F_{\eta}^{-}\right)$.

## Proof of Theorem 1.1.

Since $\nu_{x}\left(\partial\left(\Omega_{1}\right)\right)=0$ and any circle with center in $\Lambda(\Gamma)$ has measure zero by Lemma 2.5. we may choose $V_{\epsilon}$ so that $\nu_{x}\left(\partial\left(\Omega_{1, \epsilon}^{+}\left(\xi_{x}\right)\right)\right)=\nu_{x}\left(\partial\left(\Omega_{1, \epsilon}^{-}\left(\xi_{x}\right)\right)\right)=0$.

By Lemma 3.1. Theorem 2.7 and Lemma 3.3, we have

$$
\begin{aligned}
& \limsup _{T} \frac{N_{T}^{C}\left(\hat{\Omega}_{1}, \hat{\Omega}_{2}\right)}{e^{\delta_{\Gamma} T}} \leq \limsup _{T, \eta} \frac{1}{e^{\delta_{\Gamma} T}} \int_{B_{0}\left(T+\epsilon, \Omega_{1, \epsilon}^{+}\right)} F_{\eta}^{+}\left(b_{0}\right) d \rho_{\ell}\left(b_{0}\right) \\
& =\underset{\eta}{\limsup } \frac{(1+O(\eta)) \nu_{x}\left(\Omega_{1}\left(\xi_{x}\right)\right)}{\delta_{\Gamma} \cdot\left|m_{\Gamma}^{\mathrm{BMS}}\right|} \cdot \limsup _{\eta} m_{\Gamma}^{\mathrm{BR}}\left(F_{\eta}^{+}\right) \\
& =\frac{\nu_{x}\left(\hat{\Omega}_{1}\right) \nu_{y}\left(\hat{\Omega}_{2}\right)}{\delta_{\Gamma} \cdot\left|m_{\Gamma}^{\mathrm{BMS}}\right|}
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
& \liminf _{T,} \frac{N_{T}\left(\hat{\Omega}_{1}, \hat{\Omega}_{2}\right)}{e^{\delta_{\Gamma} T}} \geq \liminf _{T, \eta} \frac{1}{e^{\delta_{\Gamma} T}} \int_{B_{0}\left(T-\epsilon, \Omega_{1, \epsilon}^{-}\right)} F_{\eta}^{-}\left(b_{0}\right) d \rho_{\ell}\left(b_{0}\right) \\
& =\liminf _{\eta} \frac{(1+O(\eta)) \nu_{x}\left(\Omega_{1}\left(\xi_{x}\right)\right)}{\delta_{\Gamma} \cdot\left|m_{\Gamma}^{\mathrm{BMS}}\right|} \cdot \liminf _{\eta} m_{\Gamma}^{\mathrm{BR}}\left(F_{\eta}^{-}\right) \\
& =\frac{\nu_{x}\left(\hat{\Omega}_{1}\right) \nu_{y}\left(\hat{\Omega}_{2}\right)}{\delta_{\Gamma} \cdot\left|m_{\Gamma}^{\mathrm{BMS}}\right|}
\end{aligned}
$$

Since $\left|N_{T}-N_{T}(C)\right| \leq \# \Gamma \cap K\left\{a_{t}: 0 \leq t \leq C\right\} K$ is a finite number independent of $T$, the above proves that

$$
N_{T}\left(\hat{\Omega}_{1}, \hat{\Omega}_{2}\right) \sim e^{\delta T} \cdot \frac{\nu_{x}\left(\hat{\Omega}_{1}\right) \nu_{y}\left(\hat{\Omega}_{2}\right)}{\delta_{\Gamma} \cdot\left|m_{\Gamma}^{\text {BMS }}\right|} .
$$

## A. Appendix by François Maucourant


#### Abstract

The purpose of this note is to show how one can recover a result in the spirit of Lim and Oh from a Theorem of Roblin. The following is part of the author's PhD Thesis [12, with some minor modifications, and some of these ideas have also been used in [9], but in the case of lattices in higher rank Lie groups.


Let $(X, d)$ be a CAT(-1) space, and $\Gamma$ a discrete, non-elementary subgroup of isometries of $X$. Denote by $\partial X$ the visual boundary of $X, \bar{X}=X \cup \partial X, \delta$ the critical exponent of $\Gamma$, which is assumed finite, $\left\{\nu_{x}\right\}$ the Patterson-Sullivan density for $\Gamma$, and $m_{\Gamma}^{\mathrm{BMS}}$ the associated Bowen-Margulis-Sullivan measure. We shall assume that the length spectrum is non-arithmetic, and that $m_{\Gamma}^{\mathrm{BMS}}$ is of finite mass; remark that all these hypotheses are satisfied in the case of geometrically finite groups on hyperbolic spaces. First, let us state Roblin's Theorem.

Theorem 1. [17, Theorem 4.1.1] Let $f$ a continuous function from $\bar{X}^{2}$ to $\mathbf{R}$, and $(x, y) \in X^{2}$. Then

$$
\lim _{T \rightarrow+\infty} \frac{\delta\left\|m_{\Gamma}^{\mathrm{BMS}}\right\|}{e^{\delta T}} \sum_{\gamma \in \Gamma, d(x, \gamma y) \leq T} f\left(\gamma y, \gamma^{-1} x\right)=\int_{\partial X^{2}} f(\xi, \eta) d \nu_{x}(\xi) d \nu_{y}(\eta)
$$

We shall prove here:
Theorem 2. Let $f$ be a continuous function on $\bar{X}^{2}$, and $\zeta \in \partial X$.

$$
\lim _{T \rightarrow+\infty} \frac{\delta\left\|m_{\Gamma}^{\mathrm{BMS}}\right\|}{e^{\delta T}} \sum_{\gamma \in \Gamma, d(x, \gamma y) \leq T} f\left(\gamma \zeta, \gamma^{-1} x\right)=\int_{\partial X^{2}} f(\xi, \eta) d \nu_{x}(\xi) d \nu_{y}(\eta)
$$

A simplified version (where $f$ does not depend on the second coordinate) appeared in [12. The argument divides into three steps: first, we show a quantitative estimate for the recurrence of the action of $\Gamma$ on the set of geodesics, of independent interest. Second, we show that the quantity on the left-hand side above does not depend too much on $\zeta$. Third, we integrate over $\zeta$ to be able to apply Roblin's Theorem.

## 1: At most linear recurrence on the set of geodesics

Define $\mathcal{G}=(\partial X)^{2}-$ diag to be the set of bi-infinite oriented geodesics on $X$, and let $S X$ be the set of isometric embedding of $\mathbf{R}$ to $X$. The geodesic flow $\left(g^{t}\right)_{t \in \mathbf{R}}$ is the time-shift $g^{t} f(s)=f(s+t)$, and the canonical projection $\pi: S X \rightarrow X$ is the map $\pi(f)=f(0)$. We shall make the usual identification

$$
S X=\mathcal{G} \times \mathbf{R}
$$

and this can be done in such a way that $\pi((\xi, \eta), 0))$ is the point of the geodesic from $\xi$ to $\eta$ closest to a fixed reference point $o \in X$. In such coordinates, the geodesic flow is just $g^{t}((\xi, \eta), s)=((\xi, \eta), s+t)$, whereas the action of $\Gamma$ on $S X$ defines a cocyle $c: \Gamma \times \mathcal{G} \rightarrow \mathbf{R}$, such that for any $((\xi, \eta), t) \in S X=\mathcal{G} \times \mathbf{R}$, we have

$$
\gamma((\xi, \eta), t)=((\gamma \xi, \gamma \eta), t+c(\gamma,(\xi, \eta)))
$$

Note that $|c(\gamma,(\xi, \eta))|$ is the distance between the projections of $o$ and $\gamma^{-1} o$ on the geodesic from $\xi$ to $\eta$, and recall (see [1, Corollary 5.6]) that in $\operatorname{CAT}(0)$
spaces, projection on a closed convex set is uniquely defined and 1-Lipschitz, so the following inequality holds for any $\gamma \in \Gamma,(\xi, \eta) \in \mathcal{G}$ :

$$
|c(\gamma,(\xi, \eta))| \leq d(o, \gamma o)
$$

Proposition 1. Let $K$ be a compact subset of $\mathcal{G}$, and $(x, y) \in X^{2}$. Then there exists $C_{K}>0$ and $T_{x, y}>0$ such that for any $(\xi, \eta) \in \mathcal{G}$, and any $T \geq T_{x, y}$,

$$
|\{\gamma \in \Gamma: d(\gamma x, y) \leq T,(\gamma \xi, \gamma \eta) \in K\}| \leq C_{K} T
$$

Proof. For $v \in \Gamma \backslash S X$, define

$$
f(\Gamma v)=\sum_{\gamma \in \Gamma} 1_{K \times[0,1]}(\gamma v)
$$

Since $\Gamma$ is discrete and acts properly on $S X$, and $K \times[0,1]$ is a compact subset of $S X$, it follows that $f$ is uniformly bounded by some constant $C_{0}$ depending only on $K$. Choose $T_{x, y}=d(o, x)+d(o, y)+1, T \geq T_{x, y}$ and let $\gamma \in \Gamma$ such that $(\gamma \xi, \gamma \eta) \in K$, and $d(y, \gamma x) \leq T$. Define $v=((\xi, \eta), 0)$, then $g^{-c(\gamma,(\xi, \eta))}(\gamma v) \in K \times\{0\}$. Thus,

$$
\int_{-c(\gamma,(\xi, \eta))}^{-c(\gamma,(\xi, \eta))+1} 1_{K \times[0,1]}\left(\gamma g^{t} v\right) d t=1
$$

and so, since $|c(\gamma,(\xi, \eta))| \leq d(o, \gamma o) \leq d(x, \gamma y)+d(o, x)+d(o, y) \leq T+T_{x, y}-1$,

$$
1 \leq \int_{-T-T_{x, y}}^{T+T_{x, y}} 1_{K \times[0,1]}\left(\gamma g^{t} v\right) d t
$$

Summing over all such $\gamma$, we obtain

$$
|\{\gamma \in \Gamma: d(\gamma x, y) \leq T,(\gamma \xi, \gamma \eta) \in K\}| \leq \int_{-T-T_{x, y}}^{T+T_{x, y}} f\left(g^{t} \Gamma v\right) d t
$$

and the right hand side is bounded by $2\left(T+T_{x, y}\right) C_{0} \leq 4 C_{0} T$.

## 2: Second and third steps

Let $f$ be a continuous function on $\bar{X}^{2}$. Define

$$
F(\zeta, x, T)=\frac{1}{\left|\Gamma x \cap B_{T}(y)\right|} \sum_{\gamma \in \Gamma, d(x, \gamma y) \leq T} f\left(\gamma \zeta, \gamma^{-1} x\right)
$$

Let $\epsilon>0$, then since $f$ is uniformly continuous, there exists a neighborhood $U$ of the diagonal in $\partial X^{2}$ such that for any $(\xi, \eta) \in U$ and any $z \in X,|f(\xi, z)-f(\eta, z)| \leq \epsilon$. Let $K$ be the complement of $U$, which is a compact subset of $\mathcal{G}$. So

$$
\begin{aligned}
\mid F(\xi, x, T)-F & (\eta, x, T) \left\lvert\, \leq \frac{1}{\left|\Gamma x \cap B_{T}(y)\right|}\right. \\
& *\left(\sum_{\gamma \in \Gamma, d(x, \gamma y) \leq T,(\gamma \xi, \gamma \eta) \in U} \epsilon+\sum_{\gamma \in \Gamma, d(x, \gamma y) \leq T,(\gamma \xi, \gamma \eta) \in K} 2\|f\|_{\infty}\right)
\end{aligned}
$$

By Proposition 1, the last sum contains at most $O(T)$ terms, so for sufficiently large T,

$$
|F(\xi, x, T)-F(\eta, x, T)| \leq 2 \epsilon
$$

This proves that $F(\zeta, x, T)$ does not depend too much on $\zeta$ for large $T$, so for any $\zeta$, its value is close to the integral with respect to any probability measure. Fix $y$, it will then be sufficient to prove that the function

$$
L(T, x, y)=\int_{\partial X} F(\zeta, x, T) \frac{d \nu_{y}(\zeta)}{\left\|\nu_{y}\right\|}
$$

has limit $\frac{1}{\left\|\nu_{x}\right\| \cdot\left\|\nu_{y}\right\|} \int f d \nu_{x} \nu_{y}$ as $T \rightarrow+\infty$; indeed, recall [17] that the orbital function satisfies

$$
\left|\Gamma x \cap B_{T}(y)\right| \sim \frac{\left\|\nu_{x}\right\| \cdot\left\|\nu_{y}\right\|}{\delta\left\|m_{\Gamma}^{\mathrm{BMS}}\right\|} e^{\delta T}
$$

Define the map $g$ for any $z \in \Gamma y$ and any $x \in \bar{X}$ by:

$$
g(z, x)=\frac{1}{\left\|\nu_{y}\right\|} \int_{\partial X} f(\zeta, x) d \nu_{z}(\zeta)
$$

and extend $g$ when $z$ is in the limit set $\Lambda_{\Gamma}$, to be equal to $f(z, x)$. Then $g$ is continuous on $\overline{\Gamma y} \times \bar{X}$. By Tietze-Urysohn's Theorem, $g$ can be extended to a continuous function, still denoted by $g$, on $\bar{X}^{2}$, and moreover $\int g d \nu_{x} d \nu_{y}=\int f d \nu_{x} d \nu_{y}$. Then

$$
\begin{aligned}
L(T, x, y)= & \frac{1}{\left|\Gamma x \cap B_{T}(y)\right| \cdot \| \nu_{y}| |} \sum_{\gamma \in \Gamma, d(x, \gamma y) \leq T} \int_{\partial X} f\left(\zeta, \gamma^{-1} x\right) d \nu_{\gamma y}(\zeta) \\
& =\frac{1}{\left|\Gamma x \cap B_{T}(y)\right|} \sum_{\gamma \in \Gamma, d(x, \gamma y) \leq T} g\left(\gamma y, \gamma^{-1} x\right)
\end{aligned}
$$

and by Roblin's Theorem applied to the function $g$, we conclude that $L(T, x, y)$ has limit $\frac{1}{\left\|\nu_{x}\right\| \cdot\left\|\nu_{y}\right\|} \int f d \nu_{x} d \nu_{y}$ as $T \rightarrow+\infty$, as desired.

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[^1]:    ${ }^{1}$ The convex core $\mathcal{C}_{\Gamma} \subset \Gamma \backslash \mathbb{H}^{n}$ is defined to be the minimal convex set which contains all geodesics connecting any two points in $\Lambda(\Gamma)$.

