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On the distribution of the divisor function in arithmetic progressions

by

YOICHI MOTOHASHI (Tokyo)

§ 1. Introduction.

1.1. In 1965 Bombieri ([3]) made a great improvement on the large sieve of Linnik and Rényi, by which he obtained an astounding result on the mean-value of the remainder-term in the prime number theorem for arithmetic progressions. He has proved the inequality

$$(1.1.1)\qquad \sum_{1\leqslant q\leqslant N^{1/2}(\log N)^{-B}} \max_{y\leqslant N}\max_{(a,q)=1}\Big|\sum_{\substack{n\equiv a \bmod q\\ 1\leqslant n\leqslant y}} \Lambda(n)-\frac{y}{\varphi(q)}\Big| \ll N(\log N)^{-A},$$

where $\Lambda(n)$ denotes von Mangold's function and $B = 4A+40$ with an arbitrary positive A.

Analogues ([13], [14]) to (1.1.1) has been found for the functions

$$(1.1.2)\qquad \tau_j(n)\ (j=2,3,4),\qquad \tau^2(n)\qquad \text{and}\qquad r^2(n)$$

instead of $\Lambda(n)$, where $\tau_j(n)$ and $r(n)$ are the numbers of representations of n as a product of j integers and as a sum of two squares of integers, respectively, and we put $\tau(n) = \tau_2(n)$. As in Bombieri's result, the large sieve plays very important part also in these cases.

Recently Siebert and Wolke ([16]) have extended these analogues to certain set of multiplicative functions, also using the large sieve.

1.2. Barban ([1], [2]) has found the inequality

$$(1.2.1)\qquad \sum_{1\leqslant q\leqslant N(\log N)^{-B}} \sum_{\substack{a=1\\ (a,q)=1}}^{q}\Big(\sum_{\substack{n\equiv a \bmod q\\ 1\leqslant n\leqslant N}} \Lambda(n)-\frac{N}{\varphi(q)}\Big)^2 \ll N^2(\log N)^{-A},$$

with $B = B(A)$.

The important point of Barban's result is that the range of q in (1.2.1) is substantially wider than in the estimation (1.1.1).

Another proofs of (1.2.1) have been obtained by Davenport and Halberstam ([4]) and Gallagher ([8]), using their improvements of the large sieve. Then Montgomery ([11]) has succeeded in replacing (1.2.1) by an asymptotic equality, without using the large sieve. A special case of his result is

$$(1.2.2)\qquad \sum_{1\leqslant q\leqslant N}\sum_{\substack{a=1\\(a,q)=1}}^{q}\Bigg(\sum_{\substack{n\equiv a \bmod q\\1\leqslant n\leqslant N}}\Lambda(n)-\frac{N}{\varphi(q)}\Bigg)^2 = N^2\log N+cN^2+O\left(\frac{N^2}{(\log N)^A}\right),$$

with a numerical constant C and an arbitrarily large A. In his proof a substantial part is played by a result of Lavrik ([10]) concerning twin primes on average.

1.3. Under these circumstances it may be expected that analogues to the asymptotic equality (1.2.2), in which $\Lambda(n)$ is replaced by one of the functions of (1.1.2) would be capable of a proof.

To prove such an asymptotic formula in the case of $\tau(n)$ the divisor function, also without using the large sieve, is the main purpose of the present paper.

We shall prove firstly an analogue of Lavrik's result with the help of the Hecke–Estermann function, which will be followed by some elaborated calculations of sums over various number-theoretical functions. Here it may be interesting to remark that our analogue to Lavrik's result can be seen as an average result concerning the remainder terms of Estermann's theorem ([7]) on the Ramanujan–Ingham sum ([9])

$$\sum_{1\leqslant n\leqslant N-k}\tau(n)\tau(n+k),$$

which has an interesting application ([5], [15]) to the problem of the distribution of the divisor function in "short" segments.

1.4. Notations.

$\tau(n)$: the number of divisors of an integer n.

$\varphi(n)$: Euler's function, i.e. the number of reduced classes mod$n$.

$\zeta(s)$: Riemann's zeta-function with the variable $s=\sigma+it$.

γ: Euler's constant $=0.5722\ldots$

(m,n): the greatest common divisor of m and n.

N: sufficiently large integer.

$\sigma_{-1}(n)=\sum_{d|n}\frac{1}{d}$, where d runs over all divisors of n.

$\sigma'_{-1}(n)=\sum_{d|n}\frac{\log d}{d}$.

$\sigma''_{-1}(n)=\sum_{d|n}\frac{\log^2 d}{d}$.

ε: arbitrarily small positive constant.

Throughout the present paper, all constants in O-terms or in Vinogradov's notation "$\ll$" depend on ε at most.

§ 2. The main result.

2.1. We formulate here our main result:

THEOREM. *We have the asymptotic equality, with numericals constant*

$$\mathfrak{S}_j\ (1\leqslant j\leqslant 4),$$

$$(2.1.1)\qquad \sum_{1\leqslant l\leqslant N}\sum_{b=1}^{l}\Big\{\sum_{\substack{n\equiv b \bmod l\\1\leqslant n\leqslant N}}\tau(n)-N\big(C_1(l,b)(\log N+2\gamma-1)-2C_2(l,b)\big)\Big\}^2 = N^2(\mathfrak{S}_1\log^3N+\mathfrak{S}_2\log^2N+\mathfrak{S}_3\log N+\mathfrak{S}_4)+O\big(N^{15/8}(\log N)^2\big),$$

where

$$(2.1.2)\qquad C_1(l,b)=\frac{1}{l}\sum_{q|l}\frac{\varphi(q)}{q\varphi\left(\frac{q}{(q,b)}\right)}\mu\left(\frac{q}{(q,b)}\right),\qquad C_2(l,b)=\frac{1}{l}\sum_{q|l}\frac{\varphi(q)\log q}{q\varphi\left(\frac{q}{(q,b)}\right)}\mu\left(\frac{q}{(q,b)}\right).$$

The constants $\mathfrak{S}_j$ $(1\leqslant j\leqslant 4)$ have complicated expressions, and so we give here the explicit values of $\mathfrak{S}_1$ and $\mathfrak{S}_2$ only, but it will be clear from our proof that one can obtain the values of $\mathfrak{S}_3$ and $\mathfrak{S}_4$ also. We have

$$\mathfrak{S}_1=\frac{1}{\pi^2},$$

$$\mathfrak{S}_2=(5\gamma-2)\frac{\pi^2}{6\zeta(3)}-\frac{5}{6}\pi^2\frac{\zeta'(3)}{\zeta^2(3)}+5\frac{\zeta'(2)}{\zeta(3)}+\frac{6}{\pi^2}\left(2\gamma-\frac{1}{2}-\frac{6}{\pi^2}\zeta'(2)\right).$$

2.2. In Section 4.1 it will be explained why we take the quantity

$$N\{C_1(l,b)(\log N+2\gamma-1)-2C_2(l,b)\}$$

as the expected value of the sum

$$\sum_{\substack{n\equiv b \bmod l\\1\leqslant n\leqslant N}}\tau(n).$$

Also we should remark that in (2.1.1) we do not put the condition $(b, l) = 1$. Actually the form (2.1.1) is the "natural" one for $\tau(n)$, and if we insert the condition $(b, l) = 1$, the calculations in what follows would be extremely complicated.

We have proved an asymptotic formula analogous to (2.1.1), in which $\tau(n)$ is replaced by $r(n)$. The procedure of the calculations is rather different in this case from that of this paper, and so we shall publish it elsewhere.

§ 3. An analogue to a result of Lavrik.

3.1. Let $D(\alpha, N)$ be the expression

$$\sum_{1\leqslant n\leqslant N} \tau(n) e^{2\pi i\alpha n},$$

then we have

$$D(\alpha, N)^2 = \sum_{|k|\leqslant N-1} V(k, N) e^{2\pi i\alpha k}, \tag{3.1.1}$$

where

$$V(k, N) = \sum_{1\leqslant n\leqslant N-|k|} \tau(n)\tau(n+|k|).$$

Let a/q be a term of the Farey series of order Ω, which is to be determined explicitly in Section 3.5.

Let us consider the size of $|D(\alpha, N)|$ with α near to a/q. We have

$$D(\alpha, N) = \sum_{1\leqslant n\leqslant N} \left\{D\left(\frac{a}{q}, n\right) - D\left(\frac{a}{q}, n-1\right)\right\} e^{2\pi i\left(\alpha - \frac{a}{q}\right)n}. \tag{3.1.2}$$

For $D(a/q, n)$ we appeal to

LEMMA 3.1.1 (Hecke–Estermann, [6]). *Let $E(s; a/q)$ be the function*

$$\sum_{n=1}^{\infty} \frac{\tau(n)}{n^s} e^{2\pi i\frac{a}{q}n} \qquad (s = \sigma + it,\ \sigma > 1).$$

Then, in the case of $(a, q) = 1$, it holds that

$$E\left(s; \frac{a}{q}\right) - q^{1-2s}\zeta^2(s)$$

is regular at every point, and moreover $E(s; a/q)$ has the functional equation

$$E\left(s; \frac{a}{q}\right) = 2(2\pi)^{2s-2}\Gamma^2(1-s)q^{1-2s}\left\{E\left(1-s; \frac{\bar a}{q}\right) - \cos\pi s E\left(1-s; -\frac{\bar a}{q}\right)\right\},$$

where $\bar a$ is defined by $a\bar a \equiv 1 \bmod q$.

Now we have

$$D\left(\frac{a}{q}, n\right) = \operatorname*{Res}_{s=1} E\left(s; \frac{a}{q}\right)\frac{n^s}{s} + \operatorname*{Res}_{s=0} E\left(s; \frac{a}{q}\right)\frac{n^s}{s} + \frac{1}{2\pi i}\int_{-\delta-iT}^{-\delta+iT} E\left(s; \frac{a}{q}\right)\frac{n^s}{s}\,ds + O\left\{\frac{n^{1+\varepsilon}}{T} + n^{\varepsilon} + \frac{1}{T}\int_{-\delta}^{1+\delta}\left|E\left(\sigma+iT; \frac{a}{q}\right)\right| n^{\sigma}\,d\sigma\right\}, \tag{3.1.3}$$

where $\delta = \{\log(qn+1)\}^{-1}$ and T is to be determined later. From the above lemma we see that

$$\operatorname*{Res}_{s=1} E\left(s; \frac{a}{q}\right)\frac{n^s}{s} = \operatorname*{Res}_{s=1} q^{1-2s}\zeta^2(s)\frac{n^s}{s} = \frac{1}{q}n(\log n + 2\gamma - 1 - 2\log q) \tag{3.1.4}$$

and

$$\left|\operatorname*{Res}_{s=0} E\left(s; \frac{a}{q}\right)\frac{n^s}{s}\right| \ll (\log(qn+1))^2. \tag{3.1.5}$$

Also we have, by the functional equation and the convexity argument,

$$\left|E\left(\sigma+iT; \frac{a}{q}\right)\right| \ll (Tq)^{1-\sigma}(\log qT)^4$$

uniformly for $-\delta \leqslant \sigma \leqslant 1+\delta$. Hence we get

$$\left|\frac{1}{2\pi i}\int_{-\delta-iT}^{-\delta+iT} E\left(s; \frac{a}{q}\right)\frac{n^s}{s}\,ds\right| \ll Tq(\log qT)^5 \tag{3.1.6}$$

and

$$\int_{-\delta}^{1+\delta}\left|E\left(\sigma+iT; \frac{a}{q}\right)\right| n^{\sigma}\,d\sigma \ll n(\log qT)^4\int_{-\delta}^{1+\delta}\left(\frac{Tq}{n}\right)^{1-\sigma} d\sigma. \tag{3.1.7}$$

Taking

$$T = \sqrt{\frac{n}{q}+1},$$

it follows from (3.1.3)–(3.1.7) that

$$D\left(\frac{a}{q}, n\right) = \frac{n}{q}(\log n + 2\gamma - 1 - 2\log q) + O\{(nq+q^2)^{1/2+\varepsilon}\}.$$

Using this formula, we have, from (3.1.2) and by the partial summation, LEMMA 3.1.2.

$$D(a,N)=\frac{1}{q}\sum_{1\le n\le N}(\log n+2\gamma-2\log q)e^{2\pi i\left(a-\frac{a}{q}\right)n}+$$
$$+O\left\{(Nq+q^2)^{1/2+\varepsilon}\left(1+\left|a-\frac{a}{q}\right|N\right)\right\}.$$

3.2. Let $F(a, a/q, N)$ be the quantity

$$\frac{1}{q}\sum_{1\le n\le N}(\log n+2\gamma-2\log q)e^{2\pi i\left(a-\frac{a}{q}\right)n},$$

then we have from Lemma 3.1.2

$$\left|D(a,N)-F\left(a,\frac{a}{q},N\right)\right|\ll(Nq+q^2)^{1/2+\varepsilon}\left(1+\left|a-\frac{a}{q}\right|N\right).\tag{3.2.1}$$

We now introduce the following quantity

$$G_\Delta(a,N)=\sum_{1\le q\le\Delta}\sum_{\substack{a=1\\(a,q)=1}}^{q}\left|F\left(a,\frac{a}{q},N\right)\right|^2,\tag{3.2.2}$$

where Δ is to be determined explicitly in Section 3.5 and meantime it is required to satisfy

$$4\Delta\le\Omega.\tag{3.2.3}$$

We have

$$G_\Delta(a,N)=\sum_{|k|\le N-1}e^{2\pi iak}\sum_{1\le q\le\Delta}\frac{1}{q^2}W_q(k,N)\sum_{\substack{a=1\\(a,q)=1}}^{q}e^{-2\pi i\frac{a}{q}k},\tag{3.2.4}$$

where

$$\begin{aligned}W_q(k,N)&=\sum_{1\le n\le N-|k|}\log n\log(n+|k|)+\\&\quad+2(\gamma-\log q)\sum_{1\le n\le N-|k|}\log n(n+|k|)+4(\gamma-\log q)^2(N-|k|)\\&=T_1(k,N)+2(\gamma-\log q)T_2(k,N)+4(\gamma-\log q)^2(N-|k|),\end{aligned}\tag{3.2.5}$$

say.

For the innermost sum in (3.2.4) we have

$$\sum_{\substack{a=1\\(a,q)=1}}^{q}e^{-2\pi i\frac{a}{q}k}=\frac{\varphi(q)}{\varphi\left(\frac{q}{(q,|k|)}\right)}\mu\left(\frac{q}{(q,|k|)}\right)=f(q,|k|),\tag{3.2.6}$$

say.

Thus we write (3.2.4) as

$$\begin{aligned}G_\Delta(a,N)&=\sum_{|k|\le N-1}e^{2\pi iak}\sum_{1\le q\le\Delta}\frac{f(q,k)}{q^2}W_q(k,N)\\&=\sum_{|k|\le N-1}S_\Delta(k,N)e^{2\pi iak},\end{aligned}\tag{3.2.7}$$

say.

Now we have from (3.1.1) and (3.2.7)

$$\sum_{|k|\le N-1}\big(V(k,N)-S_\Delta(k,N)\big)^2=\int_0^1\big||D(a,N)|^2-G_\Delta(a,N)\big|^2da.\tag{3.2.8}$$

3.3. In this section we shall estimate the integral of (3.2.8) by a simpler version of the trigonometrical method of I. M. Vinogradov along the line of Lavrik ([10]).

Let

$$\frac{a'}{q'},\ \frac{a}{q},\ \frac{a''}{q''}$$

be consecutive terms of the Farey series of order Ω, and let $C(a/q)$ be the interval $\left[\frac{a'+a}{q'+q},\frac{a+a''}{q+q''}\right]$. Then $C(a/q)$ contains a/q, and for its length $|C(a/q)|$ we have the inequality

$$\left|C\left(\frac{a}{q}\right)\right|\le\frac{2}{q\Omega}.\tag{3.3.1}$$

Now we have

$$\begin{aligned}U(N)&=\int_0^1\big||D(a)|^2-G_\Delta(a)\big|^2da\\&=\sum_{1\le q\le\Omega}\sum_{\substack{a=1\\(a,q)=1}}^{q}\int_{C(a/q)}\big||D(a)|^2-G_\Delta(a)\big|^2da\\&\le 2\sum_{1\le q\le\Omega}\sum_{\substack{a=1\\(a,q)=1}}^{q}\int_{C(a/q)}\left||D(a)|^2-\left|F\left(a,\frac{a}{q},N\right)\right|^2\right|^2da+\\&\quad+2\sum_{1\le q\le\Omega}\sum_{\substack{a=1\\(a,q)=1}}^{q}\int_{C(a/q)}\left|G_\Delta(a)-\left|F\left(a,\frac{a}{q},N\right)\right|^2\right|^2da\\&=2U_1(N)+2U_2(N),\end{aligned}\tag{3.3.2}$$

say.

Here we have from (3.2.1)

$$\left| |D(a, N)|^2 - \left|F\left(a, \frac{a}{q}, N\right)\right|^2 \right|^2 \ll (Nq+q^2)^{1+2\varepsilon}\left(1+\left|a-\frac{a}{q}\right|^2 N^2\right)\left(|D(a, N)|^2 + \left|F\left(a, \frac{a}{q}, N\right)\right|^2\right).$$

Thus for a in the interval $C(a/q)$ we have, by (3.3.1)

$$\left| |D(a, N)|^2 - \left|F\left(a, \frac{a}{q}, N\right)\right|^2 \right|^2 \ll \left\{N\Omega+\Omega^2+\frac{N^3}{\Omega^2}\right\}^{1+2\varepsilon}\left(|D(a, N)|^2 + \left|F\left(a, \frac{a}{q}, N\right)\right|^2\right),$$

from which we get

(3.3.3) $U_1(N)$

$$\ll \left\{N\Omega+\Omega^2+\frac{N^3}{\Omega^2}\right\}^{1+2\varepsilon}\left\{\int_0^1 |D(a, N)|^2 da + \sum_{1\leqslant q\leqslant\Omega}\sum_{\substack{a=1\\(a,q)=1}}^{q}\int_0^1 \left|F\left(a, \frac{a}{q}, N\right)\right|^2 da\right\}$$

$$\ll \left\{N\Omega+\Omega^2+\frac{N^3}{\Omega^2}\right\}^{1+2\varepsilon} N\log^3 N.$$

For $U_2(N)$ we have, by the definition of $G_\Delta(a, N)$

(3.3.4) $$U_2(N) \ll \sum_{1\leqslant q\leqslant\Omega}\sum_{\substack{a=1\\(a,q)=1}}^{q}\int_{C(a/q)}\left|\sum_{1\leqslant q'\leqslant\Delta}\sum_{\substack{a'=1\\(a',q')=1\\a'q\neq aq'}}^{q'}\left|F\left(a, \frac{a}{q}, N\right)\right|^2\right|^2 da +$$

$$+\sum_{\Delta<q\leqslant\Omega}\sum_{\substack{a=1\\(a,q)=1}}^{q}\int_{C(a/q)}\left|F\left(a, \frac{a}{q}, N\right)\right|^4 da$$

$$= U_3(N)+U_4(N),$$

say.

It is easy to see that

(3.3.5) $$U_4(N) \ll \frac{(N\log N)^4}{\Omega}\sum_{\Delta<q\leqslant\Omega}\frac{1}{q^4} \ll \frac{(N\log N)^4}{\Omega\Delta^3}.$$

For the estimation of $U_3(N)$, we remark that

$$F\left(a, \frac{a'}{q'}, N\right) = \frac{1}{q'}(\log N+2\gamma-2\log q')\sum_{1\leqslant n\leqslant N}e^{2\pi i\left(a-\frac{a'}{q'}\right)n} - $$

$$-\frac{1}{q'}\int_1^N \frac{1}{\xi}\left(\sum_{1\leqslant n\leqslant\xi}e^{2\pi i\left(a-\frac{a'}{q'}\right)n}\right)d\xi,$$

and thus we have

$$\left|F\left(a, \frac{a'}{q'}, N\right)\right| \ll \frac{\log N}{q'\left|\sin\pi\left(a-\frac{a'}{q'}\right)\right|}.$$

Now $F(a, a'/q', N)$ has the period 1 as a function of a, and the distance between a/q and one of $a'/q' \pm 1$ is not larger than $1/2$. Hence the sum $U_3(N)$ does not exceed the constant-multiple of the quantity

$$\Delta^2\log^4 N\sum_{1\leqslant q\leqslant\Omega}\sum_{\substack{a=1\\(a,q)=1}}^{q}\int_{C(a/q)}\sum_{1\leqslant q'\leqslant\Delta}\sum_{\substack{a'=-q'\\(a',q')=1\\0<\left|\frac{a'}{q'}-\frac{a}{q}\right|\leqslant\frac{1}{2}}}^{2q'}\frac{da}{q'^4\left|\sin\pi\left(a-\frac{a'}{q'}\right)\right|^4}.$$

But then it is easy to see that, for sufficiently large N, we have for a in $C(a/q)$

$$\frac{1}{2}\left|\frac{a}{q}-\frac{a'}{q'}\right| \leqslant \left|a-\frac{a'}{q'}\right| \leqslant \frac{3}{4},$$

since we have (3.2.3).

Thus we have

$$U_3(N) \ll \Omega^2\Delta^2\log^4 N\sum_{1\leqslant q\leqslant\Omega}\sum_{\substack{a=1\\(a,q)=1}}^{q}\sum_{1\leqslant q'\leqslant\Delta}\sum_{\substack{a'=-q'\\(a',q')=1\\a'q\neq aq'}}^{2q'}\frac{1}{|aq'-qa'|^4}$$

$$= \Omega^2\Delta^2\log^4 N\sum_{u=1}^{\infty}\frac{t(u)}{u^4},$$

where $t(u)$ is the number of the integral solutions of the equation

$$|aq'-qa'| = u$$

in the range of the summation. Obviously we have

$$t(u) \ll \Delta^2\Omega$$

which gives

(3.3.6) $$U_3(N) \ll \Delta^4\Omega^3\log^4 N.$$

Collecting (3.2.8) and (3.3.2)–(3.3.6), we get

LEMMA 3.3.1.

$$\sum_{0\leqslant k\leqslant N-1}\big(V(k,N)-S_\Delta(k,N)\big)^2 \ll N^{3\varepsilon}\Big(N^2\Omega+\Omega^2N+\frac{N^4}{\Omega^2}+\frac{N^4}{\Omega\Delta^3}+\Omega^3\Delta^4\Big).$$

3.4. Now let us enter into the calculation of $S_\Delta(k,N)$. From (3.2.7) and the expression (3.2.5) of $W_q(k,N)$, we see that for the calculation of $S_\Delta(k,N)$ we need to treat the following sums:

$$P_1(k,\Delta)=\sum_{1\leqslant q\leqslant\Delta}\frac{1}{q^2}f(q,k),$$

(3.4.1) $$P_2(k,\Delta)=\sum_{1\leqslant q\leqslant\Delta}\frac{\log q}{q^2}f(q,k),$$

$$P_3(k,\Delta)=\sum_{1\leqslant q\leqslant\Delta}\frac{\log^2 q}{q^2}f(q,k),$$

and

$$T_1(k,N),\qquad T_2(k,N).$$

Let $P_k(s)$ $(k\geqslant 0)$ be the function

$$\sum_{q=1}^{\infty}\frac{1}{q^{2+s}}f(q,k),$$

then, by the usual way of expressing the sum by the Euler product, we can easily find that

$$P_k(s)=\frac{\sigma_{-1}(s+1,k)}{\zeta(s+2)}\quad\text{ for }\quad k\neq 0$$

and

$$P_0(s)=\frac{\zeta(s+1)}{\zeta(s+2)},$$

where

$$\sigma_{-1}(s,k)=\sum_{d|k}d^{-s}.$$

Then without difficulty we can get, by the usual procedure of complex integration,

LEMMA 3.4.1. *For $k>0$ we have*

$$P_1(k,\Delta)=\frac{6}{\pi^2}\sigma_{-1}(k)+O\Big(\frac{\tau(k)}{\Delta}(\log\Delta)^{10}\Big),$$

$$P_2(k,\Delta)=\frac{6}{\pi^2}\sigma'_{-1}(k)+\frac{36}{\pi^4}\sigma_{-1}(k)\zeta'(2)+O\Big(\frac{\tau(k)}{\Delta}(\log\Delta)^{12}\Big),$$

$$P_3(k,\Delta)=\frac{6}{\pi^2}\sigma''_{-1}(k)+\frac{72}{\pi^4}\sigma'_{-1}(k)\zeta'(2)+\Big(\frac{432}{\pi^6}(\zeta'(2))^2-\frac{36}{\pi^4}\zeta''(2)\Big)\sigma_{-1}(k)+O\Big(\frac{\tau(k)}{\Delta}(\log\Delta)^{15}\Big).$$

From (3.2.5) and (3.2.7) and the above lemma, we obtain

LEMMA 3.4.2. *For $k>0$ we have*

$$S_\Delta(k,N)=w_1(k)T_1(k,N)+w_2(k)T_2(k,N)+w_3(k)(N-k)+O\Big(\frac{\tau(k)}{\Delta}N(\log N)^{17}\Big),$$

where

$$w_1(k)=\frac{6}{\pi^2}\sigma_{-1}(k),$$

$$w_2(k)=\frac{12}{\pi^2}\Big(\gamma-\frac{6}{\pi^2}\zeta'(2)\Big)\sigma_{-1}(k)-\frac{12}{\pi^2}\sigma'_{-1}(k),$$

$$w_3(k)=\Big(\frac{24}{\pi^2}\gamma^2-\frac{288}{\pi^4}\gamma\zeta'(2)+\frac{1728}{\pi^6}(\zeta'(2))^2-\frac{144}{\pi^4}\zeta''(2)\Big)\sigma_{-1}(k)+\Big(\frac{288}{\pi^4}\zeta'(2)-\frac{48}{\pi^2}\gamma\Big)\sigma'_{-1}(k)+\frac{24}{\pi^2}\sigma''_{-1}(k).$$

Now it is easy to see that for $k>0$

$$T_2(k,N)=(N-k)\log(N-k)+N\log N-k\log k-2(N-k)+O(\log N).$$

Thus by Lemma 3.4.2 we have, for $k>0$,

(3.4.2) $$T_2(k,N)w_2(k)$$
$$=\frac{12}{\pi^2}\Big(\gamma-\frac{6}{\pi^2}\zeta'(2)\Big)\{N\sigma_{-1}(k)\log(N-k)-k\sigma_{-1}(k)\log(N-k)+N(\log N-2)\sigma_{-1}(k)+2k\sigma_{-1}(k)-k\sigma_{-1}(k)\log k\}-$$
$$-\frac{12}{\pi^2}\{N\sigma'_{-1}(k)\log(N-k)-k\sigma'_{-1}(k)\log(N-k)+N(\log N-2)\sigma'_{-1}(k)+2k\sigma'_{-1}(k)-k\sigma'_{-1}(k)\log k\}+O\big(\tau(k)\log^2N\big).$$

For the sum $T_1(k,N)$ $(k>0)$ we have

$$T_1(k,N)=(N-k)\log N\log(N-k)-\int_1^{N-k}\xi\Big\{\frac{\log(\xi+k)}{\xi}+\frac{\log\xi}{\xi+k}\Big\}d\xi+O(\log^2N)$$

$$= N\log N\log(N-k)-(N-k)\big(\log(N-k)-2\big)-N\log N+k\log k-$$
$$-k\int_1^{N-k}\frac{\log(\xi+k)}{\xi}\,d\xi+O(\log^2 N).$$

The last integral is equal to

$$\sum_{m=0}^{\infty}k^m\int_1^{N-k}\frac{\log(\xi+k)}{(\xi+k)^{m+1}}\,d\xi$$
$$=\log N\log(N-k)+\tfrac12(\log^2 k-\log^2 N)+$$
$$+\sum_{m=1}^{\infty}\frac{1}{m^2}\left(\left(\frac{k}{N}\right)^m-\left(\frac{k}{k+1}\right)^m\right)+O\left(\frac{\log N}{k}\right)$$
$$=\log N\log(N-k)+\tfrac12(\log^2 k-\log^2 N)-\frac{\pi^2}{6}+\sum_{m=1}^{\infty}\frac{1}{m^2}\left(\frac{k}{N}\right)^m+O\left(\frac{\log N}{k}\right),$$

since we have

$$\sum_{m=1}^{\infty}\frac{1}{m^2}\left(\frac{k}{k+1}\right)^m=\frac{\pi^2}{6}+\sum_{1\leqslant m\leqslant k}\left\{\left(\frac{k}{1+k}\right)^m-1\right\}\frac{1}{m^2}+O\left(\frac1k\right)=\frac{\pi^2}{6}+O\left(\frac{\log k}{k}\right).$$

Hence we have

$$T_1(k,N)=(N-k)\log N\log(N-k)-(N-k)\big(\log(N-k)-2\big)+$$
$$+\frac{k}{2}(\log^2 N-\log^2 k)-N\log N+k\log k+\frac{\pi^2}{6}k-$$
$$-k\sum_{m=1}^{\infty}\frac{1}{m^2}\left(\frac{k}{N}\right)^m+O(\log^2 N).$$

Thus, from Lemma 3.4.2, we get, for $k>0$,

(3.4.3) $$T_1(k,N)w_1(k)=\frac{6}{\pi^2}N(\log N-1)\sigma_{-1}(k)\log(N-k)-$$
$$-\frac{6}{\pi^2}(\log N-1)k\sigma_{-1}(k)\log(N-k)-\frac{3}{\pi^2}\sigma_{-1}(k)k\log^2 k+$$
$$+\frac{6}{\pi^2}\sigma_{-1}(k)k\log k+\frac{6}{\pi^2}\left(\frac12\log^2 N+\frac{\pi^2}{6}-2\right)k\sigma_{-1}(k)-$$
$$-\frac{6}{\pi^2}N(\log N-2)\sigma_{-1}(k)-\frac{6}{\pi^2}k\sigma_{-1}(k)\sum_{m=1}^{\infty}\frac{1}{m^2}\left(\frac{k}{N}\right)^m+$$
$$+O\big(\tau(k)\log^2 N\big).$$

3.5. Now let $S^*(k,N)$ be the quantity

$$T_1(k,N)w_1(k)+T_2(k,N)w_2(k)+(N-k)w_3(k).$$

Then, from Lemmas 3.3.1 and 3.4.2, we get the inequality

$$\sum_{1\leqslant k\leqslant N-1}\big(V(k,N)-S^*(k,N)\big)^2$$
$$\ll N^{3\varepsilon}\left(N^2\Omega+\Omega^2N+\frac{N^4}{\Omega^2}+\frac{N^4}{\Omega\Delta^3}+\Omega^3\Delta^4+\frac{N^3}{\Delta^2}\right).$$

We now take, for example,

$$\Omega=N^{12/19},\quad \Delta=N^{4/19}.$$

Then the requirement (3.2.3) is satisfied, and we have proved

LEMMA 3.5.1. *The inequality*

$$\sum_{1\leqslant k\leqslant N-1}\big(V(k,N)-S^*(k,N)\big)^2\ll N^{11/4}$$

holds for sufficiently large N.

§ 4. Proof of the theorem.

4.1. Before entering into the detailed discussion of the left hand side of (2.1.1), we explain why the term

(4.1.1) $$N\{C_1(l,b)(\log N+2\gamma-1)-2C_2(l,b)\}$$

appears in our theorem:

The sum

(4.1.2) $$\sum_{\substack{n\equiv b \bmod l\\ 1\leqslant n\leqslant N}}\tau(n)$$

may be computed by the application of the usual complex integral procedure to the function

$$R(s;l,b)=\sum_{n\equiv b\bmod l}\frac{\tau(n)}{n^s}.$$

Now we have

$$R(s;l,b)=\frac1l\sum_{q|l}\sum_{\substack{a=1\\(a,q)=1}}^{q}e^{-2\pi i\frac{a}{q}b}E\left(s;\frac{a}{q}\right),$$

where $E(s;a/q)$ is the function defined in Lemma 3.1.1. Hence by the same lemma we see that the function

$$R(s;l,b)-\frac1l\Big\{\sum_{q|l}q^{1-2s}f(q,b)\Big\}\zeta^2(s),$$

with $f(q, b)$ of (3.2.6), is regular at every point. Therefore

$$\operatorname*{Res}_{s=1} R(s; l, b)\frac{N^s}{s}$$

is equal to the quantity (4.1.1), and we may expect that this is the main term of (4.1.2).

4.2. Now let $Q(N)$ denote the sum of the left side of (2.1.1). Then

$$
\begin{aligned}
(4.2.1)\qquad Q(N) = &\sum_{1\le l\le N}\ \sum_{\substack{n_1\equiv n_2 \bmod l\\ 1\le n_1, n_2\le N}} \tau(n_1)\tau(n_2) - \\
&-2N(\log N+2\gamma-1)\sum_{1\le l\le N}\ \sum_{1\le b\le l} C_1(l, b) \sum_{\substack{n\equiv b \bmod l\\ 1\le n\le N}} \tau(n) + \\
&+4N\sum_{1\le l\le N}\ \sum_{1\le b\le l} C_2(l, b) \sum_{\substack{n\equiv b \bmod l\\ 1\le n\le N}} \tau(n) + \\
&+N^2(\log N+2\gamma-1)^2 \sum_{1\le l\le N}\ \sum_{1\le b\le l} \big(C_1(l, b)\big)^2 - \\
&-4N^2(\log N+2\gamma-1)\sum_{1\le l\le N}\ \sum_{1\le b\le l} C_1(l, b)C_2(l, b) + \\
&+4N^2\sum_{1\le l\le N}\ \sum_{1\le b\le l}\big(C_2(l, b)\big)^2 \\
= {}&Q_1(N) - 2N(\log N+2\gamma-1)Q_2(N) + 4NQ_3(N) + \\
&+N^2(\log N+2\gamma-1)^2Q_4(N) - 4N^2(\log N+2\gamma-1)Q_5(N) + \\
&+4N^2Q_6(N),
\end{aligned}
$$

say.

4.3. In this section we compute the sums $Q_j(N)$ $(4\le j\le 6)$. For these sums we remark the following simple fact:

$$
(4.3.1)\qquad
\begin{aligned}
C_1(l, b) &= \frac{1}{l^2}\sum_{1\le d\le l}(l, d)\, e^{2\pi i \frac{d}{l} b}, \\
C_2(l, b) &= \frac{1}{l^2}\sum_{1\le d\le l}(l, d)\log\frac{1}{(l, d)}\, e^{2\pi i \frac{d}{l} b}.
\end{aligned}
$$

Thus, since $C_1(l, b)$ is real, we have

$$
\begin{aligned}
(4.3.2)\qquad Q_4(N) &= \sum_{1\le l\le N}\ \sum_{1\le b\le l}|C_1(l, b)|^2 \\
&= \sum_{1\le l\le N}\frac{1}{l^4}\sum_{1\le b\le l}\Big|\sum_{1\le d\le l}(l, d)\, e^{2\pi i \frac{d}{l} b}\Big|^2
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{1\le l\le N}\frac{1}{l^3}\sum_{1\le d\le l}(l, d)^2 \\
&= \sum_{1\le l\le N}\frac{1}{l^3}\sum_{q|l} q^2\varphi\Big(\frac{l}{q}\Big) = \sum_{1\le q\le N}\frac{\varphi(q)}{q^2}\sum_{\substack{1\le l\le N\\ q|l}}\frac{1}{l} \\
&= (\log N+\gamma)\sum_{q=1}^{\infty}\frac{\varphi(q)}{q^3} - \sum_{q=1}^{\infty}\frac{\varphi(q)}{q^3}\log q + O\Big(\frac{\log N}{N}\Big).
\end{aligned}
$$

On the same way we have

$$
(4.3.3)\qquad
\begin{aligned}
Q_5(N) &= \sum_{1\le l\le N}\frac{1}{l^3}\sum_{1\le d\le l}(l, d)^2\log\frac{l}{(l, d)} \\
&= (\log N+\gamma)\sum_{q=1}^{\infty}\frac{\varphi(q)}{q^3}\log q - \sum_{q=1}^{\infty}\frac{\varphi(q)}{q^3}\log^2 q + O\Big(\frac{\log^2 N}{N}\Big), \\
Q_6(N) &= \sum_{1\le l\le N}\frac{1}{l^3}\sum_{1\le d\le l}(l, d)^2\Big(\log\frac{l}{(l, d)}\Big)^2 \\
&= (\log N+\gamma)\sum_{q=1}^{\infty}\frac{\varphi(q)}{q^3}\log^2 q - \sum_{q=1}^{\infty}\frac{\varphi(q)}{q^3}\log^3 q + O\Big(\frac{\log^3 N}{N}\Big).
\end{aligned}
$$

Here we have

$$\sum_{n=1}^{\infty}\frac{\varphi(n)}{n^{s+3}} = \frac{\zeta(s+2)}{\zeta(s+3)},$$

and thus we get

$$
(4.3.4)\qquad
\begin{aligned}
\sum_{q=1}^{\infty}\frac{\varphi(q)}{q^3} &= \frac{\pi^2}{6\zeta(3)}, \\
\sum_{q=1}^{\infty}\frac{\varphi(q)}{q^3}\log q &= \frac{\pi^2\zeta'(3)}{6\zeta^2(3)} - \frac{\zeta'(2)}{\zeta(3)}.
\end{aligned}
$$

From (4.3.2)–(4.3.4) we obtain

LEMMA 4.3.1. *We have, with numerical constants* g_1 *and* g_2,

$$
\begin{aligned}
&N^2(\log N+2\gamma-1)^2Q_4(N) - 4N^2(\log N+2\gamma-1)Q_5(N) + 4N^2Q_6(N) \\
&= \frac{\pi^2}{6\zeta(3)}N^2\log^3 N + \Big\{(5\gamma-2)\frac{\pi^2}{6\zeta(3)} - \frac{5\pi^2\zeta'(3)}{6\zeta^2(3)} + 5\frac{\zeta'(2)}{\zeta(3)}\Big\}N^2\log^2 N + \\
&\qquad + g_1N^2\log N + g_2N^2 + O(N\log^4 N).
\end{aligned}
$$

4.4. We have

$$Q_2(N) = \sum_{1\le l\le N}\sum_{1\le b\le l} C_1(l,b)\sum_{\substack{n\equiv b \bmod l\\ 1\le n\le N}}\tau(n) = \sum_{1\le l\le N}\sum_{1\le n\le N}\tau(n)C_2(l,n).$$

Since we have (4.3.1). Also we have

$$Q_3(N) = \sum_{1\le l\le N}\sum_{1\le n\le N}\tau(n)C_2(l,n).$$

Then we have, by the definition of $C_1(l,n)$,

$$\begin{aligned}Q_2(N) &= \sum_{1\le l\le N}\frac{1}{l}\sum_{1\le n\le N}\tau(n)\sum_{q|l}\frac{\varphi(q)}{q\varphi\left(\frac{q}{(q,n)}\right)}\mu\left(\frac{q}{(q,n)}\right)\\ &= \sum_{1\le n\le N}\tau(n)\sum_{1\le q\le N}\frac{\varphi(q)}{q^2\varphi\left(\frac{q}{(q,n)}\right)}\mu\left(\frac{q}{(q,n)}\right)\left(\log\frac{N}{q}+\gamma+O\left(\frac{q}{N}\right)\right)\\ &= (\log N+\gamma)\sum_{1\le n\le N}P_1(n,N)\tau(n)-\sum_{1\le n\le N}P_2(n,N)\tau(n)+\\ &\qquad +\left(\frac{1}{N}\sum_{1\le n\le N}\tau(n)\sum_{1\le q\le N}\frac{\varphi(q)}{q\varphi\left(\frac{q}{(q,n)}\right)}\right),\end{aligned}$$

where $P_1(n,N)$ and $P_2(n,N)$ are defined by (3.4.1). The inner-most sum in the error-term is less than

$$\sum_{1\le q\le N}\frac{(q,n)}{q}\le\sum_{d|n}\sum_{\substack{1\le q\le N\\ d|q}}\frac{d}{q}\ll\tau(n)\log N.$$

Thus we have

$$Q_2(N) = (\log N+\gamma)\sum_{1\le n\le N}P_1(n,N)\tau(n)-\sum_{1\le n\le N}P_2(n,N)\tau(n)+ +O(\log^4 N), \tag{4.4.1}$$

and analogously

$$Q_3(N) = (\log N+\gamma)\sum_{1\le n\le N}P_2(n,N)\tau(n)-\sum_{1\le n\le N}P_3(n,N)\tau(n)+ +O(\log^5 N). \tag{4.4.2}$$

Hence, from Lemma 3.4.1 we get

LEMMA 4.4.1.

$$Q_2(N) = \frac{6}{\pi^2}\left(\log N+\gamma-\frac{6}{\pi^2}\zeta'(2)\right)\sum_{1\le n\le N}\tau(n)\sigma_{-1}(n)-\frac{6}{\pi^2}\sum_{1\le n\le N}\tau(n)\sigma'_{-1}(n)+ +O\left((\log N)^{16}\right),$$

$$\begin{aligned}Q_3(N) &= \frac{36}{\pi^4}\left(\zeta'(2)\log N+\gamma\zeta'(2)-\frac{12}{\pi^2}(\zeta'(2))^2+\zeta''(2)\right)\sum_{1\le n\le N}\tau(n)\sigma_{-1}(n)+\\ &\quad+\frac{6}{\pi^2}\left(\log N+\gamma-\frac{12}{\pi^2}\zeta'(2)\right)\sum_{1\le n\le N}\tau(n)\sigma'_{-1}(n)-\\ &\quad-\frac{6}{\pi^2}\sum_{1\le n\le N}\tau(n)\sigma''_{-1}(n)+O\left((\log N)^{18}\right).\end{aligned}$$

4.5. Now for $Q_1(N)$ we have

$$\begin{aligned}Q_1(N) &= 2\sum_{1\le l\le N-1}\sum_{1\le u\le (N-1)/l}\sum_{1\le n\le N-ul}\tau(n)\tau(n+ul)+N\sum_{1\le n\le N}\tau^2(n)\\ &= 2\sum_{1\le k\le N-1}V(k,N)\tau(k)+N\sum_{1\le n\le N}\tau^2(n).\end{aligned} \tag{4.5.1}$$

Here we have

$$\sum_{1\le n\le N}\tau^2(n) = \frac{1}{\pi^2}N\log^3 N+\frac{6}{\pi^2}\left(2\gamma-\frac{1}{2}-\frac{6}{\pi^2}\zeta'(2)\right)N\log^2 N+ +g_3 N\log N+g_4 N+O(N^{3/4}), \tag{4.5.2}$$

with numerical constants g_3 and g_4.

We have, using Lemma 3.5.1,

$$\begin{aligned}\sum_{1\le k\le N-1}V(k,N)\tau(k) &= \sum_{1\le k\le N-1}S^*(k,N)\tau(k)+\\ &\quad+O\left\{\left(\sum_{1\le k\le N-1}\tau^2(k)\right)^{1/2}\left(\sum_{1\le k\le N-1}\left(V(k,N)-S^*(k,N)\right)^2\right)^{1/2}\right\}\\ &= \sum_{1\le k\le N-1}S^*(k,N)\tau(k)+O(N^{15/8}\log^2 N).\end{aligned} \tag{4.5.3}$$

4.6. Thus we see from Lemma 3.4.2 and formulas (3.4.2) and (3.4.3) that in order to compute the last sum of (4.5.3) we need to compute the following 14 sums, and moreover these are enough also to compute $Q_2(N)$ and $Q_3(N)$, as is seen from Lemma 4.4.1. The required sums are

$$H_1(x) = \sum_{1\le k\le x} \tau(k)\sigma_{-1}(k),$$

$$H_2(x) = \sum_{1\le k\le x} \tau(k)\sigma'_{-1}(k),$$

$$H_3(x) = \sum_{1\le k\le x} \tau(k)\sigma''_{-1}(k),$$

$$H_4(x) = \sum_{1\le k\le x} k\tau(k)\sigma_{-1}(k),$$

$$H_5(x) = \sum_{1\le k\le x} k\tau(k)\sigma'_{-1}(k),$$

$$H_6(x) = \sum_{1\le k\le x} k\tau(k)\sigma''_{-1}(k),$$

(4.6.1) $$H_7(N) = \sum_{1\le k\le N-1} \tau(k)\sigma_{-1}(k)k\log k,$$

$$H_8(N) = \sum_{1\le k\le N-1} \tau(k)\sigma_{-1}(k)k\log^2 k,$$

$$H_9(N) = \sum_{1\le k\le N-1} \tau(k)\sigma'_{-1}(k)k\log k,$$

$$H_{10}(N) = \sum_{1\le k\le N-1} \tau(k)\sigma_{-1}(k)\log(N-k),$$

$$H_{11}(N) = \sum_{1\le k\le N-1} \tau(k)\sigma_{-1}(k)k\log(N-k),$$

$$H_{12}(N) = \sum_{1\le k\le N-1} \tau(k)\sigma'_{-1}(k)\log(N-k),$$

$$H_{13}(N) = \sum_{1\le k\le N-1} \tau(k)\sigma'_{-1}(k)k\log(N-k)$$

and

$$K(N) = \sum_{1\le k\le N-1} \tau(k)\sigma_{-1}(k)k\sum_{m=1}^{\infty}\frac{1}{m^2}\left(\frac{k}{N}\right)^m.$$

Here, before entering into the computations of these sums, we give usefull

LEMMA 4.6.1. *We have, uniformly for any integer d,*

$$\sum_{1\le m\le y}\tau(dm) = y\sum_{q|d}\frac{\varphi(q)}{q}(\log dy+2\gamma-1-2\log q)+O\left(\tau^2(d)y^{1/2}\log^2 y\right).$$

For the proof let us consider the function

$$\sum_{m=1}^{\infty}\frac{\tau(dm)}{m^s}.$$

It is easy to see that this function is equal to

$$\zeta^2(s)\prod_{p|d}\left(1+d_p\left(1-\frac{1}{ps}\right)\right),$$

where p runs over all prime factors of d and d_p denotes the exponent of p in the prime-power decomposition of d. Thus, by the argument of Section 4.1,

$$\zeta^2(s)\prod_{p|d}\left(1+d_p\left(1-\frac{1}{ps}\right)\right)-d^{s-1}\sum_{q|d}q^{1-2s}\zeta^2(s)\varphi(q)$$

is regular at every point. Hence we have

$$\sum_{1\le m\le y}\tau(dm) = \operatorname*{Res}_{s=1}\left\{d^{s-1}\sum_{q|d}\varphi(q)q^{1-2s}\zeta^2(s)\frac{y^s}{s}\right\}+$$

$$+\frac{1}{2\pi i}\int_{1/2-iT}^{1/2+iT}\zeta^2(s)\prod_{p|d}\left(1+d_p\left(1-\frac{1}{ps}\right)\right)\frac{y^s}{s}\,ds+$$

$$+O\left\{\frac{y^{1+\varepsilon}}{T}+y^{\varepsilon}+\frac{y^{\varepsilon}}{T}\int_{1/2}^{1}|\zeta(\sigma+iT)|^2\left|\prod_{p|d}\left(1+d_p\left(1-\frac{1}{ps}\right)\right)\right|y^{\sigma}\,d\sigma\right\}.$$

Since we have

$$\left|\prod_{p|d}\left(1+d_p\left(1-\frac{1}{ps}\right)\right)\right|\le\tau^2(d)\qquad(\tfrac{1}{2}\le\sigma),$$

$$\int_{-T}^{T}|\zeta^2(\tfrac{1}{2}+it)|\frac{dt}{|t|+1}\ll\log^3 T$$

and

$$|\zeta(\sigma+iT)| \ll T^{(1-\sigma)/3}\log^5 T;$$

taking $T = y$ we complete the proof.

4.7. We now compute the first sum of (4.6.1). We have

$$H_1(x) = \sum_{1\leqslant k\leqslant x}\tau(k)\sum_{d|k}\frac{1}{d} = \sum_{1\leqslant d\leqslant x}\frac{1}{d}\sum_{1\leqslant m\leqslant x/d}\tau(dm)$$

$$= \sum_{1\leqslant d\leqslant x}\frac{x}{d^2}\sum_{q|d}\frac{\varphi(q)}{q}(\log x+2\gamma-1-2\log q)+$$

$$+O\Big\{\sum_{1\leqslant d\leqslant x}\frac{x^{1/2}}{d^{3/2}}\tau^2(d)\log^2 x\Big\},$$

since we have Lemma 4.6.1. Thus we have

$$H_1(x) = x(\log x+2\gamma-1)\sum_{1\leqslant q\leqslant x}\frac{\varphi(q)}{q^3}\sum_{1\leqslant u\leqslant x/q}\frac{1}{u^2} -$$

$$-2x\sum_{1\leqslant q\leqslant x}\frac{\varphi(q)}{q^3}\log q\sum_{1\leqslant u\leqslant x/q}\frac{1}{u^2}+O(x^{1/2}\log^2 x)$$

$$= \frac{\pi^4}{36\zeta(3)}x(\log x+2\gamma-1)+\frac{\pi^2}{3}\left(\frac{\zeta'(2)}{\zeta(3)}-\frac{\pi^2}{6\zeta^2(3)}\zeta'(3)\right)x+$$

$$+O(x^{1/2}\log^2 x),$$

since we have (4.3.4).

Analogously we can treat $H_2(x)$ and $H_3(x)$, and we get

LEMMA 4.7.1.

$$H_1(x) = \lambda_1 x\log x+\lambda_2 x+O(x^{1/2}\log^2 x),$$
$$H_2(x) = \lambda_3 x\log x+\lambda_4 x+O(x^{1/2}\log^2 x),$$
$$H_3(x) = \lambda_5 x\log x+\lambda_6 x+O(x^{1/2}\log^2 x)$$

with numerical constants λ_j $(1\leqslant j\leqslant 6)$.

Here we have

$$\lambda_1 = \frac{\pi^4}{36\zeta(3)},$$

$$\lambda_2 = \frac{\pi^2}{6}\left\{(2\gamma-1)\frac{\pi^2}{6\zeta(3)}+2\frac{\zeta'(2)}{\zeta(3)}-\frac{\pi^2}{3\zeta^2(3)}\zeta'(3)\right\},$$

$$\lambda_3 = \frac{\pi^4\zeta'(3)}{36\zeta^2(3)}-\frac{\pi^2\zeta'(2)}{3\zeta(3)}.$$

From this result and by the partial summation we can prove easily

LEMMA 4.7.2.

$$H_4(x) = \frac{\lambda_1}{2}x^2\log x+\tfrac{1}{2}\left(\lambda_2+\frac{\lambda_1}{2}\right)x^2+O(x^{3/2}\log^2 x),$$

$$H_5(x) = \frac{\lambda_3}{2}x^2\log x+\tfrac{1}{2}\left(\lambda_4+\frac{\lambda_3}{2}\right)x^2+O(x^{3/2}\log^2 x),$$

$$H_6(x) = \frac{\lambda_5}{2}x^2\log x+\tfrac{1}{2}\left(\lambda_6+\frac{\lambda_5}{2}\right)x^2+O(x^{3/2}\log^2 x).$$

From this lemma and again by the partial summation we can prove easily

LEMMA 4.7.3.

$$H_7(N) = \frac{\lambda_1}{2}N^2\log^2 N+\frac{\lambda_2}{2}N^2\log N-\frac{\lambda_2}{4}N^2+O(N^{3/2}\log^2 N),$$

$$H_8(N) = \frac{\lambda_1}{2}N^2\log^3 N+\tfrac{1}{2}\left(\lambda_2-\frac{\lambda_1}{2}\right)N^2\log^2 N-\tfrac{1}{2}\left(\lambda_2-\frac{\lambda_1}{2}\right)N\log N+$$

$$+\tfrac{1}{4}\left(\lambda_2-\frac{\lambda_1}{2}\right)N^2+O(N^{3/2}\log^2 N),$$

$$H_9(N) = \frac{\lambda_3}{2}N^2\log^2 N+\frac{\lambda_4}{2}N^2\log N-\frac{\lambda_4}{4}N^2+O(N^{3/2}\log^2 N).$$

We now compute $H_j(N)$ $(10\leqslant j\leqslant 13)$. By Lemma 4.7.1 and the method of partial summation, we get

$$H_{10}(N) = \lambda_1\int_1^{N-1}\frac{\xi}{N-\xi}\log\xi\,d\xi+\lambda_2\int_1^{N-1}\frac{\xi}{N-\xi}\,d\xi+O(N^{1/2}\log^3 N)$$

$$= \lambda_1\sum_{m=1}^{\infty}\frac{1}{N^m}\int_1^{N-1}\xi^m\log\xi\,d\xi+\lambda_2 N(\log N-1)+O(N^{1/2}\log^3 N).$$

Here we have

$$\sum_{m=1}^{\infty}\frac{1}{N^m}\int_1^{N-1}\xi^m\log\xi\,d\xi = N\sum_{m=1}^{\infty}\frac{1}{m+1}\left(1-\frac{1}{N}\right)^{m+1}\log(N-1)-$$

$$-N\sum_{m=1}^{\infty}\frac{1}{(m+1)^2}\left(1-\frac{1}{N}\right)^{m+1}+O(1)$$

$$= N\left(\log^2 N-\log N-\frac{\pi^2}{6}+1\right)+O(\log N).$$

Analogously we can treat other sums, using Lemma 4.7.1 or Lemma 4.7.2. Hence we get

LEMMA 4.7.4.

$$H_{10}(N) = \lambda_1 N\log^2 N+(\lambda_2-\lambda_1)N\log N+\left(\lambda_1\left(1-\frac{\pi^2}{6}\right)-\lambda_2\right)N+ +O(N^{1/2}\log^3 N),$$

$$H_{11}(N) = \frac{\lambda_1}{2}N^2\log^2 N+\tfrac{1}{2}(\lambda_2-\lambda_1)N^2\log N+\left\{\frac{\lambda_1}{4}\left(1-\frac{\pi^2}{3}\right)-\frac{3}{4}\lambda_2\right\}N^2+ +O(N^{3/2}\log^3 N),$$

$$H_{12}(N) = \lambda_3 N\log^2 N+(\lambda_4-\lambda_3)N\log N+\left(\lambda_3\left(1-\frac{\pi^2}{6}\right)-\lambda_4\right)N+ +O(N^{1/2}\log^3 N),$$

$$H_{13}(N) = \frac{\lambda_3}{2}N^2\log^2 N+\tfrac{1}{2}(\lambda_4-\lambda_3)N^2\log N+\left\{\frac{\lambda_3}{4}\left(1-\frac{\pi^2}{3}\right)-\frac{3}{4}\lambda_4\right\}N^2+ +O(N^{3/2}\log^3 N).$$

Finally we compute the sum $K(N)$. We have

$$K(N) = \sum_{m=1}^{\infty}\frac{N^{-m}}{m^2}\sum_{1\leqslant k\leqslant N-1}k^{m+1}\tau(k)\sigma_{-1}(k).$$

By Lemma 4.7.1 we get

$$\sum_{1\leqslant k\leqslant N-1}k^{m+1}\tau(k)\sigma_{-1}(k)$$
$$=\frac{(N-1)^{m+2}}{m+2}\lambda_1\log(N-1)+\left\{\lambda_2\frac{1}{m+2}+\lambda_1\frac{m+1}{(m+2)^2}\right\}(N-1)^{m+2}+ +O\left(\frac{m+1}{m+3/2}(N-1)^{m+3/2}\log^2 N\right).$$

Thus we have

$$K(N) = \lambda_1 N^2\log N\sum_{m=1}^{\infty}\frac{1}{m^2(m+2)}+\lambda_2 N^2\sum_{m=1}^{\infty}\frac{1}{m^2(m+2)}+ +\lambda_1 N^2\sum_{m=1}^{\infty}\frac{m+1}{m^2(m+2)^2}+O(N^{3/2}\log^2 N),$$

which gives

LEMMA 4.7.5.

$$K(N) = \left(\frac{\pi^2}{12}-\frac{3}{8}\right)\lambda_1 N^2\log N+\left(\frac{5}{16}\lambda_1+\left(\frac{\pi^2}{12}-\frac{3}{8}\right)\lambda_2\right)N^2+ +O(N^{3/2}\log^2 N).$$

4.8. Now from Lemma 4.4.1 and Lemma 4.7.1 we have

$$(4.8.1)\quad Q_2(N) = \frac{6}{\pi^2}\left(\log N+\gamma-\frac{6}{\pi^2}\zeta'(2)\right)H_1(N)-\frac{6}{\pi^2}H_2(N)+O\big((\log N)^{15}\big)$$
$$= \frac{6}{\pi^2}\lambda_1 N\log^2 N+\frac{6}{\pi^2}\left\{\left(\gamma-\frac{6}{\pi^2}\zeta'(2)\right)\lambda_1+\lambda_2-\lambda_3\right\}N\log N+ +g_5 N+O(N^{1/2}\log^3 N),$$

and

$$(4.8.2)\quad Q_3(N) = \frac{36}{\pi^4}\left(\zeta'(2)\log N+\gamma\zeta'(2)-\frac{12}{\pi^2}(\zeta'(2))^2+\zeta''(2)\right)H_1(N)+ +\frac{6}{\pi^2}\left(\log N+\gamma-\frac{12}{\pi^2}\zeta'(2)\right)H_2(N)-\frac{6}{\pi^2}H_3(N)+O\big((\log N)^{18}\big)$$
$$= \frac{6}{\pi^2}\left(\frac{6}{\pi^2}\zeta'(2)\lambda_1+\lambda_3\right)N\log^2 N+g_6 N\log N+g_7 N+ +O(N^{1/2}\log^3 N),$$

with numerical constants g_5, g_6, g_7.

From (3.4.2) and Lemmas 4.7.1–4.7.4 we have

$$(4.8.3)\quad \sum_{1\leqslant k\leqslant N-1}\tau(k)T_2(k,N)w_2(k)$$
$$= \frac{12}{\pi^2}\left(\gamma-\frac{6}{\pi^2}\zeta'(2)\right)\{NH_{10}(N)-H_{11}(N)+N(\log N-2)H_1(N)+ +2H_4(N)-H_7(N)\}-$$
$$-\frac{12}{\pi^2}\{NH_{12}(N)-H_{13}(N)+N(\log N-2)H_2(N)+2H_5(N)-H_9(N)\}+ +O\big(N(\log N)^5\big)$$
$$= \frac{24}{\pi^2}\left\{\left(\gamma-\frac{6}{\pi^2}\zeta'(2)\right)\lambda_1-\lambda_3\right\}N^2\log^2 N+g_8 N^2\log N+g_9 N^2+ +O(N^{3/2}\log^5 N),$$

with numerical constants g_8 and g_9.

From (3.4.3) and Lemmas 4.7.1–4.7.5 we have

$$(4.8.4)\quad \sum_{1\leqslant k\leqslant N-1} \tau(k)T_1(k,N)w_1(k)$$
$$= \frac{6}{\pi^2}N(\log N-1)H_{10}(N)-\frac{6}{\pi^2}(\log N-1)H_{11}(N)-\frac{3}{\pi^2}H_9(N)+$$
$$+\frac{6}{\pi^2}H_7(N)+\frac{6}{\pi^2}\left(\frac{1}{2}\log^2 N+\frac{\pi^2}{6}-2\right)H_4(N)-\frac{6}{\pi^2}N(\log N-2)H_1(N)-$$
$$-\frac{6}{\pi^2}k(N)+O(N\log^5 N)$$
$$= \frac{3}{\pi^2}\lambda_1 N^2\log^3 N+\frac{6}{\pi^2}\left(\frac{\lambda_2}{2}-\frac{3}{2}\lambda_1\right)N^2\log^2 N+g_{10}N^2\log N+$$
$$+g_{11}N^2+O(N^{3/2}\log^5 N),$$

with numerical constants g_{10} and g_{11}.

Thus, from (4.8.3) and (4.8.4), we have

$$\sum_{1\leqslant k\leqslant N-1} \tau(k)S^*(k,N)$$
$$= \frac{3}{\pi^2}\lambda_1 N^2\log^3 N+\frac{6}{\pi^2}\left(4\left(\gamma-\frac{6}{\pi^2}\zeta'(2)-\frac{3}{8}\right)\lambda_1+\frac{\lambda_2}{2}-4\lambda_3\right)N^2\log^2 N+$$
$$+g_{12}N^2\log N+g_{13}N^2+O(N^{3/2}\log^5 N),$$

with numerical constants g_{12} and g_{13}. And, from this formula and (4.5.1)–(4.5.3), we get

$$Q_1(N) = \frac{1}{\pi^2}(1+6\lambda_1)N^2\log^3 N+$$
$$+\frac{6}{\pi^2}\left\{8\left(\gamma-\frac{6}{\pi^2}\zeta'(2)-\frac{3}{8}\right)\lambda_1+\lambda_2-8\lambda_3+2\gamma-\frac{1}{2}-\frac{6}{\pi^2}\zeta'(2)\right\}N^2\log^2 N+$$
$$+g_{14}N^2\log N+g_{15}N^2+O(N^{15/8}\log^2 N),$$

with numerical constants g_{14} and g_{15}.

Finally collecting the formulas (4.2.1), (4.8.1), (4.8.2), (4.8.5) and Lemma 4.3.1, we obtain

$$Q(N) = \left(\frac{\pi^2}{6\zeta(3)}-\frac{6}{\pi^2}\lambda_1+\frac{1}{\pi^2}\right)N^2\log^3 N+$$
$$+\left\{(5\gamma-2)\frac{\pi^2}{6\zeta(3)}-\frac{5\pi^2\zeta'(3)}{6\zeta^2(3)}+5\frac{\zeta'(2)}{\zeta(3)}+\frac{6}{\pi^2}\left(2\gamma-\frac{1}{2}-\frac{6}{\pi^2}\zeta'(2)\right)+\right.$$
$$\left.+\frac{12}{\pi^2}\left(\gamma-\frac{1}{2}-\frac{6}{\pi^2}\zeta'(2)\right)\lambda_1-\frac{6}{\pi^2}\lambda_2-\frac{12}{\pi^2}\lambda_3\right\}N^2\log^2 N+$$
$$+\mathfrak{S}_3N^2\log N+\mathfrak{S}_4N^2+O(N^{15/8}\log^2 N).$$

4.9. Inserting the values of λ_1, λ_2 and λ_3 from Lemma 4.7.1 into the right hand side of the above formula, we have

$$\frac{12}{\pi^2}\left(\gamma-\frac{1}{2}-\frac{6}{\pi^2}\zeta'(2)\right)\lambda_1-\frac{6}{\pi^2}\lambda_2-\frac{12}{\pi^2}\lambda_3 = 0,$$

by which we complete the proof of our theorem.

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MATHEMATICAL INSTITUTE OF THE HUNGARIAN ACADEMY OF SCIENCES
Budapest, Hungary

DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE AND ENGINEERING
NIHON UNIVERSITY, Tokyo, Japan

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