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ON THE DISTRIBUTION OF THE EIGENVALUES OF A CLASS OF INDEFINITE EIGENVALUE PROBLEMS

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Abstract. We prove detailed asymptotic estimates for the distribution of the eigenvalues of linear boundary eigenvalue problems of arbitrary order n with indefinite weight function generalizing well known results for the case $n = 2$.

1. Introduction. We consider eigenvalue problems of the form

$$\ell(y) = y^{(n)} + \sum_{\nu=2}^{n} f_\nu(x) y^{(n-\nu)} = \lambda r(x) y, \quad x \in [0,1] \tag{1.1}$$

$$U_\nu(y) = U_{\nu 0}(y) + U_{\nu 1}(y) = 0, \quad \nu = 1, \cdots, n, \tag{1.2}$$

where $r : [0,1] \to \mathbb{R}\backslash\{0\}$ is a step function; $f_\nu \in L[0,1]$, $2 \le \nu \le n$, and where the boundary conditions are normalized; the latter means that

$$\begin{aligned} U_{\nu 0}(y) &= \alpha_\nu y^{(k_\nu)}(0) + \sum_{\mu=0}^{k_\nu - 1} \alpha_{\nu\mu} y^{(\mu)}(0), \\ U_{\nu 1}(y) &= \beta_\nu y^{(k_\nu)}(1) + \sum_{\mu=0}^{k_\nu - 1} \beta_{\nu\mu} y^{(\mu)}(1), \end{aligned} \tag{1.3}$$

$$|\alpha_\nu| + |\beta_\nu| > 0 \quad \text{for} \quad \nu = 1, \cdots, n,$$

$$n - 1 \ge k_1 \ge k_2 \ge \cdots \ge k_n \ge 0 \quad \text{with} \quad k_\nu > k_{\nu+2} \quad \text{for} \quad \nu = 1, \cdots, n-2.$$

A central role in our paper is played by the assumption that the boundary conditions (1.2) are regular; cf. Definition 2 and Definition 7, where the definition of Birkhoff-regularity for definite problems (Naimark [12, p. 56]) is generalized in a natural

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manner. In section 3, we determine the distribution of the eigenvalues of regular problems (1.1), (1.2). Corresponding results have been obtained previously only for special classes of problems. Langer [10] has derived formulas for the case $\ell(y) := y''$ and $r(x) = (x - x_0)^a$, and Mingarelli [11] has shown that (1.1), (1.2) with $\ell(y) := y'' + qy$ and with separated boundary conditions has two sequences λ_n^+, λ_n^- of eigenvalues with the asymptotic distribution

$$\lambda_n \sim \mp n^2\pi^2 / \big(\int_0^1 \sqrt{r_\pm(t)}dt\big)^2, \tag{1.4}$$

($r_\pm(t) := \max\{\pm r(t), 0\}$). Fleckinger and Lapidus [8] and Faierman [7] have proved an asymptotic formula for the eigenvalues of the Laplacian with an indefinite weight function (compare [8] for further references).

2. Preliminaries. Let $m \in \mathbb{N}$, $0 = a_0 < a_1 < \cdots < a_{m+1} = 1$, $I_0 = [0, a_1]$, $I_\nu = (a_\nu, a_{\nu+1}]$, $1 \le \nu \le m$ and let the step function r be defined by $r(x) = r_\nu \in \mathbb{R}\backslash\{0\}$, $0 \le \nu \le m$. We assume that $k_0 := k_1 + \cdots + k_n$ is minimal with respect to all equivalent boundary conditions (1.2). By $V(x_1, \cdots, x_n)$ we denote the Vandermonde determinant of $x_1, \cdots, x_n \in \mathbb{C}$ and by $\omega_{\nu 1}, \cdots, \omega_{\nu n}$, $1 \le \nu \le m$, we denote the n-th roots of r_ν. Further we set $\lambda = \rho^n$ and we consider a fixed sector $S \in \{S_0, \cdots, S_{2n-1}\}$ where

$$S_\nu = \Big\{\rho \in \mathbb{C} \,\Big|\, \frac{\nu\pi}{n} \le \arg\rho \le \frac{(\nu+1)\pi}{n}\Big\}, \quad 0 \le \nu \le 2n-1.$$

We enumerate the n-th roots $\omega_{\nu j}$ of r_ν such that for $\rho \in S$

$$\operatorname{Re}(\rho\omega_{\nu 1}) \le \operatorname{Re}(\rho\omega_{\nu 2}) \le \cdots \le \operatorname{Re}(\rho\omega_{\nu n}), \quad 0 \le \nu \le m.$$

If $n = 2\mu$ we have $\operatorname{Re}(\rho\omega_{\nu j}) \le 0$ for $1 \le j \le \mu$ and $\operatorname{Re}(\rho\omega_{\nu j}) \ge 0$ for $\mu + 1 \le j \le n$. For $x \in I_\nu$, $0 \le \nu \le m$ and $\rho \in S$, (1.1) has a fundamental system $y_{\nu 1}(\cdot, \rho), \cdots, y_{\nu n}(\cdot, \rho)$ of solutions satisfying (cf. [12], §4.5)

$$y_{\nu j}^{(\alpha)}(x, \rho) := \big(\frac{\partial}{\partial x}\big)^\alpha y_{\nu j}(x, \rho) = (\rho\omega_{\nu j})^\alpha e^{\rho\omega_{\nu j}(x - a_\nu)}[1] \tag{2.1}$$

for $0 \le \alpha \le n-1$, $0 \le \nu \le m$, $1 \le j \le n$, $(x, \rho) \in I_\nu \times S$. Here and henceforth we use the abbreviation

$$[a] = a + O(1/\rho), \quad a \in \mathbb{C}, \quad \rho \to \infty.$$

For fixed x each function $y_{\nu j}(x, \cdot)$ is holomorphic in S. According to [12, p. 48], the asymptotic estimates (2.1) remain valid if we replace S by a translated sector $c + S$ with $c \in \mathbb{C}$.

3. The asymptotic distribution of the eigenvalues. $\lambda = \rho^\nu$ represents an eigenvalue of (1.1), (1.2) if and only if there exists a non trivial function $y(\cdot, \rho)$

$$y(x, \rho) = \sum_{\nu=0}^{m}\sum_{j=1}^{n} c_{\nu j}(\rho) y_{\nu j}(x, \rho)$$

satisfying

$$U_\nu(y) = 0, \quad 1 \le \nu \le n,$$

and

$$\Big(\frac{\partial}{\partial x}\Big)^\alpha \big[y(a_{\nu^+},\rho) - y(a_{\nu^-},\rho)\big] = 0, \quad 0 \le \alpha \le n-1, \quad 1 \le \nu \le m.$$

Therefore, $\lambda = \rho^n$, $\rho \in S$, represents an eigenvalue of (1.1), (1.2) if and only if ρ is a root of the characteristic determinant Δ :

$$\Delta(\rho) = \det \begin{bmatrix} D_{00} & \cdots & D_{0m} \\ \vdots & & \vdots \\ D_{m0} & \cdots & D_{mm} \end{bmatrix}$$

where

$$D_{00} = \begin{bmatrix} U_{10}(y_{01}) & \cdots & U_{10}(y_{0n}) \\ \vdots & & \vdots \\ U_{n0}(y_{01}) & \cdots & U_{n0}(y_{0n}) \end{bmatrix},$$

$$D_{0m} = \begin{bmatrix} U_{11}(y_{m1}) & \cdots & U_{11}(y_{mn}) \\ \vdots & & \vdots \\ U_{n1}(y_{m1}) & \cdots & U_{n1}(y_{mn}) \end{bmatrix},$$

$$D_{\nu+1,\nu} = -\begin{bmatrix} y_{\nu 1}(a_{\nu+1^-},\rho) & \cdots & y_{\nu n}(a_{\nu+1^-},\rho) \\ \vdots & & \vdots \\ y_{\nu 1}^{(n-1)}(a_{\nu+1^-},\rho) & \cdots & y_{\nu n}^{(n-1)}(a_{\nu+1^-},\rho) \end{bmatrix},$$

$$D_{\nu+1,\nu+1} = \begin{bmatrix} y_{\nu+1,1}(a_{\nu+1^+},\rho) & \cdots & y_{\nu+1,n}(a_{\nu+1^+},\rho) \\ \vdots & & \vdots \\ y_{\nu+1,1}^{(n-1)}(a_{\nu+1^+},\rho) & \cdots & y_{\nu+1,n}^{(n-1)}(a_{\nu+1^+},\rho) \end{bmatrix},$$

$D_{\nu j} = \Omega_{nn}$ for all remaining ν, j. From (2.1) we infer

$$D_{00} = \begin{bmatrix} [\alpha_1](\rho\omega_{01})^{k_1} & \cdots & [\alpha_1](\rho\omega_{0n})^{k_1} \\ \vdots & & \vdots \\ [\alpha_n](\rho\omega_{01})^{k_n} & \cdots & [\alpha_n](\rho\omega_{0n})^{k_n} \end{bmatrix},$$

$$D_{0m} = \begin{bmatrix} [\beta_1](\rho\omega_{m1})^{k_1} e^{\rho\omega_{m1}(1-a_m)} & \cdots & [\beta_1](\rho\omega_{mn})^{k_1} e^{\rho\omega_{mn}(1-a_m)} \\ \vdots & & \vdots \\ [\beta_n](\rho\omega_{m1})^{k_n} e^{\rho\omega_{m1}(1-a_m)} & \cdots & [\beta_n](\rho\omega_{mn})^{k_n} e^{\rho\omega_{mn}(1-a_m)} \end{bmatrix},$$

$$D_{\nu+1,\nu} = -\begin{bmatrix} e^{\rho\omega_{\nu 1}(a_{\nu+1}-a_\nu)}[1] & \cdots & e^{\rho\omega_{\nu n}(a_{\nu+1}-a_\nu)}[1] \\ \vdots & & \vdots \\ (\rho\omega_{\nu 1})^{n-1} e^{\rho\omega_{\nu 1}(a_{\nu+1}-a_\nu)}[1] & \cdots & (\rho\omega_{\nu n})^{n-1} e^{\rho\omega_{\nu n}(a_{\nu+1}-a_\nu)}[1] \end{bmatrix},$$

and

$$D_{\nu+1,\nu+1} = \begin{bmatrix} [1] & \cdots & [1] \\ \vdots & & \vdots \\ (\rho\omega_{\nu+1,1})^{n-1}[1] & \cdots & (\rho\omega_{\nu+1,n})^{n-1}[1] \end{bmatrix}.$$

For $1 \leq \sigma \leq n$ we introduce the notation

$$\theta(x_1, \cdots, x_\sigma; x_{\sigma+1}, \cdots, x_n) = \det \begin{bmatrix} \alpha_1 x_1^{k_1} & \cdots & \alpha_1 x_\sigma^{k_1} & \beta_1 x_{\sigma+1}^{k_1} & \cdots & \beta_1 x_n^{k_1} \\ \vdots & & \vdots & \vdots & & \vdots \\ \alpha_n x_1^{k_n} & \cdots & \alpha_n x_\sigma^{k_n} & \beta_n x_{\sigma+1}^{k_n} & \cdots & \beta_n x_n^{k_n} \end{bmatrix}$$

and we set

$$T_1 := \left\{ \tau = (t_0, \cdots, t_m) \in \{0,1\}^{m+1} \,\middle|\, t_\nu = 1, \text{ if } \operatorname{sgn} r_\nu = -1, \ 0 \leq \nu \leq m \right\}$$

$$T_2 := \left\{ \tau = (t_0, \cdots, t_m) \in \{0,1\}^{m+1} \,\middle|\, t_\nu = 1, \text{ if } \operatorname{sgn} r_\nu = 1, \ 0 \leq \nu \leq m \right\}.$$

Part I. Let $n = 2\mu$. Using the preceding estimates we develop $\Delta(\rho)$ with respect to the minors of the first n rows, subsequently we develop the corresponding complementary minors with respect to its first n rows etc.; this procedure yields for $\rho \in S\backslash\{0\}$

$$\Delta(\rho) = \rho^{k_0 + m\frac{n(n-1)}{2}} \Delta_1(\rho) \tag{3.1}$$

with

$$\begin{aligned} \Delta_1(\rho) = \sum_{\tau=(t_0,\cdots,t_m)\in T_1\cup T_2} & (-1)^{N(\tau)} \big[\theta(\omega_{01}, \cdots, \omega_{0,\mu-1}, \\ & \omega_{0,\mu+1-t_0}; \omega_{m,\mu+t_m}, \omega_{m,\mu+2}, \cdots, \omega_{mn})\big] e^{\rho E_\tau^1} \\ & \times \prod_{k=0}^{m-1} V(\omega_{k,\mu+t_k}, \omega_{k,\mu+2}, \cdots, \omega_{kn}, \omega_{k+1,1}, \cdots, \omega_{k+1,\mu-1}, \omega_{k+1,\mu+1-t_{k+1}}) \\ =: & \sum_{\tau\in T_1\cup T_2} (-1)^{N(\tau)} [\theta_\tau^1] e^{\rho E_\tau^1} V_\tau^1, \end{aligned} \tag{3.2}$$

where $N(\tau) \in \mathbb{N}$ for $\tau \in T_1 \cup T_2$ and where

$$E_\tau^1 := \sum_{k=0}^{m} (a_{k+1} - a_k)\left(\omega_{k,\mu+t_k} + \textstyle\sum_{j=\mu+2}^{n} \omega_{k_j}\right).$$

The analogous result holds for sectors of the form $S+c$, $c \in \mathbb{C}$. We note that only the terms with maximal growth for $|\rho| \to \infty$, $\rho \in S$, have been included explicitly within the sum (3.2); all remaining terms are subsummed within the square brackets.

In the next part of the section we assume that $S := S_0$— this implies that the enumeration of the numbers ω_{kj} and the definition of θ_τ^1, V_τ^1, E_τ^1 is determined accordingly. Further we assume that $n = 4k$— in the case $n = 4k + 2$ we can proceed similarly.

With $\tau_0 := (1, \cdots, 1)$ we obtain

$$\operatorname{Re}(\rho E^1_{\tau_0}) > \operatorname{Re}(\rho E^1_\tau) \quad \text{for } \tau \in (T_1 \cup T_2)\backslash\{\tau_0\} \text{ and } \rho \in \overset{\circ}{S}_0.$$

On the boundaries of S_0 the real parts of several terms ρE^1_τ, $\tau \in T_1$ (or $\tau \in T_2$) are equal, since

$$\operatorname{Re}(\rho\omega_{k\mu}) = \operatorname{Re}(\rho\omega_{k,\mu+1}) = 0 \quad \text{if } \arg\rho = 0 \text{ and } r_k > 0$$

and

$$\operatorname{Re}(\rho\omega_{k\mu}) = \operatorname{Re}(\rho\omega_{k,\mu+1}) = 0 \quad \text{if } \arg\rho = \frac{\pi}{n} \text{ and } r_k < 0.$$

Let $\tau_j = (t^j_0, t^j_1, \cdots, t^j_m) \in T_j$, $j = 1, 2$ be defined by

$$t^1_\nu := 0 \quad \text{if } \operatorname{sign} r_\nu = 1,$$
$$t^2_\nu := 0 \quad \text{if } \operatorname{sign} r_\nu = -1,$$

then we easily infer

$$\operatorname{Re}(\rho E^1_{\tau_1}) > \operatorname{Re}(\rho E^1_\tau) \quad \text{for } \rho \in \overset{\circ}{S}_{2n-1} \text{ and } \tau \in (T_1 \cup T_2)\backslash\{\tau_1\},$$
$$\operatorname{Re}(\rho E^1_{\tau_2}) > \operatorname{Re}(\rho E^1_\tau) \quad \text{for } \rho \in \overset{\circ}{S}_1 \text{ and } \tau \in (T_1 \cup T_2)\backslash\{\tau_2\},$$

and with $0 < \epsilon < \frac{\pi}{2n}$ we have

$$\operatorname{Re}(\rho E^1_\tau) > \operatorname{Re}(\rho E^1_\sigma) \quad \text{for } \tau \in T_1,\ \sigma \in T_2\backslash\{\tau_0\},\ -\epsilon < \arg\rho < \epsilon$$
$$\operatorname{Re}(\rho E^1_\tau) < \operatorname{Re}(\rho E^1_\sigma) \quad \text{for } \tau \in T_1\backslash\{\tau_0\},\ \sigma \in T_2,\ \frac{\pi}{n} - \epsilon < \arg\rho < \frac{\pi}{n} + \epsilon.$$

Hence, almost all eigenvalues λ_k of (1.1), (1.2) are contained in $\{\lambda \in \mathbb{C} \mid |\arg\lambda| < \epsilon \text{ or } |\pi - \arg\lambda| < \epsilon\}$.

Using the method described in [12, §4] we infer from the preceding estimates that the n-th roots $\rho_k = \lambda_k^{1/n}$ of the eigenvalues are staisfying one of the equations

$$\sum_{\tau\in T_1} (-1)^{N(\tau)}\theta^1_\tau V^1_\tau e^{\rho E^1_\tau}[1] = 0 \quad \text{if } \; -\epsilon \le \arg\rho \le \epsilon$$

or (3.3)

$$\sum_{\tau\in T_2} (-1)^{N(\tau)}\theta^1_\tau V^1_\tau e^{\rho E^1_\tau}[1] = 0 \quad \text{if } \; \pi - \epsilon \le \arg\rho \le \pi + \epsilon$$

$(0 < \epsilon < \frac{\pi}{2n}$ and $n = 4\kappa)$.

Dividing the sums (3.3) by $\exp\{\rho \sum_{k=0}^m (a_{k+1} - a_k) \sum_{j=\mu+2}^n \omega_{kj}\}$ we get exponential sums of the form

$$e^{\rho c} \sum_{k=0}^{m_j} c^{(j)}_k e^{i\rho\alpha^{(j)}_k \beta}[1] = 0, \quad j = 1, 2, \quad c \in \mathbb{C}, \tag{3.4}$$

where $\beta \in \{e^{-\frac{i\pi}{n}}, 1, e^{\frac{i\pi}{n}}\}$, $\alpha_0^{(j)} < \alpha_1^{(j)} < \cdots < \alpha_{m_j}^{(j)}$ and $c_k^{(j)} \in \mathbb{C}$ for $j = 1, 2$.

For example we have for $n = 4\kappa$ and $j = 1$

$$\begin{aligned} \alpha_{m_1}^{(1)} - \alpha_0^{(1)} &= -i(E_{\tau_1}^1 - E_{\tau_0}^1) = -i \sum_{\substack{k=0 \\ r_k>0}}^{m} (a_{k+1} - a_k)(\omega_{k\mu} - \omega_{k,\mu+1}) \\ &= -i \sum_{\substack{k=0 \\ r_k>0}}^{m} (a_{k+1} - a_k) r_k^{1/n} 2i = 2R_+, \end{aligned} \tag{3.5}$$

and for $n = 4\kappa$ and $j = 2$ we get

$$\begin{aligned} \alpha_{m_2}^{(2)} - \alpha_0^{(2)} &= -ie^{i\frac{\pi}{n}}(E_{\tau_0}^1 - E_{\tau_2}^1) \\ &= -ie^{i\frac{\pi}{n}} \sum_{\substack{k=0 \\ r_k<0}}^{m} (a_{k+1} - a_k)(\omega_{k,\mu+1} - \omega_{k\mu}) \\ &= -ie^{i\frac{\pi}{n}} 2ie^{-i\frac{\pi}{n}} \sum_{\substack{k=0 \\ r_k<0}}^{m} (a_{k+1} - a_k) r_k^{1/n} = 2R_-. \end{aligned} \tag{3.6}$$

In the case $n = 4\kappa + 2$ we obtain similar formulas but the role of R_+ and R_- has to be interchanged.

The coefficients $c_0^{(j)}$, $c_{m_j}^{(j)}$ of the relevant exponential terms of (3.4) can be determined explicitly. On account of (3.2) there are nonvanishing constants $k_\alpha^{(i)}$ defined by products of Vandermonde determinants, such that the following relations hold:

(i) For $n = 2\mu$ we get

a) If $r_0 > 0$, $r_m > 0$ (3.7)

$$\begin{aligned} c_0^{(1)} &= \theta(\omega_{01}, \cdots, \omega_{0\mu}; \omega_{m\mu+1}, \cdots, \omega_{mn}) k_1^{(1)} =: k_1^{(1)}\theta_1, \\ c_{m_1}^{(1)} &= \theta(\omega_{01}, \cdots, \omega_{0\mu-1}, \omega_{0\mu+1}; \omega_{m\mu}, \omega_{m\mu+2}, \cdots, \omega_{mn}) k_2^{(1)} =: k_2^{(1)}\theta_2, \\ c_0^{(2)} &= k_3^{(2)}\theta_1, \quad c_{m_2}^{(2)} = c_0^{(1)}, \end{aligned}$$

b) if $r_0 > 0$, $r_m < 0$

$$\begin{aligned} c_0^{(1)} &= k_4^{(1)}\theta_1, \quad c_{m_1}^{(1)} = \theta(\omega_{01}, \cdots, \omega_{0\mu-1}, \omega_{0\mu+1}; \omega_{m\mu+1}, \cdots, \omega_{mn}) k_5^{(1)} =: k_5^{(1)}\theta_3 \\ c_0^{(2)} &= \theta(\omega_{01}, \cdots, \omega_{0\mu}; \omega_{m\mu}, \omega_{m\mu+2}, \cdots, \omega_{mn}) k_6^{(2)} =: k_6^{(2)}\theta_4, \quad c_{m_2}^{(2)} = c_0^{(1)}, \end{aligned}$$

c) if $r_0 < 0$, $r_m > 0$

$$c_0^{(1)} = k_7^{(1)}\theta_1, \quad c_{m_1}^{(1)} = k_8^{(1)}\theta_4, \quad c_0^{(2)} = k_9^{(2)}\theta_3, \quad c_{m_2}^{(2)} = c_0^{(1)},$$

d) if $r_0 < 0$, $r_m < 0$

$$c_0^{(1)} = k_{10}^{(1)}\theta_1, \quad c_{m_1}^{(1)} = k_{11}^{(1)}\theta_1, \quad c_0^{(2)} = k_{12}^{(2)}\theta_2, \quad c_{m_2}^{(2)} = c_0^{(1)}.$$

(ii) For $n = 4\kappa + 2$ we get

a) if $r_0 > 0$, $r_m > 0$

$$c_0^{(1)} = k_1^{(1)}\theta_1, \quad c_{m_1}^{(1)} = k_2^{(1)}\theta_1, \quad c_0^{(2)} = k_3^{(2)}\theta_2, \quad c_{m_2}^{(2)} = c_0^{(1)},$$

b) if $r_m < 0 < r_0$

$$c_0^{(1)} = k_4^{(1)}\theta_1, \quad c_{m_1}^{(1)} = k_5^{(1)}\theta_4, \quad c_0^{(2)} = k_6^{(2)}\theta_3, \quad c_{m_2}^{(2)} = c_0^{(1)},$$

c) if $r_0 < 0 < r_m$

$$c_0^{(1)} = k_7^{(1)}\theta_1, \quad c_{m_1}^{(1)} = k_8^{(1)}\theta_3, \quad c_0^{(2)} = k_9^{(2)}\theta_4, \quad c_{m_2}^{(2)} = c_0^{(1)},$$

d) if $r_0 < 0$, $r_m < 0$

$$c_0^{(1)} = k_{10}^{(1)}\theta_1, \quad c_{m_1}^{(1)} = k_{11}^{(1)}\theta_2, \quad c_0^{(2)} = k_{12}^{(2)}\theta_1, \quad c_{m_2}^{(1)} = c_0^{(1)}.$$

Remark 1. The determinants θ_i in (3.5) are not independent. Compare [12, p. 59] and [13]. Substituting $\phi_0 = \frac{2\pi}{n}k_0$ where $k_0 = k_1 + \cdots + k_n$ we get

(i) For a) $\theta_2 = \pm\theta_1 e^{i\phi_0}$, for b) $\theta_3 = \pm\theta_4 e^{i\phi_0}$, for c) $\theta_4 = \pm\theta_3 e^{i\phi_0}$ and for d) $\theta_1 = \pm\theta_2 e^{i\phi_0}$. The proofs are analogous to [12, p. 60] or [13, p. 11].

(ii) Let additionally $\alpha_i, \beta_i \in \mathbb{R}$, $1 \le i \le n$, then we get for b) $\theta_3 = \pm\bar{\theta}_1$ and for c) $\theta_4 = \pm\bar{\theta}_1$.

Definition 2. *For $n = 2\mu$ (1.1), (1.2) is called regular if $r_0 r_m > 0$ and $\theta_1 \neq 0$ or $r_0 r_m < 0$ and $\theta_1 \neq 0 \neq \theta_3$.*

Remark 3. (i) For α_i, $\beta_i \in \mathbb{R}$, $1 \le i \le k$, the assumption $\theta_1 \neq 0$ is sufficient for the regularity of (1.1), (1.2). If the boundary conditions (1.2) are separated and therefore Birkhoff-regular in the sense of [12, §4], then the determinants θ_i in (3.7) are products of two nonvanishing determinants (cf. [12, p. 96]), and (1.1), (1.2) is regular.

(ii) Definition 2 is independent of the sectors used in the definition of $\theta_1, \cdots, \theta_4$.

We define

$$R_+ := \sum_{\substack{k=0 \\ r_k > 0}}^{m} (a_{k+1} - a_k)|r_k|^{1/n} = \int_0^1 \sqrt[n]{r_+(t)}\, dt,$$

$$R_- := \sum_{\substack{k=0 \\ r_k < 0}}^{m} (a_{k+1} - a_k)|r_k|^{1/n} = \int_0^1 \sqrt[n]{r_-(t)}\, dt,$$

and we assume without loss of generality $R_+ \ge R_- > 0$ (otherwise we substitute $\lambda \to -\lambda$).

The distribution of the zeros of exponential sums is well-known; if we have exponential sums of the special form (3.4) we can use for example the following lemma which results from [14, pp. 25–28].

Lemma 4. *Let $n_1 < n_2 < \cdots < n_p$ and $c_\nu \in \mathbb{C}$, $1 \le \nu \le p$ with $c_1 \neq 0 \neq c_p$. Then the zeros ρ_k of*

$$f : \mathbb{C} \to \mathbb{C}, \quad \rho \longmapsto \sum_{\nu=1}^{p} [c_\nu] e^{i n_\nu \rho}$$

are fulfilling the asymptotic estimates

$$\rho_k = \frac{2k\pi}{n_p - n_1}\left[1 + O(\frac{1}{|k|})\right], \qquad k \in \mathbb{Z}\backslash\{0\}.$$

From (3.6) and the preceding results we infer

Theorem 5. *For $n = 2\mu$ every regular eigenvalue problem* (1.1), (1.2) *has two sequences $(\lambda_k^{(j)})_{k\in\mathbb{N}}$, $j = 1, 2$, of eigenvalues satisfying*

$$\lambda_k^{(1)} = (-1)^{n/2}\left(\frac{k\pi}{R_+}\right)^n\left[1 + O(\frac{1}{k})\right], \qquad k \in \mathbb{N}$$

and (3.8)

$$\lambda_k^{(2)} = -(-1)^{n/2}\left(\frac{k\pi}{R_-}\right)^n\left[1 + O(\frac{1}{k})\right].$$

We note that in the definite case $R_- = 0$ our proof of Theorem 5 has to be modified only slightly. In this case the sequence $(\lambda_k^{(2)})_{k\in\mathbb{N}}$ has to be omitted.

Part II. Let $n = 2\mu - 1$, $\mu \ge 2$. For the proof of asymptotic estimates for $\Delta(\rho)$, $\rho \in S$, we use the same procedure as in the case of even order problems. In addition to the abbreviations τ_0, τ_1, τ_2, $E_\tau^1, \theta_\tau^1, V_\tau^1$ introduced in Part I, we set for $\tau = (t_0, \cdots, t_m) \in T_1 \cup T_2$

$$E_\tau^2 = \sum_{k=0}^{m}(a_{k+1} - a_k)\{\omega_{k,\mu-1+t_k} + \textstyle\sum_{j=\mu+1}^{n}\omega_{kj}\},$$

$$V_\tau^2 = \prod_{k=0}^{m-1} V\left(\omega_{k,\mu-1+t_k}, \omega_{k,\mu+1}, \cdots, \omega_{kn}, \omega_{k+1,1}, \cdots, \omega_{k+1,\mu-2}, \omega_{k+1,\mu-t_{k+1}}\right)$$

and

$$\theta_\tau^2 = \theta\left(\omega_{01}, \cdots, \omega_{0,\mu-2}, \omega_{0,\mu-t_0}; \omega_{m,\mu-1+t_m}, \omega_{m,\mu+1}, \cdots, \omega_{mn}\right).$$

Expanding $\Delta(\rho)$ we infer as with (3.2) for $\rho \in S\backslash\{0\}$

$$\Delta(\rho) = \rho^{k_0 + \frac{m\cdot n(n-1)}{2}}(-1)^N \Delta_1(\rho), \tag{3.9}$$

where

$$\Delta_1(\rho) = \sum_{j=1}^{2}\sum_{\tau\in T_1\cup T_2}(-1)^{N_j(\tau)}\theta_\tau^j V_\tau^j e^{\rho E_\tau^j}[1],$$

$$N,\ N_1(\tau),\ N_2(\tau) \in \mathbb{N},\ \ N_1(\tau_0) = N_2(\tau_0) = 0,$$

$$N_1(\tau_1) = N_2(\tau_1) = m + 1 - \sum_{j=0}^{m} t_j^1$$

and

$$N_1(\tau_2) = N_2(\tau_2) = m + 1 - \sum_{j=0}^{m} t_j^2.$$

In the following we assume for simplicity that $n = 4\kappa - 1$; in the case $n = 4\kappa + 1$ we can proceed in an analogous way.

Let $0 < \epsilon < \frac{\pi}{2n}$, $S := S_0$ and let the n-th roots $\omega_1, \cdots, \omega_n$ and $\hat{\omega}_1, \cdots, \hat{\omega}_n$ of 1 and -1 respectively be enumerated such that for $0 \leq \arg\rho \leq \frac{\pi}{2n}$

$$\operatorname{Re}(\rho\omega_1) \leq \cdots \leq \operatorname{Re}(\rho\omega_\mu) \leq 0 \leq \operatorname{Re}(\rho\omega_{\mu+1}) \leq \cdots \leq \operatorname{Re}(\rho\omega_n)$$

and

$$\operatorname{Re}(\rho\hat{\omega}_1) \leq \cdots \leq \operatorname{Re}(\rho\hat{\omega}_{\mu-1}) \leq 0 \leq \operatorname{Re}(\rho\hat{\omega}_\mu) \leq \cdots \leq \operatorname{Re}(\rho\hat{\omega}_n).$$

Using the identities $\omega_\mu = -\hat{\omega}_\mu$, $\omega_{\mu-1} = -\hat{\omega}_{\mu+1}$, $\omega_{\mu+1} = -\hat{\omega}_{\mu-1}$ we obtain from the definition of R_+, R_- and E_τ^j by subtracting

$$E := \sum_{k=0}^{m} \sum_{j=\mu+2}^{n} (a_{k+1} - a_k)\omega_{kj}$$

from E_τ^j :

$$\begin{aligned}
E_{\tau_0}^1 - E &= \omega_{\mu+1}R^+ + \hat{\omega}_{\mu+1}R_-, \qquad (3.10)\\
E_{\tau_1}^1 - E &= \omega_\mu R^+ + \hat{\omega}_{\mu+1}R_-,\\
E_{\tau_2}^1 - E &= \omega_{\mu+1}R^+ + \hat{\omega}_\mu R_-,\\
E_{\tau_0}^2 - E &= (\omega_\mu + \omega_{\mu+1})R_+ + (\hat{\omega}_\mu + \hat{\omega}_{\mu+1})R_- = E_{\tau_0}^1 - E + \omega_\mu(R_+ - R_-),\\
E_{\tau_1}^2 - E &= (\omega_{\mu-1} + \omega_{\mu+1})R_+ + (\hat{\omega}_\mu + \hat{\omega}_{\mu+1})R_- = E_{\tau_2}^1 - E + \omega_{\mu-1}(R_+ - R_-),\\
E_{\tau_2}^2 - E &= (\omega_\mu + \omega_{\mu+1})R_+ + (\hat{\omega}_{\mu-1} + \hat{\omega}_{\mu+1})R_- = E_{\tau_1}^1 - E + \omega_{\mu+1}(R_+ - R_-).
\end{aligned}$$

Case II A: $R_+ > R_-$ (and $n = 4\kappa - 1$). In this case we infer from (3.10) that

$$\operatorname{Re}(\rho E_\tau^j), \quad j \in \{1,2\}, \quad \tau \in T_1 \cup T_2 \quad (\text{and } \rho \neq 0)$$

is maximal if and only if

$$\begin{aligned}
&(i) && E_\tau^j = E_{\tau_0}^1 && \text{for } 0 < \arg\rho < \frac{\pi}{2n},\\
&(ii) && E_\tau^j = E_{\tau_0}^2 && \text{for } \frac{\pi}{2n} < \arg\rho < \frac{\pi}{n},\\
&(iii) && E_\tau^j = E_{\tau_2}^1 && \text{for } -\frac{\pi}{2n} < \arg\rho < 0, \qquad (3.11)\\
&(iv) && E_\tau^j = E_{\tau_1}^2 && \text{for } -\frac{\pi}{n} < \arg\rho < -\frac{\pi}{2n},\\
&(v) && E_\tau^j = E_{\tau_2}^2 && \text{for } \frac{\pi}{n} < \arg\rho < \frac{3\pi}{2n}.
\end{aligned}$$

Further we have for $j \in \{1,2\}$ and $\tau \in T_1 \cup T_2$

$$\operatorname{Re}\left(e^{-i\frac{\pi}{2n}} E^2_{\tau_1}\right) = \operatorname{Re}\left(e^{-i\frac{\pi}{2n}} E^1_{\tau_2}\right) > \operatorname{Re}\left(e^{-i\frac{\pi}{2n}} E^j_{\tau}\right)$$

for $(\tau, j) \notin \{(\tau_1, 2), (\tau_2, 1)\}$,

$$\begin{aligned} &\operatorname{Re}\left(E^1_\tau\right) = \operatorname{Re}\left(E^1_{\tau_0}\right), \quad \tau \in T_2, \\ &\operatorname{Re}\left(e^{i\frac{\pi}{2n}} E^1_{\tau_0}\right) = \operatorname{Re}\left(e^{i\frac{\pi}{2n}} E^2_{\tau_0}\right) > \operatorname{Re}\left(e^{i\frac{\pi}{2n}} E^j_{\tau}\right) \end{aligned} \tag{3.12}$$

for $(\tau, j) \notin \{(\tau_0, 1), (\tau_0, 2)\}$ and

$$\operatorname{Re}\left(e^{i\frac{\pi}{n}} E^2_{\tau_0}\right) = \operatorname{Re}\left(e^{i\frac{\pi}{n}} E^2_{\tau}\right), \quad \tau \in T_2.$$

If $R_+ < R_-$ we obtain an analogous result.

Case II B: $R_+ = R_-$ (and $n = 4\kappa - 1$). According to (3.10) and (3.11) we have in this case

$$E^1_{\tau_0} = E^2_{\tau_0}, \quad E^1_{\tau_1} = E^2_{\tau_2} \quad \text{and} \quad E^1_{\tau_2} = E^2_{\tau_1}, \tag{3.13}$$

and $\operatorname{Re}(\rho E^j_\tau)$, $j \in \{1,2\}$, $\tau \in T_1 \cup T_2$ is maximal if and only if

$$\begin{aligned} &\text{(i)} \qquad E^j_\tau = E^1_{\tau_0} = E^2_{\tau_0} \quad \text{for } 0 < \arg\rho < \frac{\pi}{n} \\ \text{and} \\ &\text{(ii)} \qquad E^j_\tau = E^1_{\tau_2} = E^2_{\tau_1} \quad \text{for } -\pi < \arg\rho < 0. \end{aligned} \tag{3.14}$$

Further we infer from the definition of E^1_τ and E^2_τ that $\operatorname{Re} E^1_\tau = \operatorname{Re} E^2_\sigma$ and $\operatorname{Re}\left(e^{i\frac{\pi}{n}} E^1_\sigma\right) = \operatorname{Re}\left(e^{i\frac{\pi}{n}} E^2_\tau\right)$ for $\sigma \in T_1$ and $\tau \in T_2$.

Now we assume again that the coefficients of the dominant exponential terms of $\Delta(\rho)$, $\rho \in S_0 \cup S_{2n-1}$, do not vanish.

Definition 6. a) *For* $n = 4\kappa - 1$ *problem* (1.1), (1.2) *is called regular if*

(i) $\theta^1_{\tau_0}, \theta^2_{\tau_0}, \theta^1_{\tau_2}, \theta^2_{\tau_1} \neq 0$ *and* $R_+ \neq R_-$

or

(ii) $\theta^1_{\tau_0} U^1_{\tau_0} + \theta^2_{\tau_0} U^2_{\tau_0} \neq 0 \neq \theta^1_{\tau_2} U^1_{\tau_2} + (-1)^{\sum_{k=0}^m (t^1_k - t^2_k)} \theta^2_{\tau_1} U^2_{\tau_1}$ *and* $R_+ = R_-$.

b) *For* $n = 4\kappa + 1$ *problem* (1.1), (1.2) *is called regular if*

(iii) $\theta^1_{\tau_0}, \theta^2_{\tau_0}, \theta^1_{\tau_1}, \theta^2_{\tau_2} \neq 0$ *and* $R_+ \neq R_-$

or

(iv) $\theta^1_{\tau_0} U^1_{\tau_0} + \theta^2_{\tau_0} U^2_{\tau_0} \neq 0 \neq \theta^1_{\tau_1} U^1_{\tau_1} + (-1)^{\sum_{k=0}^m (t^1_k - t^2_k)} \theta^2_{\tau_2} U^2_{\tau_2}$ *and* $R_+ = R_-$.

Remark 7. Definition 7 is independent of the sector S_0 used for the definition of the constants θ^j_τ, U^j_τ.

It is possible to derive relations between the nonvanishing constants in Definition 6; we omit details (cf. Remark 1).

Theorem 8. *For $n = 2\mu - 1$, $\mu \geq 2$ every regular boundary eigenvalue problem has a countably infinite set of eigenvalues.*

a) *If $R := R_+ = R_-$, then there are two sequences $(\lambda_k^{(j)})$, $j = 1,2$ of eigenvalues satisfying for $k \in \mathbb{N}$*

$$\lambda_k^{(j)} = (-1)^j \left(\frac{k\pi}{R\cos(\pi/2n)}\right)^n \{1 + O(\frac{1}{k})\}, \quad j = 1,2. \tag{3.15}$$

b) *If $R_+ \neq R_-$ and $R_0 := \min\{R_+, R_-\} > 0$, then there are four sequences $(\lambda_k^{(j)})$, $3 \leq j \leq 6$, of eigenvalues satisfying for $k \in \mathbb{N}$*

$$\lambda_k^{(j)} = (-1)^j \left(\frac{k\pi}{R_0\cos(\pi/2n)}\right)^n \{1 + O(\frac{1}{k})\}, \quad j = 3,4., \tag{3.16}$$

$$\lambda_k^{(j)} = i(-1)^j \left(\frac{2k\pi}{|R_+ - R_-|}\right)^n \{1 + \frac{\xi_j}{k} + O(\frac{1}{k^2})\}, \quad j = 5,6. \tag{3.17}$$

The constants ξ_5, ξ_6 can be evaluated explicitly and almost all eigenvalues $\lambda_k^{(j)}$, $j = 5,6$, are simple.

Proof: Using the preceding estimates for $\Delta(\rho)$ we obtain (3.15) and (3.16) as with the proof of Theorem 5.

For the proof of (3.17) we discuss exemplarily the case $n = 4\kappa - 1$, $R_+ \neq R_-$. In this case we infer from (3.11), (3.12) that (1.1), (1.2) has four sequences $(\lambda_k^{(j)})$, $3 \leq j \leq 6$, of eigenvalues. By $(\lambda_k^{(6)})$ we denote the sequence having the positive imaginary axis as asymptote; let $\lambda_k^{(6)} = \{\rho_k^{(6)}\}^n$ where

$$\frac{\pi}{2n} - \epsilon \leq \arg \rho_k^{(6)} \leq \frac{\pi}{2n} + \epsilon, \quad (k \geq K).$$

According to (3.9)–(3.12) $\rho_k^{(6)}$ must be the solution of an equation of the form

$$0 = \theta_{\tau_0}^1 V_{\tau_0}^1 e^{\rho E_{\tau_0}^1}[1] + \theta_{\tau_0}^2 V_{\tau_0}^2 e^{\rho E_{\tau_0}^2} = \theta_{\tau_0}^2 V_{\tau_0}^2 e^{\rho E_{\tau_0}^1} \left\{ \frac{\theta_{\tau_0}^1 V_{\tau_0}^1}{\theta_{\tau_0}^2 V_{\tau_0}^2}[1] + e^{\rho(E_{\tau_0}^2 - E_{\tau_0}^1)} \right\}.$$

Since $E_{\tau_0}^2 - E_{\tau_0}^1 = \omega_\mu(R_+ - R_-)$ this equation is equivalent to

$$e^{\rho\omega_\mu(R_+ - R_-)} = \exp\left\{ -ie^{-i\frac{\pi}{2n}}\rho(R_+ - R_-)\right\} = [A] \tag{3.18}$$

with $A = -(\theta_{\tau_0}^1 V_{\tau_0}^1)/(\theta_{\tau_0}^2 V_{\tau_0}^2)$. The solutions of (3.18) with $\frac{\pi}{2n} - \epsilon \leq \rho_{|k|}^{(6)} \leq \frac{\pi}{2n} + \epsilon$ satisfy

$$\begin{aligned} \rho_{|k|}^{(6)} &:= \frac{e^{i\frac{\pi}{2n}}}{-i(R_+ - R_-)} \left\{ 2k\pi i + \ln_0 A + O(\frac{1}{|k|}) \right\} \\ &= -e^{i\frac{\pi}{2n}} \frac{2k\pi}{R_+ - R_-} \left\{ 1 + \frac{\ln_0 A}{2k\pi i} + O(\frac{1}{k^2}) \right\}, \end{aligned} \tag{3.19}$$

where $k \in \mathbb{N}$ for $R_+ < R_-$ and $-k \in \mathbb{N}$ for $R_+ > R_-$. (3.19) implies (3.17) for $j = 6$ with

$$\xi_6 = \pm \frac{n \ln_0 A}{2\pi i}.$$

In all remaining cases we prove (3.17) similarly.

Remark 9. (i) The method used in this paper can also be applied for the discussion of boundary eigenvalue problems (1.1), (1.2) with a piecewise continuous weight function r with $|r(x)| \geq c > 0$ for $x \in [0,1]$. In this more general situation one can assume without loss of generality that the coefficient of $y^{(n-1)}$ in $\ell(y)$ is zero (if the coefficients f_ν are sufficiently smooth).

(ii) For $k \geq K$ the multiplicity of the eigenvalues is bounded by $\#(T_1 U T_2) - 1$ (in formula (3.15)) or by $\#T_1 - 1$ or $\#T_2 - 1$ (in Theorem 5 or formula (3.16)).

If we have for example

$$r(x) = \begin{cases} -a < 0 & \text{for } 0 \leq x \leq x_1 \\ b > 0 & \text{for } x_1 < x \leq 1, \end{cases}$$

then almost all eigenvalues of (1.1), (1.2) are simple and satisfy asymptotic estimates of the form

$$\lambda_k = \lambda_k^0 \{1 + \frac{a}{k} + O(\frac{1}{k^2})\}.$$

(iii) Just as in the case of definite problems it is possible to weaken the hypothesis of regularity by assuming that the coefficients of the dominant terms in the expansion of $\Delta_1(\rho)$ have the form

$$\sum_{k=0}^{s} \left(\frac{\gamma_k}{\rho^k} + O(\frac{1}{\rho^{s+1}}) \right), \qquad s \in \mathbb{N} \text{ fixed},$$

where $\sum_{k=0}^{s} |\lambda_k| > 0$. Details will be discussed elsewhere.

(iv) In the case $n = 1$ the asymptotic behaviour of the eigenvalues of regular problems (1.1), (1.2) can be determined easily. We omit details.

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