ON THE DISTRIBUTION OF THE NUMBER OF SUCCESSES IN INDEPENDENT TRIALS

By J. N. Darroch

University of Adelaide

0. Summary. The distribution of the number of successes in n independent trials is "bell-shaped". The expected number of successes, μ say, either determines the most probable number of successes uniquely or restricts it to the pair of integers nearest to μ.

1. The shape of the distribution. Consider a set of n independent trials and let $p_j(= 1 - q_j)$ denote the probability of success in the jth trial, $j = 1, 2, \cdots, n$. Also let $\mathbf{p}_k = (p_1, p_2, \cdots, p_k)$, $k = 1, 2, \cdots, n$, and let $P_r(\mathbf{p}_k) = P(r$ successes in the first k trials$)$, $r = -2, -1, 0, 1, \cdots, k, k + 1, k + 2$. Thus we have trivially $P_{-2}(\mathbf{p}_k) = P_{-1}(\mathbf{p}_k) = P_{k+1}(\mathbf{p}_k) = P_{k+2}(\mathbf{p}_k) = 0$.

In this section it will be assumed that $0 < p_j < 1$, $j = 1, 2, \cdots, n$ with no essential loss of generality.

Lemma 1.

$$P_2(\mathbf{p}_2) - P_1(\mathbf{p}_2) \geqq 0 \Rightarrow P_1(\mathbf{p}_2) - P_0(\mathbf{p}_2) > 0.$$

Proof. We have to prove that $A \cap B$ is empty where

$$A = \{\mathbf{p}_2 : P_2(\mathbf{p}_2) - P_1(\mathbf{p}_2) \geqq 0\}, \qquad B = \{\mathbf{p}_2 : P_1(\mathbf{p}_2) - P_0(\mathbf{p}_2) \leqq 0\}.$$

Let $\alpha = p_1 + p_2$ and $\beta = p_1 p_2$. Then $\alpha^2 \geqq 4\beta$. Now in A, $3\beta - \alpha \geqq 0$ and therefore $3\alpha^2 - 4\alpha \geqq 0$ which implies that $\alpha \geqq 4/3$; while in B, $3\beta - 2\alpha + 1 \geqq 0$ and therefore $3\alpha^2 - 8\alpha + 4 \geqq 0$ which implies that $\alpha \leqq 2/3$. Hence $A \cap B$ is empty.

Theorem 1.

$$P_{r+1}(\mathbf{p}_n) - P_r(\mathbf{p}_n) \geqq 0 \Rightarrow P_r(\mathbf{p}_n) - P_{r-1}(\mathbf{p}_n) > 0, \qquad r = 1, 2, \cdots, n - 1.$$

Proof. We have

$$P_r(\mathbf{p}_k) = q_k P_r(\mathbf{p}_{k-1}) + p_k P_{r-1}(\mathbf{p}_{k-1}); \tag{1}$$

and hence

$$\begin{aligned} P_{r+1}(\mathbf{p}_k) - P_r(\mathbf{p}_k) &= q_k(P_{r+1}(\mathbf{p}_{k-1}) - P_r(\mathbf{p}_{k-1})) \\ &\quad + p_k(P_r(\mathbf{p}_{k-1}) - P_{r-1}(\mathbf{p}_{k-1})) \quad r = 1, 2, \cdots, k - 1; k = 2, 3, \cdots, n. \end{aligned} \tag{2}$$

Now assume that

$$P_{r+1}(\mathbf{p}_{k-1}) - P_r(\mathbf{p}_{k-1}) \geqq 0 \Rightarrow P_r(\mathbf{p}_{k-1}) - P_{r-1}(\mathbf{p}_{k-1}) > 0, \quad r = 1, 2, \cdots, k - 2. \tag{3}$$

Received 15 November 1963; revised 2 February 1964.

Then by (2) and (3)

$$P_{r+1}(\mathbf{p}_k) - P_r(\mathbf{p}_k) \geqq 0 \Rightarrow P_r(\mathbf{p}_{k-1}) - P_{r-1}(\mathbf{p}_{k-1}) > 0$$
$$\Rightarrow P_{r-1}(\mathbf{p}_{k-1}) - P_{r-2}(\mathbf{p}_{k-1}) > 0,$$

and hence $P_{r+1}(\mathbf{p}_k) - P_r(\mathbf{p}_k) \geqq 0 \Rightarrow P_r(\mathbf{p}_k) - P_{r-1}(\mathbf{p}_k) > 0$, $r = 1, 2, \cdots, k - 1$. Referring to Lemma 1, the proof of Theorem 1 by induction on k is now complete.

Note that an equivalent formulation of Theorem 1 is

$$P_r(\mathbf{p}_n) - P_{r-1}(\mathbf{p}_n) \leqq 0 \Rightarrow P_{r+1}(\mathbf{p}_n) - P_r(\mathbf{p}_n) < 0 \quad r = 1, 2, \cdots n - 1.$$

Therefore Theorem 1 states that the probabilities $P_{-1}(\mathbf{p}_n)$, $P_0(\mathbf{p}_n)$, $P_1(\mathbf{p}_n)$, $\cdots$, $P_n(\mathbf{p}_n)$, $P_{n+1}(\mathbf{p}_n)$ strictly increase and then strictly decrease, except that there may be at most two equal maxima.

LEMMA 2.

$$P_2(\mathbf{p}_2) - 2P_1(\mathbf{p}_2) + P_0(\mathbf{p}_2) \geqq 0 \quad \textit{and} \quad P_1(\mathbf{p}_2) - P_0(\mathbf{p}_2) > 0$$
$$\Rightarrow P_1(\mathbf{p}_2) - 2P_0(\mathbf{p}_2) + P_{-1}(\mathbf{p}_2) > 0.$$

PROOF. We have to prove that $A \cap B \cap C$ is empty where

$$A = \{\mathbf{p}_2 : P_2(\mathbf{p}_2) - 2P_1(\mathbf{p}_2) + P_0(\mathbf{p}_2) \geqq 0\}$$
$$B = \{\mathbf{p}_2 : P_1(\mathbf{p}_2) - P_0(\mathbf{p}_2) > 0\}$$
$$C = \{\mathbf{p}_2 : P_1(\mathbf{p}_2) - 2P_0(\mathbf{p}_2) + P_{-1}(\mathbf{p}_2) \leqq 0\}.$$

Now in A, $6\beta - 3\alpha + 1 \geqq 0$ and in B $2\alpha - 3\beta - 1 > 0$. Therefore in $A \cap B$, $\alpha > 1$. In C, $-3\alpha + 4\beta + 2 \geqq 0$ and therefore $\alpha^2 - 3\alpha + 2 \geqq 0$ which implies that $\alpha \leqq 1$. Therefore $A \cap B \cap C$ is empty.

THEOREM 2.

$$P_{r+1}(\mathbf{p}_n) - 2P_r(\mathbf{p}_n) + P_{r-1}(\mathbf{p}_n) \geqq 0 \quad \textit{and} \quad P_r(\mathbf{p}_n) - P_{r-1}(\mathbf{p}_n) > 0$$
$$\Rightarrow P_r(\mathbf{p}_n) - 2P_{r-1}(\mathbf{p}_n) + P_{r-2}(\mathbf{p}_n) > 0, \quad r = 1, 2, \cdots, n.$$

PROOF. From (1) we have

$$\begin{aligned} P_{r+1}(\mathbf{p}_k) - 2P_r(\mathbf{p}_k) + P_{r-1}(\mathbf{p}_k) &= q_k(P_{r+1}(\mathbf{p}_{k-1}) - 2P_r(\mathbf{p}_{k-1}) + P_{r-1}(\mathbf{p}_{k-1})) \\ &\quad + p_k(P_r(\mathbf{p}_{k-1}) - 2P_{r-1}(\mathbf{p}_{k-1}) + P_{r-2}(\mathbf{p}_{k-1})); \quad r = 1, 2, \cdots, k. \end{aligned} \tag{4}$$

The proof of Theorem 2 uses (4) and Lemma 2 and proceeds by induction in much the same way as the proof of Theorem 1. As the details are straightforward but tedious to write out, we omit them.

Note that an equivalent formulation of Theorem 2 is

$$P_r(\mathbf{p}_n) - 2P_{r-1}(\mathbf{p}_n) + P_{r-2}(\mathbf{p}_n) \leqq 0 \quad \text{and} \quad P_r(\mathbf{p}_n) - P_{r-1}(\mathbf{p}_n) > 0$$
$$\Rightarrow P_{r+1}(\mathbf{p}_n) - 2P_r(\mathbf{p}_r) + P_{r-1}(\mathbf{p}_n) < 0,$$

since if a, b and c are statements, a and $b \Rightarrow c$ is equivalent to not c and $b \Rightarrow$ not a.

When the above two formulations of Theorem 2 are applied to the number of failures instead of to the number of successes, two further formulations are obtained, namely:

$$P_r(\mathbf{p}_n) - 2P_{r-1}(\mathbf{p}_n) + P_{r-2}(\mathbf{p}_n) \geqq 0 \quad \text{and} \quad P_r(\mathbf{p}_n) - P_{r-1}(\mathbf{p}_n) < 0$$
$$\Rightarrow P_{r+1}(\mathbf{p}_n) - 2P_r(\mathbf{p}_n) + P_{r-1}(\mathbf{p}_n) > 0,$$

and

$$P_{r+1}(\mathbf{p}_n) - 2P_r(\mathbf{p}_n) + P_{r-1}(\mathbf{p}_n) \leqq 0 \quad \text{and} \quad P_r(\mathbf{p}_n) - P_{r-1}(\mathbf{p}_n) < 0$$
$$\Rightarrow P_r(\mathbf{p}_n) - 2P_{r-1}(\mathbf{p}_n) + P_{r-2}(\mathbf{p}_n) < 0.$$

Therefore Theorem 2 states that the probabilities $P_{-2}(\mathbf{p}_n), P_{-1}(\mathbf{p}_n), P_0(\mathbf{p}_n), \cdots, P_n(\mathbf{p}_n), P_{n+1}(\mathbf{p}_n), P_{n+2}(\mathbf{p}_n)$ first increase convexly and then concavely, and then they decrease concavely and then convexly. The convexity and concavity are strict everywhere except that there may be at most one set of three consecutive probabilities which increase linearly and at most one set of three which decrease linearly.

Thus the distribution of the number of successes may be described as *bell-shaped.*

2. The most probable number of successes. In this section we examine the extent to which the expected number of successes, $\sum_j p_j = \mu$ say, determines the most probable number of successes. The argument rests heavily on the following result due to Hoeffding ([2] Corollary 2.1; David [1] gave an alternative solution of a special case of Hoeffding's problem).

HOEFFDING'S THEOREM. Given μ, the maximum and minimum of any linear function of the probabilities $P_0(\mathbf{p}_n), P_1(\mathbf{p}_n), \cdots, P_n(\mathbf{p}_n)$ are attained when the probabilities $p_1, p_2, \cdots, p_n$ take on, at most, three different values, only one of which is distinct from 0 and 1.

Writing $\mathbf{p}_n = \mathbf{p}$, define the integers $l = l(\mathbf{p})$ and $m = m(\mathbf{p})$ by

$$P_l(\mathbf{p}) - P_{l-1}(\mathbf{p}) > 0, \qquad P_{l+1}(\mathbf{p}) - P_l(\mathbf{p}) \leqq 0$$

and

$$P_m(\mathbf{p}) - P_{m-1}(\mathbf{p}) \geqq 0, \qquad P_{m+1}(\mathbf{p}) - P_m(\mathbf{p}) < 0.$$

Also let $n_1(\mathbf{p}), n_0(\mathbf{p})$ respectively denote the numbers of unit and zero components of $\mathbf{p}$.

Next let

$$G(\mu) = \left\{\mathbf{p}: \sum_{j=1}^{n} p_j = \mu, 0 \leqq p_j \leqq 1, j = 1, 2, \cdots, n\right\}$$

and define

$$l^*(\mu) = \min_{\mathbf{p}\varepsilon G(\mu)} l(\mathbf{p}), \qquad m^*(\mu) = \max_{\mathbf{p}\varepsilon G(\mu)} m(\mathbf{p})$$
$$n_1^*(\mu) = \max_{\mathbf{p}\varepsilon G(\mu)} n_1(\mathbf{p}), \qquad n_1^*(\mu) = \max_{\mathbf{p}\varepsilon G(\mu)} n_0(\mathbf{p}).$$

Our main task in this section is to find $l^*(\mu)$ and $m^*(\mu)$.

THEOREM 3. Let $[x]$ denote the integral part of x. Then $l^*(\mu) = \mu$ when μ is integral; $l^*(\mu) = \mu + (\mu - [\mu])/(n - [\mu]) - 1$ when $\mu + (\mu - [\mu])/(n - [\mu])$ is integral but μ is not; and $l^*(\mu) = [\mu + (\mu - [\mu])/(n - [\mu])]$ otherwise. Also $m^*(\mu) = \mu$ when μ is integral; and $m^*(\mu) = ([\mu] + 1 - \mu)/([\mu] + 1)] + 1$ otherwise.

PROOF. Let

$$S(\mathbf{p}) = \{r : n_1(\mathbf{p}) \leqq r \leqq l(\mathbf{p})\} = \{r : P_r(\mathbf{p}) - P_{r-1}(\mathbf{p}) > 0\}. \tag{5}$$

Now form the intersection $S^*(\mu) = \bigcap_{\mathbf{p} \varepsilon G(\mu)} S(\mathbf{p})$. Noting that $G(\mu)$ is closed we deduce from (5) that

$$\begin{aligned} S^*(\mu) &= \{r : n_1^*(\mu) \leqq r, r \leqq l^*(\mu)\} \\ &= \{r : \min_{\mathbf{p} \varepsilon G(\mu)} (P_r(\mathbf{p}) - P_{r-1}(\mathbf{p})) > 0\} \end{aligned} \tag{6}$$

By Hoeffding's theorem,

$$\min_{\mathbf{p} \varepsilon G(\mu)} (P_r(\mathbf{p}) - P_{r-1}(\mathbf{p})) = \min_{\mathbf{p} \varepsilon H(\mu)} (P_r(\mathbf{p}) - P_{r-1}(\mathbf{p})),$$

where $H(\mu)$ is the subset of $G(\mu)$ containing those $\mathbf{p}$ whose components are all 0, 1 or π say. When $\mathbf{p} \varepsilon H(\mu)$ we shall write $n_1(\mathbf{p}) = \nu_1$, $n_0(\mathbf{p}) = \nu_0$, $\pi = \pi(\mu, \nu_1, \nu_0)$ where

$$0 < \pi < 1, \qquad \nu_1 + (n - \nu_1 - \nu_0)\pi = \mu. \tag{7}$$

In order to derive $l^*(\mu)$ suppose first that $\mu \neq [\mu]$. Then it is readily seen that $n_1^*(\mu) = [\mu]$, $n_0^*(\mu) = n - [\mu] - 1$. Therefore

$$0 \leqq \nu_1 \leqq [\mu], \qquad 0 \leqq \nu_0 \leqq n - [\mu] - 1. \tag{8}$$

If $\mathbf{p} \varepsilon H(\mu)$, $P_r(\mathbf{p})$ is the binomial probability of obtaining $r - \nu_1$ successes in $n - \nu_1 - \nu_0$ trials when the probability of success is π. Consequently

$$\begin{aligned} P_r(\mathbf{p}) - P_{r-1}(\mathbf{p}) > 0 &\Leftrightarrow 0 \leqq r - \nu_1 < (n - \nu_1 - \nu_0 + 1)\pi \\ &\Leftrightarrow \nu_1 \leqq r < \mu + \pi(\mu, \nu_1, \nu_0) \end{aligned}$$

by (7) and therefore

$$\min_{\mathbf{p} \varepsilon H(\mu)} (P_r(\mathbf{p}) - P_{r-1}(\mathbf{p})) > 0 \Leftrightarrow [\mu] \leqq r < \mu + \min_{\nu_1, \nu_0} \pi(\mu, \nu_1, \nu_0).$$

Now from (7) and (8),

$$\min_{\nu_1, \nu_0} \pi(\mu, \nu_1, \nu_0) = \min_{\nu_1, \nu_0} \frac{\mu - \nu_1}{n - \nu_1 - \nu_0} = \min_{\nu_1} \frac{\mu - \nu_1}{n - \nu_1} = \frac{\mu - [\mu]}{n - [\mu]}.$$

Thus when $\mu \neq [\mu]$, $l^*(\mu)$ is the largest integer which is strictly less than $\mu + (\mu - [\mu])/(n - [\mu])$.

It remains to find $l^*(\mu)$ when $\mu = [\mu]$, in which case $n_1^*(\mu) = \mu$, $n_0^*(\mu) = n - \mu$. When $\nu_1 = \mu$, $\nu_0 = n - \mu$,

$$P_\mu(\mathbf{p}) = 1 \quad \text{and} \quad P_r(\mathbf{p}) = 0, \qquad r \neq \mu.$$

Otherwise, when $0 \leqq \nu_1 \leqq \mu - 1$ and $0 \leqq \nu_0 \leqq n - \mu - 1$, by definition

$$\min_{\nu_1, \nu_0} \pi(\mu, \nu_1, \nu_0) > 0.$$

It readily follows that $l^*(\mu) = \mu$.

The proof of Theorem 3 is completed by remarking that the formula for $m^*(\mu)$ can be obtained by applying the formula for $l^*(\mu)$ to the distribution of the number of failures.

Theorem 3 shows that $l^*(\mu)$ and $m^*(\mu)$ are always equal to either $[\mu]$ or $[\mu] + 1$, subject of course to $l^*(\mu) \leqq m^*(\mu)$. When $l^*(\mu) = m^*(\mu)$ it follows that $l(\mathbf{p}) = m(\mathbf{p}) = l^*(\mu)$ for all $\mathbf{p} \varepsilon G(\mu)$, and therefore that the maximum probability is unique and occurs at $l^*(\mu)$ for all $\mathbf{p} \varepsilon G(\mu)$. Theorem 4 determines the conditions for this to happen and it is convenient to express these conditions in terms of the value of $\delta(\mu) = \mu - [\mu]$, the fractional part of μ.

THEOREM 4. *Given μ, the most probable number of successes is equal to*

μ, *if* $\delta(\mu) = 0$,

$[\mu] + 1$, *if* $(n - [\mu])/(n - [\mu] + 1) < \delta(\mu) < 1$,

$[\mu]$, *if* $0 \leqq \delta(\mu) < 1/([\mu] + 2)$,

one or both of $[\mu]$, $[\mu] + 1$ *if* $1/([\mu] + 2) \leqq \delta(\mu) \leqq (n - [\mu])/(n - [\mu] + 1)$.

PROOF. Referring to Theorem 3 it is sufficient to point out that $l^*(\mu) = [\mu] + 1$ if and only if $\mu + (\mu - [\mu])/(n - [\mu]) > [\mu] + 1$ in which case $\delta(\mu) > (n - [\mu])/(n - [\mu] + 1)$. Also that, by a similar argument $m^*(\mu) = [\mu]$ if and only if $\delta(\mu) < 1/([\mu] + 2)$.

Acknowledgment. I wish to thank the referee for pointing out that some improvement was necessary in my original proof of Theorem 3.

REFERENCES

[1] DAVID, H. A. (1960). A conservative property of binomial tests. *Ann. Math. Statist.* **31** 1205–1207.

[2] HOEFFDING, WASSILY (1956). On the distribution of the number of successes in independent trials. *Ann. Math. Statist.* **27** 713–721.