ON THE DISTRIBUTION OF THE QUOTIENT OF TWO CHANCE VARIABLES

By J. H. Curtiss

Cornell University

1. **Introduction.** Although the quotient of two chance variables appears frequently in mathematical statistics, the methods used in the literature to derive the distributions of quotients have usually been special ones devised for the particular variables under consideration, and in no way indicative of the general result. It is the purpose of this paper to study the distribution of the quotient of two variables for itself alone, with attention first to the question of existence, and then to the accurate derivation of a number of general formulas for the frequency function and d.f.[1] The principal formulas which we shall derive may be described briefly as follows (the numerals refer to the equation numbers in the text):

(3.1). The frequency function of the quotient of two variables which have an absolutely continuous joint probability function.

(4.11), (4.12). The d.f. of the quotient of a pair of arbitrary independent variables, expressed in terms of the d.f.'s of these variables.

(5.2). The d.f. of the quotient of a pair of arbitrary independent variables, expressed in terms of the c.f.'s[2] of these variables.

(6.4). The limiting form of the d.f. of a quotient of two sums of arbitrary identical independent variables.

(7.1). A formula analogous to (3.1) for the product of two chance variables.

(7.2). A formula analogous to (4.11) for the product of two chance variables.

2. **The existence of the quotient distribution.** The function $Z = X/Y$ is a continuous function of X and Y, finite and uniquely defined for all points (X, Y) such that $Y \neq 0$. Therefore if $P\{Y = 0\} = 0$, the pr.f.[3] $P(S)$ of the joint distribution of X and Y determines a probability distribution for Z (see [1, pp. 12–13]). To avoid irrelevant difficulties, we shall assume in the sequel that $P\{Y = 0\} = 0$ unless definite statement is made to the contrary. This assumption involves no real restriction on our work, for in situations in which, a priori, the assumption is not fulfilled, we can always replace the distribution

[1] I.e., distribution function. The underlying axioms, terminology, and abbreviations in this paper are uniform with those of Cramér's book [1]. For the definition of d.f., see [1, p. 11].

[2] I.e., characteristic functions. See [1, p. 23].

[3] I.e., probability function; [1, p. 9].

of Y by the conditional distribution of Y relative to the hypothesis that $Y \neq 0$. In such cases, then, the distribution of Z which we are about to study is to be interpreted as a conditional distribution relative to this hypothesis.

We shall suppose that the space of X is the x-axis, that of Y, the y-axis, and that of Z, the z-axis. It is quite readily seen that the set of points in the (x, y) plane which corresponds to the set $Z \leqq z$ consists of

(i) the infinite region[4] in the upper half-plane which is bounded by the negative x axis and by the line $x = zy$;

(ii) the infinite region in the lower half-plane bounded by the positive x-axis and the line $x = zy$;

(iii) the line $x = zy$ except for the origin.

Denoting this set by S_z, we have

$$H(z) = \int_{S_z} dP(S) = P(S_z),$$

where $H(z)$ is the d.f. of Z. The present paper, from the viewpoint of analysis, is simply a study of the Lebesgue-Stieltjes integral appearing in this equation.

3. The continuous case. Suppose first that $P(S)$ is absolutely continuous. This means that the joint distribution of X and Y has a frequency function $\varphi(x, y)$, which is defined almost everywhere, is non-negative, and has the property that $P(S) = \int_S \varphi(x, y)\, dx\, dy$. In general, this integral must be taken in the Lebesgue sense, but of course if the discontinuities of φ form a set of two-dimensional measure zero, and if the Jordan content of any bounded portion of the boundary of S is zero, then this integral is just an ordinary improper double Riemann integral.[5] In particular, these conditions are fulfilled if φ is continuous everywhere and if $S = S_z$.

The transformation $x = uv$, $y = v$, gives a continuous one-to-one map of S_z onto a set $\mathfrak{S}_z$ of the (u, v) plane which consists of the closed half-plane lying to the left of the line $u = z$, but with the u-axis deleted. The Jacobian of the transformation has the absolute value $|v|$. By the theorem for change of variables in Lebesgue integrals [4, pp. 653–655], we have

$$H(z) = \int_{S_z} \varphi(x, y)\, dx\, dy = \int_{\mathfrak{S}_z} |v|\, \varphi(uv, v)\, du\, dv.$$

By Fubini's Theorem [6, pp. 203–208], the last integral can be expressed as a repeated integral. Integrating first with respect to v, we obtain this result

THEOREM 3.1: *If the joint variable (X, Y) has the frequency function $\varphi(x, y)$, then*

$$H(z) = \int_{-\infty}^{z} \left[\int_{-\infty}^{+\infty} |v|\, \varphi(uv, v)\, dv \right] du,$$

---

[4] I.e., *open* connected set.

[5] See [4, pp. 476–478: p. 575].

and consequently $H(z)$ is an absolutely continuous function of z. The frequency function of the distribution of Z exists almost everywhere, and is given by the formula

$$h(z) = F'(z) = \int_{-\infty}^{+\infty} |v| \varphi(zv, v)\, dv. \tag{3.1}$$

We remark that if X and Y are independent, so that $\varphi(x, y) = f(x)\cdot g(y)$, where f and g are respectively the frequency functions of X and Y, then (3.1) may be written in the form

$$h(z) = \int_{-\infty}^{+\infty} |v| f(zv)g(v)\, dv. \tag{3.2}$$

This case was considered recently by Huntington [5], with the additional restrictions that $g(y) = 0$, $y < 0$, and that $f(x)$ and $g(y)$ be continuous.

All the familiar special quotient distributions of applied mathematical statistics, such as Student's t and Fisher's z, may conveniently and rigorously be derived by means of (3.1) and (3.2); in each case the required result follows immediately after an obvious change of variables in the integrand. We pause here only to point out explicitly the result obtained when X and Y have a normal joint distribution with variances σ_X^2, σ_Y^2, and correlation coefficient ρ. If the means $E(X)$ and $E(Y)$ are not equal to zero, it is apparently impossible to evaluate (3.1) in closed form; this case has been studied in some detail by Geary [3] and by Fieller [2]. But if $E(X) = E(Y) = 0$, then

$$h(z) = \frac{\sigma_X \sigma_Y \sqrt{1-\rho^2}}{\pi} \cdot \frac{1}{\sigma_Y^2\left(z - \rho\dfrac{\sigma_X}{\sigma_Y}\right)^2 + \sigma_X^2(1-\rho^2)},$$

which is the frequency function of a Cauchy distribution with mode at the point $z = \rho\sigma_X/\sigma_Y$, the value of the regression coefficient of X on Y. If X and Y are independent, then $\rho = 0$, and the frequency function becomes

$$h(z) = \frac{\sigma_X \sigma_Y}{\pi} \cdot \frac{1}{\sigma_Y^2 z^2 + \sigma_X^2}. \tag{3.3}$$

4. **The quotient of two arbitrary independent variables.** We shall henceforth drop the restriction that $P(S)$ be absolutely continuous, but shall suppose instead that X and Y are independent chance variables with one-dimensional distributions of the most general type, except that the distribution of Y will be subject to the restriction that $P\{Y = 0\} = 0$.

We denote the d.f. of X by $F(x)$, that of Y by $G(y)$, and, as usual, that of Z

by $H(z)$. It is to be noticed that the condition $P\{Y = 0\} = 0$ implies that $G(y)$ is continuous at the point $y = 0$. Let

$$f(t) = \int_{-\infty}^{+\infty} e^{itx}\, dF(x)$$

$$g^+(t) = \int_0^{\infty} e^{ity}\, dG(y) \tag{4.1}$$

$$g^-(t) = \int_{-\infty}^{0} e^{ity}\, dG(y).$$

Clearly

$$H(z) = P\{X - zY \leqq 0;\, Y > 0\} + P\{X - zY \geqq 0;\, Y < 0\}. \tag{4.2}$$

We introduce the functions

$$\Gamma_1(u) = P\{X - zY \leqq u;\, Y > 0\} = [1 - G(0)]\cdot P\{X - zY \leqq u \mid Y > 0\},$$ [6]

$$\gamma_1(t) = \int_{-\infty}^{+\infty} e^{itu}\, d\Gamma_1(u),$$

$$\Gamma_2(u) = P\{zY - X \leqq u;\, Y < 0\} = G(0)\cdot P\{zY - X \leqq u \mid Y < 0\},$$

$$\gamma_2(t) = \int_{-\infty}^{+\infty} e^{itu}\, d\Gamma_2(u), \tag{4.3}$$

$$\Gamma(u) = \Gamma_1(u) + \Gamma_2(u)$$

$$\gamma(t) = \int_{-\infty}^{+\infty} e^{itu}\, d\Gamma(u) = \gamma_1(t) + \gamma_2(t).$$

By (4.2) and (4.3),

$$H(z) = \Gamma(0). \tag{4.4}$$

We shall now evaluate $\Gamma_1(u)$ and $\Gamma_2(u)$ in terms of $F(x)$ and $G(y)$, and also $\gamma_1(t)$ and $\gamma_2(t)$ in terms of $f(t)$, $g^+(t)$, and $g^-(t)$.

Let us assume for a moment that $P\{Y > 0\} \neq 0$; that is, that $G(0) < 1$. The conditional distribution of Y relative to the hypothesis that $Y > 0$ then has the d.f.

$$G_1(y) = \begin{cases} \dfrac{G(y) - G(0)}{1 - G(0)}, & y \geqq 0, \\ 0, & y < 0. \end{cases} \tag{4.5}$$

The d.f. of $-zY$ relative to this hypothesis is $G_1(-y/z)$ if $z < 0$, and $1 - G_1[(-y/z) - 0]$ if $z > 0$.

[6] By $P(A \mid b)$ is meant the conditional probability of the event A relative to the hypothesis b.

It is well known that the corresponding d.f. of the sum $X + (-zY)$ is given by a convolution of the d.f.'s of X and $(-zY)$.[7] In the present case, this result takes the form

$$(4.6)\quad P\{X - zY \leqq u \mid Y > 0\} = \begin{cases} \int_{-\infty}^{+\infty} F(u-v)\, dG_1\left(-\frac{v}{z}\right), & z < 0, \\ \int_{-\infty}^{+\infty} F(u-v)\, d\left[1 - G_1\left(-\frac{v}{z} - 0\right)\right], & z > 0. \end{cases}$$

Referring to the definition of these Lebesgue-Stieltjes integrals [4, pp. 662–663], we see that the change of variables $w = -v/z$ yields the equations

$$(4.7)\qquad P\{X - zY \leqq u \mid Y > 0\} = \begin{cases} \int_0^{\infty} F(u + zw)\, dG_1(w), & z < 0, \\ \int_0^{\infty} F(u + zw)\, dG_1(w - 0), & z > 0. \end{cases}$$

Now the definition of the variation of $G_1(y)$ [4, pp. 341–342] used in forming these Lebesgue-Stieltjes integrals makes no distinction between the variation of $G_1(y)$ and that of $G_1(y - 0)$ over any bounded set contained in an interval of integration $a \leq y < \infty$, provided that $G_1(y)$ is continuous at a in the two-sided sense. Since $G_1(y)$ is continuous at $y = 0$ in this sense, it is possible to replace $G_1(w - 0)$ by $G_1(w)$ in the second of the two integrals in (4.7).

Equation (4.7) is clearly true for $z = 0$ as well as for all other values of z. Referring to (4.5) and (4.3), we see that

$$\Gamma_1(u) = \int_0^{\infty} F(u + zw)\, dG(w), \quad \text{all } z.$$

The c.f. of the convolution (4.6) is the product of the c.f.'s of X and of the conditional distribution of $-zY$ [1, p. 36]. This product is $f(t) \cdot \int_0^{\infty} e^{-itzy}\, dG_1(y)$. Thus by (4.5), (4.3), and (4.1),

$$(4.8)\qquad \gamma_1(t) = [1 - G(0)]\left[f(t) \cdot \int_0^{\infty} e^{-itzy}\, dG_1(y)\right] = f(t) g^{+}(-tz).$$

We have established (4.7) and (4.8) under the condition that $P\{Y > 0\} \neq 0$. However, it is obvious that they are trivially true if $P\{Y > 0\} = 0$.

We turn now to $\Gamma_2(u)$. Supposing that $P\{Y < 0\} \neq 0$, the conditional distribution of Y relative to the hypothesis that $Y < 0$, has the d.f.

$$G_2(y) = \begin{cases} \frac{G(y)}{G(0)}, & y < 0, \\ 1, & y \geqq 0. \end{cases}$$

[7] See [1, pp. 35–36]; also [7].

The conditional distribution of zY has the d.f. $G_2(y/z)$ for $z > 0$, and $1 - G_2[(y/z) - 0]$ for $z < 0$. The d.f. of $-X$ is $1 - F(-x - 0)$. Thus

$$P\{zY - X \leqq u \mid Y < 0\} = \begin{cases} \int_{-\infty}^{+\infty} \{1 - F[-(u - v) - 0]\}\, d\left[1 - G_2\left(\frac{v}{z} - 0\right)\right], & z < 0, \\ \int_{-\infty}^{+\infty} \{1 - F[-(u - v) - 0]\}\, dG\left(\frac{v}{z}\right), & z > 0, \end{cases}$$

$$= 1 - \int_{-\infty}^{0} F(zw - u - 0)\, dG_2(w).$$

Evidently the first and last members of this equation are equal for $z = 0$ as well as for all other values of z. From (4.3) we obtain

$$\Gamma_2(u) = G(0) - \int_{-\infty}^{0} F(zw - u - 0)\, dG(w), \quad \text{all } z.$$

Also, as before,

$$\gamma_2(t) = f(-t)g^{-}(zt).$$

Obviously, the last two equations are still true if $P\{Y < 0\} = 0$.

To summarize, we have shown that

$$\text{(4.9)} \quad \Gamma(u) = G(0) + \int_0^{\infty} F(u + zw)\, dG(w) - \int_{-\infty}^{0} F(zw - u - 0)\, dG(w), \quad \text{all } z;$$

$$\text{(4.10)} \quad \gamma(t) = f(t)g^{+}(-zt) + f(-t)g^{-}(zt).$$

Referring now to (4.4) and letting $u = 0$ in (4.9), we are able to state the following theorem:

THEOREM 4.1: *If X and Y are independent chance variables with respective d.f.'s $F(x)$ and $G(y)$, the d.f. of the quotient X/Y is given by the formula*

$$\text{(4.11)} \quad H(z) = G(0) + \int_0^{\infty} F(zw)\, dG(w) - \int_{-\infty}^{0} F(zw - 0)\, dG(w)$$

for all values of z.

We shall not attempt to make a careful study of the above formula, such as the studies which certain writers have made of convolutions. However, it does seem desirable to place on record here certain remarks concerning it of a more or less superficial character. For convenience in later reference, we state these remarks in the form of four lemmas.

LEMMA 4.1: *Let M_1 be the set of all values of z such that if $z \epsilon M_1$, the set of discontinuity points of $F(zw)$ on the w-axis has a point in common with the point spectrum of $G(w)$. Then if $z \epsilon C(M_1)$,[8] the integrals $\int_0^{\infty} F(zw \pm 0)\, dG(w)$,*

[8] By $C(M_1)$ we mean the complement of M_1 with respect to the z-axis.

$\int_{-\infty}^{0} F(zw \pm 0)\, dG(w)$, *are Riemann-Stieltjes integrals and consequently the integrands can be replaced by* $F(zw)$ *without altering the values of the integrals.*

The lemma follows immediately from the definitions of Riemann-Stieltjes and Lebesgue-Stieltjes integrals.

LEMMA 4.2: *The set* M_1 *is denumerable.*

The proof can easily be supplied by the reader.

LEMMA 4.3: *Let* M_2 *be the set of all values of* z *such that if* $z \in M_2$, $\Gamma(u)$ *is discontinuous at* $u = 0$. *Then* $M_2 \subset M_1$.

To prove this statement, we first observe that $\Gamma(u)$ is a genuine d.f. [1, p. 11]. For obviously $\Gamma(-\infty) = 0$, $\Gamma(+\infty) = 1$, and since $\Gamma_1(u)$ and $\Gamma_2(u)$ are both products of d.f.'s into constants, these two functions, and therefore $\Gamma(u)$, must be continuous from the right. It is this last property of $\Gamma(u)$ which is needed for our present purposes; in particular, we have the relation $\lim_{u\to+0}\Gamma(u) = \Gamma(0) = H(z)$. On the other hand, by the general convergence theorem for Lebesgue-Stieltjes integrals [4, pp. 663–664], we have

$$\lim_{u\to-0} \Gamma(u) = G(0) + \int_0^{\infty} F(zw - 0)\, dG(w) - \int_{-\infty}^{0} F(zw)\, dG(w).$$

If z be chosen so that this integral and the ones in (4.11) are all Riemann-Stieltjes integrals, the expression $(zw - 0)$, wherever it appears, may be replaced by zw without changing the values of the integrals. Thus for such a value of z, $\Gamma(+0) = \Gamma(-0)$. According to Lemma 4.1, we can be sure that at least if $z \in C(M_1)$, the integrals here will be Riemann-Stieltjes integrals, so our proposition is proved.

Since $H(z_1 + 0)$ is equal to $\Gamma(+0)$ with $z = z_1$, and $H(z_1 - 0)$ is equal to $\Gamma(-0)$ with $z = z_1$, we have the following result:

LEMMA 4.4: *The set* M_2 *is the set of discontinuity points of* $H(z)$.

By using the alternate form of the convolutions used to derive (4.9), we obtain a representation of $\Gamma(u)$ somewhat more complicated than that appearing in (4.9). The corresponding formula for $H(z)$ is as follows:

$$(4.12)\quad H(z) = \begin{cases} G(0)[1 - F(-0)] - G(0)F(0) + \int_{-\infty}^{0} G\left(\frac{v}{z}\right) dF(v) \\ \qquad - \int_0^{\infty} G\left(\frac{v}{z} - 0\right) dF(v - 0), & z < 0; \\ F(0)[1 - G(0)] + G(0)[1 - F(-0)], & z = 0; \\ 1 + G(0)[1 - F(-0)] - G(0)F(0) + \int_{-\infty}^{0} G\left(\frac{v}{z}\right) dF(v - 0) \\ \qquad - \int_0^{\infty} G\left(\frac{v}{z} - 0\right) dF(v), & z > 0. \end{cases}$$

5. **Representation of $H(z)$ by characteristic functions..** A simple algebraic formula connecting the c.f. of Z with those of X and Y is not available. However, there exists an interesting representation of $H(z)$ in terms of the functions $f(t)$, $g^{+}(t)$, and $g^{-}(t)$. The result may be stated as follows:

THEOREM 5.1:[9] *Let the distributions of the independent variables X and Y have finite first absolute moments, and let the integral*

$$(5.1)\qquad \left(\int_{-\infty}^{-1} + \int_{1}^{\infty}\right) \frac{| f(t)g^{+}(-zt) + f(-t)g^{-}(zt) |}{t} dt$$

be finite for each value of z. Let $\Delta(u)$ be any d.f. with a finite first absolute moment, and let $\left(\int_{-\infty}^{-1} + \int_{1}^{\infty}\right) \left| \frac{\delta(t)}{t} \right| dt$ be finite, where $\delta(t)$ is the c.f. of $\Delta(u)$. Then

$$(5.2)\qquad H(z) = \Delta(0) - \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{f(t)g^{+}(-zt) + f(-t)g^{-}(zt) - \delta(t)}{t} dt.$$

If the integral obtained by formal differentiation under the integral sign with respect to z in (5.2) is uniformly convergent in a certain interval I, then the frequency function $h(z)$ of the distribution of z exists in that interval and is given by the formula

$$h(z) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} [f(t)g^{+\prime}(-zt) - f(-t)g^{-\prime}(zt)]\, dt, \qquad z \in I.$$

We remark that the condition (5.1) will be satisfied for all values of z if $f(t)$ alone satisfies a similar condition, inasmuch as $| g^{+}(t) | \leqq 1$, $| g^{-}(t) | \leqq 1$. Important special cases of the theorem arise when $\Delta(u)$ is replaced by $F(u)$ or $G(u)$, and when $\Delta(u)$ is so chosen that $\Delta(0) = 0$.

Our proof of the theorem will depend on a rather general result due to Cramér [1, Theorem 12], which we shall restate here in the special form applicable to the problem at hand.

LEMMA 5.1: *Let $R(u)$ be a function of bounded variation over the infinite interval $-\infty < u < \infty$, let $\lim_{u\to-\infty} R(u) = \lim_{u\to+\infty} R(u) = 0$, and let $r(t) = \int_{-\infty}^{+\infty} e^{itu}\, dR(u)$. If* (a) $\int_{-\infty}^{+\infty} | u |\, dR(u)$ *and* (b) $\left(\int_{-\infty}^{-1} + \int_{1}^{\infty}\right) \left| \frac{r(t)}{t} \right| dt$, *both are finite, then for every value of u,*

$$R(u) = -\frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{r(t)}{t} e^{-itu}\, dt.$$

To prove Theorem 5.1, we observe that since $\Gamma(u)$ is a d.f. (see proof of Lemma 4.3), the difference $\Gamma(u) - \Delta(u)$ is a function similar to the function $R(u)$ of the lemma. If we do let $R(u) = \Gamma(u) - \Delta(u)$, it follows at once that $r(t) = \gamma(t) - \delta(t) = f(t)\cdot g^{+}(-zt) + f(-t)g^{-}(zt) - \delta(t)$. If we can verify that this $R(u)$

[9] The theorem is due to Cramér in the case in which $G(0) = 0$, and $\Delta(u) = G(u)$. See [1, Theorem 16].

satisfies conditions (a) and (b) of the lemma, then we shall have established the relation,

$$\Gamma(u) = \Delta(u) - \frac{1}{2\pi i}\int_{\infty}^{+\infty} \frac{f(t)g^{+}(-zt) + f(-t)g^{-}(zt) - \delta(t)}{t} e^{-itu}\, dt,$$

for all values of u, and letting $u = 0$ in this equation, we shall obtain (5.2).

Condition (b) in the lemma is taken care of by (5.1) and the condition on $\delta(t)$ in Theorem 5.1. Clearly condition (a) will be satisfied if it turns out that $\Gamma(u)$ has a finite first absolute moment. Now the existence of finite first absolute moments of X and Y will insure the existence of finite first absolute moments for the conditional distributions involved in the definitions of $\Gamma_1(u)$ and $\Gamma_2(u)$, because $E \mid X - zY \mid \leqq E \mid X \mid + \mid z \mid E \mid Y \mid$. It follows quite readily from this that the first absolute moment of $\Gamma(u)$ is finite. The proof of the theorem is complete.

6. **Distributions of variable form.** We consider now the case in which the distributions of the numerator and denominator approach limiting forms.

THEOREM 6.1: *Let the independent variables X_α and Y_β have respective d.f.'s $F_\alpha(x)$ and $G_\beta(y)$ which depend upon the two parameters α and β. Let $H_{\alpha,\beta}(z)$ be the d.f. of the quotient $Z_{\alpha,\beta} = X_\alpha/Y_\beta$. If there exist two chance variables X and Y with respective distribution functions $F(x)$ and $G(y)$ such that $\lim_{\alpha\to\infty} F_\alpha(x) = F(x)$ at all points of continuity of $F(x)$, and $\lim_{\beta\to\infty} G_\beta(y) = G(y)$, at all points of continuity of $G(y)$, then*

$$\lim_{\substack{\alpha\to\infty\\ \beta\to\infty}} H_{\alpha,\beta}(z) = \lim_{\alpha\to\infty}\lim_{\beta\to\infty} H_{\alpha,\beta}(z) = \lim_{\beta\to\infty}\lim_{\alpha\to\infty} H_{\alpha,\beta}(z) = H(z) \tag{6.1}$$

at all points of continuity of $H(z)$, where $H(z)$ is the d.f. of the variable X/Y. The double limit in (6.1) *is uniform in any finite or infinite interval of continuity of $H(z)$.*

In the interpretation of the limits involved in this theorem, it is to be understood that in the hypotheses, α may tend to infinity over any unbounded set T_α of the α-axis, and β may tend to infinity over any unbounded set T_β of the β-axis, provided that in (6.1), α and β are restricted so that $\alpha \in T_\alpha$ and $\beta \in T_\beta$.

To prove the theorem, we introduce functions $f_\alpha(t)$, $g_\beta^{+}(t)$, $g_\beta^{-}(t)$, $\Gamma_{\alpha,\beta}(u)$, $\gamma_{\alpha,\beta}(t)$, which are defined by equations (4.1) and (4.3) with F, G, X, Y replaced respectively by F_α, G_β, X_α, Y_β. On the other hand, with reference to the distributions of X and Y, we employ the notation of section 4 without modification. According to the work in that section, $\Gamma(u)$ is given by (4.9) and its c.f. $\gamma(t)$ is given by (4.10). Also,

$$\gamma_{\alpha,\beta}(t) = f_\alpha(t)g_\beta^{+}(-zt) + f_\alpha(-t)g_\beta^{-}(zt).$$

But it is an immediate consequence of our hypotheses that $\lim_{\alpha\to\infty} f_\alpha(t) = f(t)$,

$\lim_{\beta\to\infty} g_\beta^+(t) = g^+(t)$, and $\lim_{\beta\to\infty} g_\beta^-(t) = g^-(t)$, all of the limits being uniform in any finite interval of values of t.[10] Thus

$$\lim_{\substack{\alpha\to\infty\\ \beta\to\infty}} \gamma_{\alpha,\beta}(t) = \lim_{\alpha\to\infty}\lim_{\beta\to\infty} \gamma_{\alpha,\beta}(t) = \lim_{\beta\to\infty}\lim_{\alpha\to\infty} \gamma_{\alpha,\beta}(t) = \gamma(t), \tag{6.2}$$

uniformly in any finite interval on the t-axis.

Consider the extreme members of (6.2). It follows immediately from a well-known general theorem[11] that $\lim_{\alpha\to\infty,\beta\to\infty} \Gamma_{\alpha,\beta}(u) = \Gamma(u)$ at all continuity points of $\Gamma(u)$. Then since $H_{\alpha,\beta}(z) = \Gamma_{\alpha,\beta}(0)$ and $H(z) = \Gamma(0)$, we find that

$$\lim_{\substack{\alpha\to\infty\\ \beta\to\infty}} H_{\alpha,\beta}(z) = H(z), \qquad z \in C(M_2),$$

where M_2 is the set defined in Lemma 4.3. By Lemma 4.4, the set M_2 is the set of discontinuity points of $H(z)$, so the equality of the first and last members of (6.1) is established at all continuity points of $H(z)$. The uniformity of the limit is due to a general property of convergent sequences of d.f.'s; see [1, p. 31].

The existence and equivalence to $H(z)$ of each of the iterated limits in (6.1) may be established by two consecutive applications of the foregoing argument, and by the use of (6.2). We leave the details to the reader.

It is to be remarked that both $H_{\alpha,\beta}(z)$ and $H(z)$ can be represented by (4.11), provided, of course, that F and G in (4.11) are replaced by F_α and G_β in the case of $H_{\alpha,\beta}$; thus our theorem essentially states that the order of the double limit and the integration is immaterial in this formula. A similar remark applies to formula (5.2).

The reader is reminded that we have tacitly been assuming that the d.f. of any variable appearing in a denominator is continuous at the origin. In case $G_\beta(y)$ does not satisfy this condition, but $G(y)$ does satisfy it, and if, as suggested in section 2, we consider $H_{\alpha,\beta}(y)$ to be the d.f. of the conditional distribution of $Z_{\alpha,\beta}$ relative to the hypothesis that $Y_\beta \neq 0$, then it can be shown rather easily that Theorem 6.1 remains true with this modified interpretation. But if $G(y)$ is discontinuous at the origin, and if $H(z)$ is interpreted as the d.f. of the conditional distribution, then (6.1) may be no longer true, as can be shown by trivial examples.

Perhaps the most important cases of variable distributions arise in the consideration of sums of independent chance variables. We accordingly present the following synthesis of Theorem 6.1 and a simple case of the Central Limit Theorem.

THEOREM 6.2: *Let U_1, U_2, $\cdots$, be a sequence of identically distributed chance variables, each with mean zero and (finite) standard deviation σ_U, and let V_1,*

[10] See [1, p. 30].

[11] See [1, Theorem 11]. The result needed here is a trivial extension of the theorem cited.

$V_2, \cdots$, be a sequence of identically distributed chance variables, each with mean zero and (finite) standard deviation σ_V. Furthermore, let the variables U_i and V_j be all independent, $i = 1, 2, \cdots, j = 1, 2, \cdots$. If m and n tend to infinity in such a way that

$$\lim_{\substack{m\to\infty \\ n\to\infty}} \sqrt{\frac{n}{m}} = k \neq 0, \tag{6.3}$$

then the d.f. of the conditional distribution of the variable

$$W_{m,n} = \frac{U_1 + U_2 + \cdots + U_m}{V_1 + V_2 + \cdots + V_n},$$

relative to the hypothesis that the denominator is different from zero, tends uniformly to the function

$$J(w) = \int_{-\infty}^{w} \frac{k\sigma_U \sigma_V}{\pi} \cdot \frac{1}{\sigma_V^2 k^2 u^2 + \sigma_U^2}\, du. \tag{6.4}$$

For if we let

$$Z_{m,n} = \frac{\dfrac{U_1 + U_2 + \cdots + U_m}{\sigma_U\sqrt{m}}}{\dfrac{V_1 + V_2 + \cdots + V_n}{\sigma_V\sqrt{n}}},$$

then $W_{m,n} = \sqrt{m/n}(\sigma_U/\sigma_V)Z_{m,n}$. The Central Limit Theorem [1, Theorem 20] states that the d.f.'s of the numerator and denominator of $Z_{m,n}$ each tend to the function $\int_{-\infty}^{x} (1/\sqrt{2\pi})e^{-t^2/2}\, dt$, which is the d.f. of a normal distribution with mean zero and variance one. By (3.3), the quotient of two variables, each of which has this d.f., has the continuous d.f. $H(z) = \int_{-\infty}^{z} (1/\pi)[1/(1 + x^2)]\, dx$. If we let $H_{m,n}(z)$ denote the d.f. of the conditional distribution of $Z_{m,n}$, relative to the hypothesis that the denominator of $Z_{m,n}$ is different from zero, then by Theorem 6.1, $\lim_{m\to\infty, n\to\infty} H_{m,n}(z) = H(z)$ uniformly in z. Now the d.f. of the conditional distribution of $W_{m,n}$ is $H_{m,n}[\sqrt{n/m}(\sigma_V/\sigma_U)w]$, and because of (6.3) and the uniformity of the limit of $H_{m,n}(z)$, this approaches $H[k(\sigma_V/\sigma_U)w]$. Differentiating the last expression with respect to w, we find that the resulting frequency function is equal to $J'(w)$; and this concludes the proof.

As an application of the theorem, let us consider the following problem. From an urn containing white and black balls in the proportion of p to $1 - p$, we shall make 100 random drawings of a single ball with replacement after each drawing. Let $W_{50,50}$ be the ratio of the deviation of the number of white balls in the first 50 drawings from the expected number, to the deviation of the number of white balls in the second 50 drawings from the expected number. What is the approximate value of w for which $P\{W_{50,50} \geqq w \mid b\} = .05$, where the hypothesis b is that the denominator of $W_{50,50}$ shall be different from zero?[12]

To answer this question, we observe that the numerator and denominator of $W_{50,50}$ can each be expressed as the sum of 50 independent identical chance variables, each with mean zero and with variance $p(1 - p)$. Thus according to Theorem 6.2, the approximate d.f. of $W_{50,50}$ is

$$J(w) = \int_{-\infty}^{w} \frac{1}{\pi} \frac{1}{1 + u^2} du = \frac{1}{2} + \frac{1}{\pi} \arctan w,$$

and the required value of w satisfies the equation $J(\infty) - J(w) = .05$. The solution of this equation (correct to one decimal place) is $w = 6.3$.

It is perhaps needless to remark that a study of the error involved in supposing $J(w)$ to be the d.f. of $W_{m,n}$ in Theorem 6.2, must necessarily precede the unreserved acceptance of numerical results obtained by means of that theorem.

7. **Products of chance variables.** We conclude this paper with a rather brief treatment of the distribution of the product of two chance variables. To preserve a notation uniform with that of the preceding sections, we shall write the product as $X = YZ$, where the d.f.'s of X, Y, and Z are to be denoted, as before, by $F(x)$, $G(y)$, and $H(z)$, respectively. The existence of $F(x)$ is readily proved by the methods of section 2. The assumption that $P\{Y = 0\} = 0$ is of course unnecessary here, and will be dropped in this section.

In the continuous case, an argument similar to the one employed in section 3 will establish the following result:

THEOREM 7.1: *If the joint variable (Y, Z) has the frequency function $\psi(y, z)$, then*

$$F(x) = \int_{-\infty}^{x} \left[\int_{-\infty}^{+\infty} \left| \frac{1}{v} \right| \psi\left(\frac{u}{v}, v\right) dv \right] du$$
$$= \int_{-\infty}^{x} \left[\int_{-\infty}^{+\infty} \left| \frac{1}{v} \right| \psi\left(v, \frac{u}{v}\right) dv \right] du,$$

and consequently $F(x)$ is an absolutely continuous function of x. The frequency function of the distribution of X exists almost everywhere, and is given by the formula

$$(7.1) \qquad f(x) = F'(x) = \int_{-\infty}^{+\infty} \left| \frac{1}{v} \right| \psi\left(\frac{x}{v}, v\right) dv = \int_{-\infty}^{+\infty} \left| \frac{1}{v} \right| \psi\left(v, \frac{x}{v}\right) dv.$$

In the discontinuous case, with Y and Z independent, we can write $X = ZY = Z/(1/Y)$ and use Theorem 4.1 to derive a formula for $F(x)$. We have:

$$F(x) = P\{X \leqq x\} = P\{Y \neq 0\}P\{X \leqq x \mid Y \neq 0\} + P\{X \leqq x; Y = 0\}.$$

[12] This hypothesis would always be fulfilled in case $50p$ is not an integer.

Excluding for a moment the trivial case in which $P\{Y \neq 0\} = 0$, let $G_1(y)$ be the d.f. of the conditional distribution of $(1/Y)$ relative to the hypothesis that $Y \neq 0$. Then

$$P\{Y \neq 0\}G_1(y) = \begin{cases} G(-0) + 1 - G\left(\frac{1}{y} - 0\right), & y > 0, \\ G(-0), & y = 0, \\ G(-0) - G\left(\frac{1}{y} - 0\right), & y < 0. \end{cases}$$

It is to be observed that $G_1(y)$ is continuous at $y = 0$. Using Theorem 4.1, we find that

$$P\{X \leqq x \mid Y \neq 0\} = G_1(0) + \int_0^\infty H(xw)\, dG_1(w) - \int_{-\infty}^0 H(xw - 0)\, dG_1(w).$$

So

$$\begin{aligned} P\{Y \neq 0\}&P\{X \leqq x \mid Y \neq 0\} \\ &= G(-0) + \int_{0+0}^\infty H(xw)\, d\left[-G\left(\frac{1}{w} - 0\right)\right] - \int_{-\infty}^{0-0} H(xw - 0)\, d\left[-G\left(\frac{1}{w} - 0\right)\right] \\ &= G(-0) + \int_{0+0}^\infty H\left(\frac{x}{v}\right) dG(v) - \int_{-\infty}^{0-0} H\left(\frac{x}{v} - 0\right) dG(v). \end{aligned}$$

This equation is trivially true if $P\{Y \neq 0\} = 0$. Also,

$$P\{X \leqq x;\, Y = 0\} = \begin{cases} 0, & x < 0, \\ G(0) - G(-0), & x \geqq 0. \end{cases}$$

Thus we obtain the following theorem:

THEOREM 7.2: *If Y and Z are independent chance variables with respective d.f.'s $G(y)$ and $H(z)$, then the d.f. of their product is given by the formula*

$$F(x) = \int_{0+0}^\infty H\left(\frac{x}{v}\right) dG(v) - \int_{-\infty}^{0-0} H\left(\frac{x}{v} - 0\right) dG(v) + \begin{cases} G(-0), & x < 0, \\ G(0), & x \geqq 0, \end{cases} \tag{7.2}$$

for all values of x.

REFERENCES

[1] H. CRAMÉR, *Random Variables and Probability Distributions*, Cambridge, 1937.

[2] E. C. FIELLER, "The distribution of the index in a normal bivariate population," *Biometrika*, Vol. 24 (1932), pp. 428–440.

[3] R. C. GEARY, "The frequency distribution of the quotient of two normal variates," *Roy. Stat. Soc. Jour.*, Vol. 93 (1930), pp. 442–446.

[4] E. W. HOBSON, *The Theory of Functions of a Real Variable*, Vol. 1, Cambridge, 1927.

[5] E. V. HUNTINGTON, "Frequency distribution of product and quotient," *Annals of Math. Stat.*, Vol. 10 (1939), pp. 195–198.

[6] H. KESTELMAN, *Modern Theories of Integration*, Oxford, 1937.

[7] A. WINTNER, "On the addition of independent distributions," *Am. Jour. Math.*, Vol. 56 (1934), pp. 8–16.