

ON THE DISTRIBUTION OF TWO RANDOM MATRICES USED IN CLASSIFICATION PROCEDURES¹

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Summary. Two classification statistics discussed in the literature can be written as functions of the elements of a $2 \cdot 2$ symmetric random matrix M . An analytic derivation is given of the distribution of M , and of a related matrix M^* , extending earlier work on distribution theory by Wald [1] and Anderson [2].

1. Introduction. A problem of classification discussed by Wald [1] and Anderson [2] may be described as follows. We have $N_1 + N_2 + 1$ independent p -dimensional chance vectors. We know that the first N_1 vectors are observations from a population π_1 , the following N_2 are observations from a population π_2 , and the last vector is an observation from a population π , where π is either π_1 or π_2 . It is assumed that the probability distribution in both π_1 and π_2 is multivariate normal with the same covariance matrix Σ ; the vector of expected values is $\mu^{(1)}$ in π_1 and $\mu^{(2)}$ in π_2 . The values of $\mu^{(1)}$, $\mu^{(2)}$, and Σ are not known. Let X denote the $p \cdot (N_1 + N_2 + 1)$ matrix of observations. On the basis of X we want to classify the last observation, $X_{N_1+N_2+1}$ as coming from π_1 or π_2 ; that is, we want to make one of the two decisions, $\pi = \pi_1$ or $\pi = \pi_2$.

When the parameter values are known, the class of Bayes solutions is easily found, resulting in pairs of classification regions of the form

$$(1) \quad T^* \leq k \quad \text{and} \quad T^* > k,$$

where

$$(2) \quad T^* = X'_{N_1+N_2+1} \Sigma^{-1} (\mu^{(1)} - \mu^{(2)}) - \frac{1}{2} (\mu^{(1)} + \mu^{(2)})' \Sigma^{-1} (\mu^{(1)} - \mu^{(2)}).$$

Both Wald and Anderson propose, therefore, the use of classification statistics derived from (2) by substituting estimates for the unknown parameter values. Wald considers principally the statistic

$$(3) \quad U = X'_{N_1+N_2+1} S^{-1} (\bar{X}^{(1)} - \bar{X}^{(2)}),$$

where

$$\bar{X}^{(1)} = (1/N_1) \sum_{t=1}^{N_1} X_t, \quad \bar{X}^{(2)} = (1/N_2) \sum_{t=N_1+1}^{N_1+N_2} x_t,$$

and

$$S = (1/(N_1 + N_2 - 2)) \cdot \left[\sum_{t=1}^{N_1} (X_t - \bar{X}^{(1)})(X_t - \bar{X}^{(1)})' + \sum_{t=N_1+1}^{N_1+N_2} (X_t - \bar{X}^{(2)})(X_t - \bar{X}^{(2)})' \right].$$

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Anderson proposes rather the statistic

$$(4) \quad W = X'_{N_1+N_2+1} S^{-1} (\bar{X}^{(1)} - \bar{X}^{(2)}) - \frac{1}{2} (\bar{X}^{(1)} + \bar{X}^{(2)})' S^{-1} (\bar{X}^{(1)} - \bar{X}^{(2)}).$$

If we let $A = (N_1 + N_2 - 2)S$ and $[(N_1 N_2)^{\frac{1}{2}} / (N_1 + N_2)^{\frac{1}{2}}] (\bar{X}^{(1)} - \bar{X}^{(2)}) = Z$, we can write $U = [(N_1 + N_2)^{\frac{1}{2}} (N_1 + N_2 - 2) / (N_1 N_2)^{\frac{1}{2}}] X'_{N_1+N_2+1} A^{-1} Z$. Under either alternative the vector variable Z has an expected value $[(N_1 N_2)^{\frac{1}{2}} / (N_1 + N_2)^{\frac{1}{2}}] (\mu^{(1)} - \mu^{(2)})$, and covariance matrix Σ . If $\pi = \pi_1$, the expected value of $X_{N_1+N_2+1}$ is $\mu^{(1)}$; if $\pi = \pi_2$, the expected value of $X_{N_1+N_2+1}$ is $\mu^{(2)}$. Thus, in either instance, the sampling distribution of U is contained as a special case of the sampling distribution of

$$(5) \quad V = k Y_1 A^{-1} Y_2,$$

where k is a known scalar, Y_1 and Y_2 are p -dimensional normal variables with expected values ζ and ξ , say, respectively, and A is a $p \times p$ symmetric matrix with a Wishart distribution involving n degrees of freedom; the 3 sets of variables are independently distributed with the same covariance matrix Σ . Further, the statistic W can be written

$$\begin{aligned} W &= (X_{N_1+N_2+1} - (1/(N_1 + N_2))(N_1 \bar{X}^{(1)} + N_2 \bar{X}^{(2)}))' S^{-1} (\bar{X}^{(1)} - \bar{X}^{(2)}) \\ &\quad + ((1/(N_1 + N_2))(N_1 \bar{X}^{(1)} + N_2 \bar{X}^{(2)}) - \frac{1}{2} (\bar{X}^{(1)} - \bar{X}^{(2)}))' S^{-1} (\bar{X}^{(1)} - \bar{X}^{(2)}) \\ &= (X_{N_1+N_2+1} - (1/(N_1 + N_2))(N_1 \bar{X}^{(1)} + N_2 \bar{X}^{(2)}))' S^{-1} (\bar{X}^{(1)} - \bar{X}^{(2)}) \\ &\quad + [(N_1 - N_2)/(2N_1 + 2N_2)] (\bar{X}^{(1)} - \bar{X}^{(2)})' S^{-1} (\bar{X}^{(1)} - \bar{X}^{(2)}). \end{aligned}$$

Or, if we let

$$\begin{aligned} &[(N_1 + N_2)^{\frac{1}{2}} / (N_1 + N_2 + 1)^{\frac{1}{2}}] \\ &\quad \cdot (X_{N_1+N_2+1} - (1/(N_1 + N_2))(N_1 \bar{X}^{(1)} + N_2 \bar{X}^{(2)})) = Z^*, \end{aligned}$$

we can write

$$\begin{aligned} W &= [(N_1 + N_2 + 1)^{\frac{1}{2}} (N_1 + N_2 - 2) / (N_1 N_2)^{\frac{1}{2}}] Z^* A^{-1} Z \\ &\quad + [(N_1 - N_2)(N_1 + N_2 - 2) / (2N_1 N_2)] Z' A^{-1} Z. \end{aligned}$$

The vector variable Z^* is normally distributed, independently of Z , with covariance matrix Σ . Under the hypothesis $\pi = \pi_1$, the expected value of Z^* is $[N_2 / (N_1 + N_2)^{\frac{1}{2}} (N_1 + N_2 + 1)^{\frac{1}{2}}] (\mu^{(1)} - \mu^{(2)})$; under the hypothesis $\pi = \pi_2$, the expected value of Z^* is $[-N_1 / (N_1 + N_2)^{\frac{1}{2}} (N_1 + N_2 + 1)^{\frac{1}{2}}] (\mu^{(1)} - \mu^{(2)})$. The sampling distribution of W under either alternative is thus a special case of the sampling distribution of

$$(6) \quad W^* = a Y_1' A^{-1} Y_2 + b Y_2' A^{-1} Y_2,$$

where a and b are known scalars, and Y_1 , Y_2 , and A are defined as before. In the case of W , the vectors ζ and ξ are proportional to $(\mu^{(1)} - \mu^{(2)})$.

Wald [1] investigated the general sampling distribution of V , and showed that the statistic can be expressed as a function of 3 variables. These variables, which

he called $m_1, m_2,$ and $m_3,$ and which become $m_{11}, m_{22},$ and m_{12} in our notation, are the elements of the symmetric matrix

$$(7) \quad M = Y'B^{-1}Y,$$

where $Y = (Y_1, Y_2)$ and $B = A + YY'$. The classification statistic V can be written

$$(8) \quad V = k \frac{m_{12}}{(1 - m_{11})(1 - m_{22}) - m_{12}^2}.$$

Wald showed geometrically that the distribution of M is a constant multiple of the product of 3 factors, the first a product of gamma- and beta-functions, the second an exponential term, and the third the expected value of a matrix of noncentral Wishart variables which was not evaluated. Anderson [2] has evaluated this product in the case when ζ and ξ are proportional.

In this paper, we give an analytic derivation of the distribution of M in the case when ζ and ξ are proportional, obtaining the constant of the distribution (which Wald and Anderson did not obtain). From the distribution of $M,$ we obtain the distribution of the related matrix

$$(9) \quad M^* = Y'A^{-1}Y.$$

It can be easily shown that

$$(10) \quad M = M^*(I + M^*)^{-1}.$$

These distributions are useful because of interest in the exact sampling distributions of U and $W.$ Further, as will be shown in a subsequent paper, an approach to the classification problem based on the principle of invariance results in a complete class of classification regions depending only on the elements of the matrix $M^*,$ or equivalently of the matrix $M,$ and on a single function of the parameters.

2. Distribution of $M.$ We can write $\rho = k_1\delta$ and $\xi = k_2\delta,$ where k_1 and k_2 are known scalars. The joint density function of Y and A is given by

$$(11) \quad p(Y, A) = \frac{|\Sigma|^{-\frac{1}{2}(n+2)} |A|^{-\frac{1}{2}(n-p-1)}}{2^{\frac{1}{2}p(n+2)} \pi^{p+p(p-1)/4} \prod_{i=1}^p \Gamma(\frac{1}{2}(n+1-i))} \cdot \exp \left\{ -\frac{1}{2}\lambda^2(k_1^2 + k_2^2) - \frac{1}{2} \text{tr} \Sigma^{-1}(A + YY') + \delta' \Sigma^{-1}(k_1 Y_1 + k_2 Y_2) \right\},$$

where $\lambda^2 = \delta' \Sigma^{-1} \delta.$ We make the transformation $B = A + YY'.$ This is a one-to-one transformation with Jacobian 1. We have

$$(12) \quad p(Y, B) = \frac{|\Sigma|^{-\frac{1}{2}(n+2)} |B - YY'|^{-\frac{1}{2}(n-p-1)}}{2^{\frac{1}{2}p(n+2)} \pi^{p+p(p-1)/4} \prod_{i=1}^p \Gamma(\frac{1}{2}(n+1-i))} \cdot \exp \left\{ -\frac{1}{2}\lambda^2(k_1^2 + k_2^2) - \frac{1}{2} \text{tr} \Sigma^{-1}B + \delta' \Sigma^{-1}(k_1 Y_1 + k_2 Y_2) \right\}.$$

There is a nonsingular matrix Ψ such that $\Psi\Sigma\Psi' = I$ and $\delta'\Psi' = (\lambda, 0, \dots, 0)$ with $\lambda \geq 0$. We make the transformation $Y^* = \Psi Y$ and $B^* = \Psi B \Psi'$. The Jacobian of the transformation is $|\Sigma|^{\frac{1}{2}(p+3)}$. Under the transformation

$$\begin{aligned} |B - YY'| &= |\Psi^{-1}B^*\Psi'^{-1} - \Psi^{-1}Y^*Y^{*\prime}\Psi'^{-1}| \\ &= |\Psi^{-1}(B^* - Y^*Y^{*\prime})\Psi'^{-1}| = |\Psi'\Psi|^{-1} |B^* - Y^*Y^{*\prime}| \\ &= |\Sigma| \cdot |B^* - Y^*Y^{*\prime}|, \end{aligned}$$

and

$$\delta'\Sigma^{-1}(k_1Y_1 + k_2Y_2) = \lambda(k_1y_{11}^* + k_2y_{12}^*).$$

Further,

$$\begin{aligned} M &= Y'B^{-1}Y = Y^{*\prime}\Psi'^{-1}(\Psi^{-1}B^*\Psi'^{-1})^{-1}\Psi^{-1}Y^* \\ &= Y^{*\prime}\Psi'^{-1}\Psi'B^{*-1}\Psi\Psi^{-1}Y^* = Y^{*\prime}B^{*-1}Y^*. \end{aligned}$$

The matrix B is positive definite with probability 1, and the matrix Ψ is nonsingular, so that the matrix B^* is positive definite with probability 1. We can write $B^* = TT'$, where T is a nonsingular triangular matrix whose elements are functions of the b_{ij}^* , chosen so that

$$t_{11} = b_{11}^{* \frac{1}{2}}, \quad t_{ij} = 0 \quad \text{for } j > i.$$

We use the matrix T to make the transformation $Y^* = TU$, where $U = (U_1, U_2)$ has the same dimensions as Y^* . The Jacobian of the transformation is $|T|^2 = |B^*|$. We have

$$\begin{aligned} |B^* - Y^*Y^{*\prime}| &= |TT' - TUU'T'| = |T(I - UU')T'| \\ &= |T|^2 \cdot |I - UU'| = |B^*| \cdot |I - U'U|, \end{aligned}$$

since $|I - UU'| = |I - U'U|$. Also

$$M = Y^{*\prime}B^{*-1}Y^* = U'T'(TT')^{-1}TU = U'T'T'^{-1}T^{-1}TU = U'U.$$

The joint density function of U and B^* is given by

$$\begin{aligned} (13) \quad p(U, B^*) &= \frac{|B^*|^{\frac{1}{2}(n-p+1)} |I - U'U|^{\frac{1}{2}(n-p-1)}}{2^{\frac{1}{2}p(n+2)} \pi^{p+(p-1)/4} \prod_{i=1}^p \Gamma(\frac{1}{2}(n+1-i))} \\ &\quad \cdot \exp \left\{ -\frac{1}{2}\lambda^2(k_1^2 + k_2^2) - \frac{1}{2} \text{tr } B^* + \lambda b_{11}^{* \frac{1}{2}}(k_1 u_{11} + k_2 u_{12}) \right\} \end{aligned}$$

The variables b_{ij}^* range over all values such that B^* is positive definite. The space of U is the set of points independent of the b_{ij}^* 's for which

$$(1 - U_1'U_1) \geq 0 \quad (1 - U_2'U_2) \geq 0 \quad |U'U| \geq 0$$

and

$$|I - U'U| \geq 0.$$

It can be shown (e.g., see [3]) that

$$\begin{aligned}
 (14) \quad & \int \dots \int_{\substack{B^*_{(1)} \text{ pos. def.} \\ -\infty \leq b^*_{1i}/b^*_{11} \leq \infty}} |B^*|^{\frac{1}{2}(n-p+1)} e^{-\frac{1}{2}\text{tr}B^*} db^*_{12} \dots db^*_{pp} \\
 & = 2^{\frac{1}{2}(p-1)(n+2)} \pi^{p(p-1)/4} b^*_{11}{}^{n+1} e^{-\frac{1}{2}b^*_{11}} \prod_{i=1}^{p-1} \Gamma(\frac{1}{2}(n+2-i)), \quad i, j = 2, \dots, p,
 \end{aligned}$$

where $B^*_{(1)} = (b^*_{ij} - b^*_{1i}b^*_{1j}/b^*_{11})$. Hence

$$\begin{aligned}
 (15) \quad p(U, b^*_{11}) &= \frac{\Gamma(\frac{1}{2}(n+1)) |I - U'U|^{\frac{1}{2}(n-p-1)} b^*_{11}{}^{n+1}}{\Gamma(\frac{1}{2}(n-p+2))\Gamma(\frac{1}{2}(n-p+1))2^{\frac{1}{2}(n+2)}\pi^p} \\
 &\quad \cdot \exp\{-\frac{1}{2}b^*_{11} - \frac{1}{2}\lambda^2(k_1^2 + k_2^2) + \lambda b^*_{11}{}^{\frac{1}{2}}(k_1 u_{11} + k_2 u_{12})\}.
 \end{aligned}$$

Expanding $\exp(\lambda b^*_{11}{}^{\frac{1}{2}}(k_1 u_{11} + k_2 u_{12}))$ in a power series and integrating with respect to b^*_{11} we obtain

$$\begin{aligned}
 (16) \quad p(U) &= \frac{\Gamma(\frac{1}{2}(n+1)) |I - U'U|^{\frac{1}{2}(n-p-1)} e^{-\frac{1}{2}\lambda^2(k_1^2 + k_2^2)}}{\Gamma(\frac{1}{2}(n-p+2))\Gamma(\frac{1}{2}(n-p+1))\pi^p} \\
 &\quad \cdot \sum_{j=0}^{\infty} \frac{\Gamma(\frac{1}{2}(n+2+j))2^{\frac{1}{2}j}\lambda^j (k_1 u_{11} + k_2 u_{12})^j}{j!}.
 \end{aligned}$$

We can construct an orthogonal matrix G as follows. Let

$$\begin{aligned}
 g_{1j} &= u_{j1} / \left(\sum_{i=1}^p u_{i1}^2\right)^{\frac{1}{2}} & j = 1, 2, \dots, p, \\
 g_{21} &= -\left(\sum_{i=2}^p u_{i1}^2\right)^{\frac{1}{2}} / \left(\sum_{i=1}^p u_{i1}^2\right)^{\frac{1}{2}} & g_{2j} = u_{11} u_{j1} / \left(\sum_{i=1}^p u_{i1}^2\right)^{\frac{1}{2}} \left(\sum_{i=2}^p u_{i1}^2\right)^{\frac{1}{2}}, \\
 & & j = 2, \dots, p.
 \end{aligned}$$

For $k = 3, \dots, p-1$

$$\begin{aligned}
 g_{kj} &= 0, \quad j = 1, \dots, k-2; & g_{k,k-1} &= -\left(\sum_{i=k}^p u_{i1}^2\right)^{\frac{1}{2}} / \left(\sum_{i=k-1}^p u_{i1}^2\right)^{\frac{1}{2}} \\
 g_{kj} &= u_{k-11} u_{j1} / \left(\sum_{i=k-1}^p u_{i1}^2\right)^{\frac{1}{2}} \left(\sum_{i=k}^p u_{i1}^2\right)^{\frac{1}{2}}, & j &= k, \dots, p; \\
 g_{pj} &= 0, \quad j = 1, \dots, p-2; & g_{p,p-1} &= -u_{p1} / (u_{p-1,1}^2 + u_{p1}^2)^{\frac{1}{2}}; \\
 g_{pp} &= u_{p-1,1} / (u_{p-1,1}^2 + u_{p1}^2)^{\frac{1}{2}}.
 \end{aligned}$$

We make the transformation $V = GU_2$. Under the transformation

$$\begin{aligned}
 |I - U'U| &= |(1 - U'_1 U_1)(1 - U'_2 U_2) - U'_1 U_2 U'_2 U_1| \\
 &= |(1 - U'_1 U_1)(1 - V'V) - v'_1 U'_1 U_1|
 \end{aligned}$$

and

$$u_{12} = v_1 u_{11} / \left(\sum_{i=1}^p u_{i1}^2\right)^{\frac{1}{2}} - v_2 \left(\sum_{i=2}^p u_{i1}^2\right)^{\frac{1}{2}} / \left(\sum_{i=1}^p u_{i1}^2\right)^{\frac{1}{2}}.$$

Now we make the following transformation. Let

$$\begin{aligned} u_{11} &= m_{11}^{\frac{1}{2}} \cos \theta_1, \\ u_{21} &= m_{11}^{\frac{1}{2}} \sin \theta_1 \cos \theta_1, \\ &\vdots \\ &\vdots \\ u_{p-1,1} &= m_{11}^{\frac{1}{2}} \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{p-2} \cos \theta_{p-1}, \\ u_{p1} &= m_{11}^{\frac{1}{2}} \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{p-2} \sin \theta_{p-1}. \end{aligned}$$

The Jacobian of the transformation is

$$\frac{1}{2} m_{11}^{\frac{1}{2}(p-2)} \sin \theta_1^{p-2} \sin \theta_2^{p-3} \cdots \sin \theta_{p-2}$$

with $0 \leq \theta_i \leq \pi$ for $i = 1, 2, \dots, p - 2$ and $0 \leq \theta_{p-1} \leq 2\pi$.

Under the transformation, $U'_1 U_1 = m_{11}$ and

$$\begin{aligned} &p(m_{11}, V, \theta) \\ &= \frac{\Gamma(\frac{1}{2}(n+1)) e^{-\frac{1}{2}\lambda^2(k_1^2+k_2^2)} m_{11}^{\frac{1}{2}(p-2)} \left((1-m_{11}) \left(1 - \sum_{i=1}^p v_i^2 \right) - m_{11} v_1^2 \right)^{\frac{1}{2}(n-p-1)}}{2\pi^p \Gamma(\frac{1}{2}(n-p+2)) \Gamma(\frac{1}{2}(n-p+1))} \\ &\quad \cdot \sin \theta_1^{p-2} \cdots \sin \theta_{p-2} \\ &\quad \cdot \sum_{j=0}^{\infty} \frac{\Gamma(\frac{1}{2}(n+2+j)) 2^{\frac{1}{2}j} \lambda^j (k_1 m_{11}^{\frac{1}{2}} \cos \theta_1 + k_2 v_1 \cos \theta_1 - k_2 v_2 \sin \theta_1)^j}{j!}. \end{aligned} \tag{17}$$

Since

$$\int_0^{1/2\pi} \sin^m \theta \cos^n \theta \, d\theta = \frac{1}{2} \frac{\Gamma(\frac{1}{2}(m+1)) \Gamma(\frac{1}{2}(n+1))}{\Gamma(\frac{1}{2}(m+n)+1)},$$

it follows that

$$\int_0^\pi \sin^{p-i-1} \theta_i \, d\theta_i = 2 \int_0^{1/2\pi} \sin^{p-i-1} \theta_i \, d\theta_i = \frac{\Gamma(\frac{1}{2}) \Gamma(\frac{1}{2}(p-i))}{\Gamma(\frac{1}{2}(p-i+1))}, \quad i = 2, \dots, p-2,$$

and

$$(\Gamma(\frac{1}{2}))^{p-3} \prod_{i=2}^{p-2} \frac{\Gamma(\frac{1}{2}(p-i))}{\Gamma(\frac{1}{2}(p-i+1))} = \frac{\pi^{\frac{1}{2}(p-3)}}{\Gamma(\frac{1}{2}(p-1))}.$$

Further, $\int_0^{2\pi} d\theta_{p-1} = 2\pi$ so that we have

$$\begin{aligned} &p(m_{11}, V, \theta_1) \\ &= \frac{\Gamma(\frac{1}{2}(n+1)) e^{-\frac{1}{2}\lambda^2(k_1^2+k_2^2)} m_{11}^{\frac{1}{2}(p-2)} \left((1-m_{11}) \left(1 - \sum_{i=1}^p v_i^2 \right) - m_{11} v_1^2 \right)^{\frac{1}{2}(n-p-1)}}{\Gamma(\frac{1}{2}(n-p+2)) \Gamma(\frac{1}{2}(n-p+1)) \Gamma(\frac{1}{2}(p-1)) \pi^{\frac{1}{2}(p+1)}} \\ &\quad \cdot \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} \frac{\Gamma(\frac{1}{2}(n+2+j+l))}{j! l!} \\ &\quad \cdot (2^{\frac{1}{2}} \lambda)^{j+l} (k_1 m_{11}^{\frac{1}{2}} + k_2 v_1)^j (-k_2 v_2)^l \sin \theta_1^{p-2+l} \cos \theta_1^j. \end{aligned} \tag{18}$$

Since $\int_0^\pi \sin^n \theta \cos^n \theta d\theta = 0$ for n odd, we obtain on integrating with respect to θ_1

$$\begin{aligned}
 & p(m_{11}, V) \\
 (19) \quad &= \frac{\Gamma(\frac{1}{2}(n+1))e^{-\frac{1}{2}\lambda^2(k_1^2+k_2^2)}m_{11}^{\frac{1}{2}(p-2)}\left((1-m_{11})(1-\sum_{i=1}^p v_i^2) - m_{11}v_1^2\right)^{\frac{1}{2}(n-p-1)}}{\Gamma(\frac{1}{2}(n-p+2))\Gamma(\frac{1}{2}(n-p+1))\Gamma(\frac{1}{2}(p-1))\pi^{\frac{1}{2}(p+1)}} \\
 & \cdot \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} \left\{ \frac{\Gamma(\frac{1}{2}(n+2+l)+j)\Gamma(\frac{1}{2}(p-1+l))\Gamma(j+\frac{1}{2})2^{j+\frac{1}{2}}\lambda^{2j+l}}{\Gamma(\frac{1}{2}(p+l)+j)(2j)!l!} \right. \\
 & \quad \left. \cdot (k_1 m_{11}^{\frac{1}{2}} + k_2 v_1)^{2j} (-k_2 v_2)^l \right\}.
 \end{aligned}$$

We partition the vector V into two parts, the first part consisting of the single element v_1 , and the second part of the $(p-1)$ -dimensional vector V^* . In a manner similar to that in which U_1 was transformed, we transform the vector V^* to a variable $m_{22.1} = V^* V^*$, and to $(p-2)$ angles. After integrating with respect to the angles, and simplifying the resulting expression, we obtain

$$\begin{aligned}
 (20) \quad & p(m_{11}, m_{22.1}, v_1) = \frac{\Gamma(\frac{1}{2}(n+1))e^{-\frac{1}{2}\lambda^2(k_1^2+k_2^2)}m_{11}^{\frac{1}{2}(p-2)}m_{22.1}^{\frac{1}{2}(p-3)}}{\Gamma(\frac{1}{2}(n-p+2))\Gamma(\frac{1}{2}(n-p+1))\Gamma(\frac{1}{2}(p-1))\Gamma(\frac{1}{2})} \\
 & \cdot ((1-m_{11})(1-m_{22.1}) - v_1^2)^{\frac{1}{2}(n-p-1)} \\
 & \cdot \sum_{j=0}^{\infty} \frac{\Gamma(\frac{1}{2}(n+2)+j)}{\Gamma(\frac{1}{2}p+j)j!} (\frac{1}{2}\lambda^2)^j (k_1^2 m_{11} + 2k_1 k_2 m_{11}^{\frac{1}{2}} v_1 + k_2^2 (v_1^2 + m_{22.1}))^j.
 \end{aligned}$$

Finally, we make the transformation

$$v_1 = m_{12}/m_{11}^{\frac{1}{2}} \quad m_{22.1} = m_{22} - m_{12}^2/m_{11}$$

and we have

$$\begin{aligned}
 (21) \quad & p(M) = \frac{\Gamma(\frac{1}{2}(n+1))e^{-\frac{1}{2}\lambda^2(k_1^2+k_2^2)} |M|^{\frac{1}{2}(p-3)} |I-M|^{\frac{1}{2}(n-p-1)}}{\Gamma(\frac{1}{2}(n-p+2))\Gamma(\frac{1}{2}(n-p+1))\Gamma(\frac{1}{2}(p-1))\Gamma(\frac{1}{2})} \\
 & \cdot \sum_{j=0}^{\infty} \frac{\Gamma(\frac{1}{2}(n+2)+j)}{\Gamma(\frac{1}{2}p+j)j!} (\frac{1}{2}\lambda^2)^j (k_1^2 m_{11} + 2k_1 k_2 m_{12} + k_2^2 m_{22})^j,
 \end{aligned}$$

with $0 \leq m_{11} \leq 1, 0 \leq m_{22} \leq 1, |M| \geq 0, |I-M| \geq 0$.

3. Distribution of M^* . Making the transformation defined by $M = M^*(I + M^*)^{-1}$, we obtain

$$\begin{aligned}
 (22) \quad & p(M^*) = \frac{\Gamma(\frac{1}{2}(n+1))e^{-\frac{1}{2}\lambda^2(k_1^2+k_2^2)} |M^*|^{\frac{1}{2}(p-3)}}{\Gamma(\frac{1}{2}(n-p+2))\Gamma(\frac{1}{2}(n-p+1))\Gamma(\frac{1}{2}(p-1))\Gamma(\frac{1}{2})} \\
 & \cdot \sum_{j=0}^{\infty} \left\{ \frac{\Gamma(\frac{1}{2}(n+2)+j)}{\Gamma(\frac{1}{2}p+j)j!} (\frac{1}{2}\lambda^2)^j \right. \\
 & \quad \left. \cdot \frac{(k_1^2 m_{11}^* + 2k_1 k_2 m_{12}^* + k_2^2 m_{22}^* + (k_1^2 + k_2^2)(m_{11}^* m_{22}^* - m_{12}^{*2}))^j}{|I + M^*|^{\frac{1}{2}(n+2)+j}} \right\}.
 \end{aligned}$$

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