ON THE DISTRIBUTIONS OF THE TIMES BETWEEN EVENTS IN A STATIONARY STREAM OF EVENTS

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1. Introduction.

Many authors - particularly in the physical and engineering sciences - have considered the problem of obtaining the distribution of the time between successive axis-crossings by a stochastic process. This interest dates back to the pioneering work of S. O. Rice [4], who developed a series expression and used a very simple approximation for the case of axis crossings by a stationary normal process.

More generally, one may consider the distribution of the time between an axis-crossing and the nth subsequent axis-crossing, or between, say, an upcrossing of the axis and the nth subsequent downcrossing. Problems of this type have been discussed by Longuet-Higgins [3], with particular reference to the normal case. In particular he obtains series expressions which are variants and generalizations of that given by Rice [4, Eqn 3.4-11]. For example, Longuet-Higgins calculates (by somewhat heuristic methods) a series for the probability density for the time between an "arbitrary" upcrossing of zero to the (r+1)st subsequent upcrossing. (A precise definition of what is meant by such a density is usually not given, in the relevant literature. This difficulty, and one method of overcoming it, will be discussed in Section 2). The series just referred to may be written in the form

$$(1)\quad f(\tau) = \sum_{i=0}^{\infty} (-)^i \binom{r+i}{i} \int \underset{0 < t_2 \ldots < t_{n-1} < \tau}{\cdots} \int W(0,t_2 \ldots t_{n-1},\tau)/W(0)\, dt_2 \ldots dt_{n-1}$$

where $W(t_1 \ldots t_n)dt_1 \ldots dt_n$ represents the "probability of an upcrossing of the axis in each of the intervals (t_i, t_i+dt_i)." For a normal stationary process, the functions $W(t_1 \ldots t_n)$ may be written in terms of the finite-dimensional distributions of the process. It is this fact which leads to the usefulness of the series expansions, when approximated by a small number of terms.

The upcrossings of the axis by, say, a stationary normal process, form a <u>stationary stream of events</u> (point process). Equation (1) then applies to

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this point process and gives the probability density for the time from an "arbitrary" event to the (r+1)st subsequent event. One of our objects in this paper is to generalize results of this form to include arbitrary stationary streams of events.

Further, it is sometimes of interest to differentiate between two or more types of events in a stationary stream and to obtain distributions for the time between an event of the first type and a subsequent event of another type. A simple example of this situation is given by the upcrossings and downcrossings of the axis by a stationary stochastic process. Again certain series expressions are available ([3]) for the desired distributions in this particular case and it will be our aim here to generalize this type of result also. An appropriate framework for such a generalization is that of a "mixture" of two streams. Briefly, a stationary stream of events will be called a mixture of two component stationary streams if the events of the component streams alternate in forming the mixture. (For properties of mixtures in this sense, see Leadbetter [2]).

The integrands occurring in the terms of (1) do not necessarily have strictly equivalent counterparts within a more general framework than the axis-crossing case. Hence it is not immediately clear what type of results should be sought in the more general framework of streams of events. However the integrals occurring in (1) are closely related to the factorial moments of the number of upcrossings of the axis in $(0,\tau)$ by, say, a stationary normal process (Cramér and Leadbetter [1] Eqn. 10.6.2). Hence it is natural to seek formulae in terms of factorial moments of the numbers of events in a given time interval for the case of a more general stream. This also has the advantage of making possible a more satisfying and rigorous development than is at present available in the literature concerned with axis-crossings. The situation for mixtures of streams is just a little more complicated, but similar results may be obtained.

In Section 2 we shall briefly describe the framework and notation to be used. Section 3 will be concerned with the case of a stationary stream of events, and Section 4 with a mixture of two such streams.

2. Framework and Notation.

It is not our purpose here to discuss the basic structure of a stationary stream of events. (A convenient account of this may be found in [1, Section 3.8], and references therein). We shall take the basic structure as given and write $N(s,t)$ for the number of events occurring in the interval (s,t). It will be assumed that the stream is regular (orderly) in the sense that $P\{N(0,t) > 1\} = o(t)$ as $t \downarrow 0$. Write $v_k = v_k(t) = P\{N(0,t) = k\}$.

We are interested in the distribution function for the time between an "arbitrary" event and the nth subsequent event. The word "arbitrary" is, however, not well defined (though often used) within this present context and hence we prefer to use a slightly different but precise definition for the distribution functions of interest. (This and related points are discussed in more detail in [2]). Specifically we write

$$F_n(t) = \lim_{\tau \to 0} P\{N(0,t) \geq n \mid N(-\tau,0) \geq 1\}. \tag{2}$$

This limit exists (see [2]) and is interpreted as the conditional probability that the nth event after time zero occurs before t, given that an event occurred "at" time zero. It is know (for example cf. [2]) that

(3) $$F_n(t) = 1 + D^+\{\lambda^{-1} \sum_{k=0}^{n-1} (n-k)v_k(t)\}$$

where D^+ denotes the right hand derivative and λ the mean number of events per unit time.

For the case of a mixture of two stationary streams, considered in Section 4 we shall use the notation of [2]. Specifically let $N(s,t)$ refer to the mixture and $N_i(s,t)$, $i = 1,2$, refer to the component streams consisting of events of "type 1" and "type 2." Write for $i = 1,2$,

(4) $$v_k^{(i)}(t) = P\{N(0,t) = k \text{ and first event after time zero is of type } i\}$$

Now

$$F_k^{(i)}(t) = \lim_{\tau \to 0} P\{N(0,t) \geq k \mid N_i(-\tau,0) \geq 1\}$$

may be interpreted as "the distribution function of the time between an arbitrary event of type i, to the kth following event, of whatever type."

The relation corresponding to (3) which we shall need in this case is

(5) $$F_{2k-1}^{(1)}(t) = 1 + \lambda_1^{-1}D^+W_{k-1}^{(1)}(t),\ k = 1,2,\ldots$$

where

(6) $$W_k^{(1)}(t) = v_{2k}^{(2)} + v_{2k-1}^{(2)} + 2(v_{2k-2}^{(2)} + v_{2k-3}^{(2)} + \ldots + k(v_2^{(2)} + v_1^{(2)}) + (k+1)v_0^{(2)} + v_{2k-1}^{(1)} + v_{2k-2}^{(1)} + 2(v_{2k-3}^{(1)} + v_{2k-4}^{(1)} + \ldots + k(v_1^{(1)}+v_0^{(1)}),$$

and $\lambda_1(=\lambda/2)$ denotes the mean number of events of type 1, per unit time. (A derivation of Equation 5 has been given in [2]). Clearly $F^{(1)}_{2k-1}(t)$ may be interpreted as the "distribution function for the time from an arbitrary type 1 event to the kth following type 2 event."

3. Series expansions for a single stream.

As noted in Section 1 we shall be interested in series expression involving the factorial moments of the number $N(0,t)$ of events in the interval $(0,t)$. We shall accordingly assume that $N(0,t)$ has finite moments of all orders - indeed that the probability generating function $P(z)(=P_t(z)) = \sum_0^{\infty} v_k(t)z^k$ exists for each t, in the region $|z| < \rho$, for some $\rho > 2$. This assumption may probably be weakened but facilitates the calculation of the results, which are thus obtained as absolutely convergent series. Let $\mu_n(t)$ denote the nth factorial moment of $N(0,t)$. That is

$$\mu_n(t)/n! = \sum_{k=n}^{\infty} \binom{k}{n} v_k(t). \tag{7}$$

The probabilities $v_k(t)$ are normally regarded as fundamental in specifying the properties of the stream of events. However we may, of course, regard the factorial moments $\mu_n(t)$ as fundamental quantities since they determine the $v_k(t)$ under the given assumptions. Specifically if $\psi(z)(=\psi_t(z))$ denotes the factorial moment generating function for $N = N(0,t)$ we have $\psi(z-1) = \mathcal{E} z^N = \sum_{r=0}^{\infty} \mu_r (z-1)^r/r!$, an expansion valid for $|z-1| < \rho-1$. But since $P(z) = \psi(z-1)$ and the circle $|z-1| < \rho-1$ includes the origin ($\rho > 2$) we have

$$v_k(t) = P^{(k)}(0)/k! = \sum_{r=k}^{\infty} (-)^{r-k} \binom{r}{k} \frac{\mu_r}{r!}\ , \tag{8}$$

the latter series converging absolutely.

Write, now, $V_n = V_n(t) = \sum_{m=n}^{\infty} v_m(t)$, $a_n = a_n(t) = \int_0^t F_n(u)du.$

Then it follows from (3) that for $n \geq 1$,

(9) $$\lambda(a_{n-1} - a_n) = - [\sum_{k=o}^{n} (n-k)v_k(u) - \sum_{k=o}^{n-1} (n-1-k)\ v_k(u)]_o^t$$

$$= - [v_o + v_1 \ldots + v_{n-1}]_o^t$$

$$= V_n(t).$$

Hence for $k \geq 1$,

$$\mu_k(t)/k! = \sum_{n=k}^{\infty} \binom{n}{k}(V_n - V_{n+1})$$

$$= \sum_{n=k}^{\infty} \binom{n-1}{k-1}\ V_n$$

$$= \lambda \sum_{n=k}^{\infty} \binom{n-1}{k-1}(a_{n-1}-a_n)$$

(10) $$= \lambda \sum_{n=k-1}^{\infty} \binom{n-1}{k-2} \int_o^t F_n(u)\ du.$$

In this calculation some obvious and easily justified series manipulations have been used. It follows by inversion of the summation and integration signs (for the positive quantities involved) that $\mu_k(t)$ is absolutely continuous, being the integral of $\mu_k'(t)$, say, where $\mu_k'(t)$ may (and will) be taken to be non decreasing in t.

From (10) we have for $t > 0$

$$\sum_{k=2}^{\infty} \frac{\mu_k}{k!}\ t^k = \lambda \sum_{n=1}^{\infty} a_n \sum_{k=2}^{n+1} \binom{n-1}{k-2} t^k$$

since the double series has positive terms, which reduces to

(11) $$\sum_{k=2}^{\infty} \frac{\mu_k}{k!}\ t^k = \lambda\ t^2 \sum_{n=1}^{\infty} a_n(1+t)^{n-1}.$$

Both sides of (11) are finite if $0 \leq t < \rho$. It follows from this that the functions

(12) $$\sum_{k=2}^{\infty} \frac{\mu_k}{k!} (z-1)^{k-2} , \quad \lambda \sum_{n=1}^{\infty} a_n z^n$$

are each regular in $|z| < \rho - 1$ and identical on the real axis between $z = 1$ and $z = \rho - 1$ $(\rho > 2)$. Thus the two functions in (12) are identical in $|z| < \rho-1$ and we obtain

$$a_n = \lambda^{-1} \sum_{k=n+1}^{\infty} (-)^{k-n-1} \binom{k-2}{n-1} \frac{\mu_k}{k!}$$

or

(13) $$\int_0^t F_n(u)\, du = \lambda^{-1} \sum_{k=n+1}^{\infty} (-)^{k-n-1} \binom{k-2}{n-1} \int_0^t \mu_k'(x) dx/k! \ .$$

The series in (13) is absolutely convergent and it follows from Fubini's Theorem that

$$\int_0^t F_n(u)\, du = \int_0^t [\lambda^{-1} \sum_{k=n+1}^{\infty} (-)^{k-n-1} \binom{k-2}{n-1} \frac{\mu_k'(x)}{k!}]\, dx$$

whence, for $n \geq 1$,

(14) $$F_n(t) = \lambda^{-1} \sum_{k=n+1}^{\infty} (-)^{k-n-1} \binom{k-2}{n-1} \frac{\mu_k'(t)}{k!} \ .$$

This equation initially holds a.e. but may be taken to hold everywhere by our choice of nondecreasing forms for the $\mu_k'(t)$, provided $\mu_k'(t)$ is also chosen to be continuous to the right at any discontinuity points. Thus Equation (14) expresses $F_n(t)$ in terms of (derivatives of) the factorial moments of $N(0,t)$.

4. Mixtures of streams.

In this section we shall consider the distribution function for the time between an "arbitrary" type 1 event and the kth following type 2 event, in a mixture of two stationary streams. That is we shall be concerned with the function $F_{2k-1}^{(1)}(t)$ defined in Section 2 and our aim will be to express this function in terms of certain factorial moments.

Define a random varialbe $M = M(0,t)$ such that $M-1$ is the number of type-one events in $(0,t)$ for which the immediately following type-2 event is also in $(0,t)$. That is $M - 1$ is the number of complete pairs of events in $(0,t)$ starting with a type-1 event. It is the factorial moments $\nu_n(t)$ of M which will be relevant in the series expression for $F_{2k-1}(t)$. Let $u_n = P\{M=n\}$. Then, with the notation of Section 2, a little calculation shows that for $n \geq 2$,

$$u_n = v^{(1)}_{2n-1} + v^{(1)}_{2n-2} + v^{(2)}_{2n} + v^{(2)}_{2n-1} \tag{15}$$

whereas for $n = 1$ an additional term $v^{(2)}_o$ must be added to the right of (15).

Making the same assumptions as in Section 3 concerning the existence of generating functions, we have in particular that

$$\frac{\nu_k(t)}{k!} = \sum_{n=k}^{\infty} \binom{n}{k} u_n$$

$$= \sum_{n=k}^{\infty} \binom{n-1}{k-1} U_n$$

where $U_n = U_n(t) = \sum_{r=n}^{\infty} u_r$. Now if $A_n = A_n(t) = \lambda_1^{-1} \sum_{s=n+1}^{\infty} U_s$ it follows that

$$\frac{\nu_k(t)}{k!} = \lambda_1 \sum_{n=k-1}^{\infty} \binom{n-1}{k-2} A_n. \tag{16}$$

Now it is shown in the appendix that

$$A_{n+1} = t + \lambda_1^{-1} W_n^{(1)}(t) - W_n^{(1)}(0)) \tag{17}$$

which, by (5) yields

$$A_{n+1} = \int_o^t F_{2n+1}(u)\, du \tag{18}$$

Equation (16) has the same form as (10) and an inversion along the same lines as that leading to (14) yeilds

$$(19) \qquad F_{2n-1}(t) = \lambda_1^{-1} \sum_{k=n+1}^{\infty} (-)^{k-n-1} \binom{k-2}{n-1} \nu_k'(t)/k! \ ,$$

subject to the same comments as made after Equation (14).

Since the (factorial) moments determine the distribution from which they are obtained under less restrictive conditions than those assumed here, it is likely that the results obtained are valid, at least in part, under weaker conditions. We hope to discuss such questions and further generalizations in a future paper.

Appendix. Proof of Equation (17).

From the definition of A_n we have,

$$A_{n+1} = \lambda_1^{-1} (\theta - \sum_{s=1}^{n+1} U_s)$$

where $\theta = \sum_1^{\infty} U_s$. Now since $U_s = 1 - \sum_{r=1}^{s-1} u_r$ it follows that

$$\sum_{s=1}^{n+1} U_s = (n+1) - (u_n + 2u_{n-1} \cdots + nu_1)$$

$$(A.1) \qquad = (n+1) - W_n^{(1)}(t) + v_o^{(2)}(t)$$

by (6) and (15). Also $\theta = \sum_1^{\infty} U_s = \sum_1^{\infty} ru_r = \mathcal{E}M$. This may be evaluated by substituting for u_n from (15). However it is perhaps a little simpler to note that if $\theta_n^{(i)}$ denotes the probability that the first event after time zero is of type i and that there are exactly n type 2 events in (0,t), then

$$\theta = 1 + \sum_1^\infty n\theta_n^{(1)} + \sum_2^\infty (n-1)\theta_n^{(2)}$$

$$= 1 + \sum_1^\infty n\theta_n - \sum_1^\infty \theta_n^{(2)}$$

where $\theta_n = \theta_n^{(1)} + \theta_n^{(2)} = P\{N_2(0,t) = n\}$. Thus $\sum_1^\infty n\theta_n = \mathcal{E}\, N_2(0,t) = \lambda_1 t$ by virtue of the fact that the mean number of events per unit time is the same for each of the component streams (cf. [2], Section 4). On the other hand, $\sum_1^\infty \theta_n^{(2)}$ is the probability that the first event after time zero is of type 2 and that at least one such event occurs before time t. Hence it may be written as $v_o^{(2)}(0) - v_o^{(2)}(t)$. Gathering these facts we see that

$$\theta = 1 + \lambda_1 t - v_o^{(2)}(0) + v_o^{(2)}(t)$$

From this relation and Equation (A.1) it follows that

$$A_{n+1} = t + \lambda_1^{-1} [W_n^{(1)}(t) - n - v_o^{(2)}(0)]$$

which yields (17) since $v_k^{(i)}(0) = 0$ for $k > 0$ and hence, by (6), $W_n^{(1)}(0) = (n+1)\, v_o^{(2)}(0) + nv_o^{(1)}(0) = n + v_o^{(2)}(0)$.

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