ON THE DIVERGENCE OF FOURIER SERIES

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1. By a well known theorem of Kolmogoroff there is a function whose Fourier series diverges almost everywhere. Actually, Kolmogoroff's proof was later generalized so that the Fourier series diverged everywhere [**2**, p. 175]; but we shall be concerned only with the almost everywhere theorem here. The proof involves rather severe restrictions on the orders of the partial sums which are shown to diverge. The following problem connected with this theorem was suggested to the author by Professor A. Zygmund. Given a sequence $\{p_\nu\}$ of positive integers increasing to ∞, can an integrable function f on $(0, 2\pi)$ be constructed so that the partial sums of its Fourier series of order p_ν diverge almost everywhere ? The object of our paper is to give an affirmative answer to this question. Let $s_p(x; f)$ denote the pth partial sum of the Fourier series of the function f at the point x.

THEOREM. *Let $\{p_\nu\}$ be a sequence of integers increasing to ∞. Then there is an integrable function f such that the sequence $s_{p_\nu}(x; f)$ diverges almost everywhere.*

2. Our construction of f is a modification of that of Kolmogoroff. In the latter f was made a sum of trigonometric polynomials of the following type:

$$\frac{1}{n+1}\sum_{k=0}^{n} K_{m_k}(x - A_k), \qquad A_k = \frac{4\pi k}{2n+1} \tag{1}$$

where K_m is the Fejer kernel of order m. The orders m_k were made to satisfy the following conditions:

(a) $m_0 \geqq n^4$; (b) $m_{k+1} > 2m_k$;

(c) $2n+1$ divides $2m_k+1$, $k=0, 1, \cdots, n$.

See [**1**, p. 70].

Given n, from our sequence $\{p_\nu\}$, we choose $(2n+1)^2$ distinct terms $p_{\nu(j)}$, $j=1, 2, \cdots, (2n+1)^2$ in their natural order so that (i) $p_{\nu(1)} > 2n^4$, (ii) $p_{\nu(j)} \leqq (2n)^{-2} p_{\nu(j+1)}$, $j=1, 2, \cdots, (2n+1)^2-1$. Dividing each number $2p_{\nu(j)}+1$ by $2n+1$ leaves a remainder s_j. Since there are only $2n+1$ possible remainders, at least $n+1$ of the numbers

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must have the same remainder, say s. Let $2p_{\nu(j)}+1=L_j+s$. If s is even, the L_j's are odd. Let $2M_j+1=L_j$ and $s=2r$. Then $p_{\nu(j)}=M_j+r$. If s is odd, the L_j's are even. We write $2p_{\nu(j)}+1=(L_j-2n-1)+(s+2n+1)$. Let $2M_j+1=L_j-2n-1$ and $2r=s+2n+1$. Then $p_{\nu(j)}=M_j+r$ with $2M_j+1$ divisible by $2n+1$. Designate the $p_{\nu(j)}$'s by q_k in their natural order, $k=0, 1, \cdots, n$: $q_k=m_k+r$ with $2m_k+1$ divisible by $2n+1$ and $0\leqq r<2n+1$. Now

$$m_0 \geqq p_{\nu(1)} - r \geqq n^4; \qquad m_{k+1} \geqq 2n^2 m_k$$

by (i) and (ii) respectively. Hence all the conditions (a), (b), and (c) above are met in constructing an acceptable function of type (1). We write

$$\phi_n(x) = \frac{e^{irx}}{n+1} \sum_{k=0}^{n} K_{m_k}(x - A_k)$$

and investigate the q_j partial sums of this trigonometric polynomial for x in the interval $J_j=(A_j+n^{-1}(\log n)^{-1/5}, A_{j+1}-n^{-1}(\log n)^{-1/5})$, $1\leqq j<n-n^{1/2}$.

Writing the Fejer kernels in exponential form, we have

$$\begin{aligned} s_{m_j+r}(x;\phi_n) &= \frac{e^{irx}}{n+1} \sum_{k=0}^{j} K_{m_k}(x - A_k) \\ &\quad + \frac{e^{irx}}{2(n+1)} \sum_{k=j+1}^{n} \sum_{l=-m_j-2r}^{m_j} \left(1 - \frac{|l|}{m_k+1}\right) e^{il(x-A_k)} \end{aligned} \tag{2}$$

it being understood that the term corresponding to $l=0$ is $1/2$. Let the first term on the right be denoted by $R_1(x)$. The second term on the right may be written as

$$\begin{aligned} R_2(x) + R_3(x) &= \frac{e^{irx}}{2(n+1)} \sum_{k=j+1}^{n} \sum_{l=-m_j-2r}^{-m_j-1} \left(1 - \frac{|l|}{m_k+1}\right) e^{il(x-A_k)} \\ &\quad + \frac{e^{irx}}{n+1} \sum_{k=j+1}^{n} s_{m_j}(x - A_k; K_{m_k}). \end{aligned}$$

By the method of [1, p. 71] it follows that

$$\begin{aligned} R_1(x) + R_3(x) &= - e^{irx} \sin [(m_j + 1/2)x] \\ &\quad \cdot \left\{ \frac{1}{n+1} \sum_{k=j+1}^{n} \frac{m_k - m_j}{m_k + 1} \, \frac{1}{2 \sin (A_k - x)/2} \right\} + G(x) \end{aligned}$$

where $|G(x)|$ is bounded by a numerical constant independent of n

for x in J_j. The factor in curly brackets of the first term on the right may be written

$$(3)\quad \begin{aligned} T(x) + U(x) = \frac{1}{n+1} \sum_{k=j+1}^{n} \frac{1}{2 \sin (A_k - x)/2} \\ - \frac{1}{n+1} \sum_{k=j+1}^{n} \frac{m_j+1}{m_k+1} \frac{1}{2 \sin (A_k - x)/2}. \end{aligned}$$

By condition (ii), $(m_j+1)(m_k+1)^{-1} \leqq n^{-2}$ for $k > j$ so that the sum on the right is bounded. Summarizing our results thus far, we have from (2) and (3)

$$(4)\quad s_{m_j+r}(x; \phi_n) = R_2(x) - e^{irx} \sin [(m_j + 1/2)x] T(x) + G(x) + H(x)$$

where $|G(x)+H(x)|$ is bounded by a constant independent of n.

In the expression defining $R_2(x)$ above, the terms involving $|l|/m_k+1$ lead to a function $I(x)$ which is less in absolute value than $4rm_j(m_{j+1}+1)^{-1} \leqq 1$ by (ii). Hence, $R_2(x) - I(x)$ is

$$\frac{ie^{irx}}{2(n+1)} \sum_{k=j+1}^{n} \frac{\exp [-i(m_j + 1/2)(x - A_k)]}{2 \sin (A_k - x)/2} - \frac{ie^{irx}}{2(n+1)} \sum_{k=j+1}^{n} \frac{\exp [-i(m_j + 1/2 + 2r)(x - A_k)]}{2 \sin (A_k - x)/2}.$$

Since $(m_j+1/2)A_k \equiv 0$, mod 2π,

$$\begin{aligned} R_2(x) = I(x) + \frac{i}{2} \exp [i(r - m_j - 1/2)x] T(x) \\ - \frac{i}{2} \exp [-i(r + m_j + 1/2)x] \left\{ \frac{1}{n+1} \sum_{k=j+1}^{n} \frac{e^{2irA_k}}{2 \sin (A_k - x)/2} \right\}. \end{aligned}$$

Let the sum in curly brackets of the third term on the right side be denoted by $T_1(x)$. Clearly $|T_1(x)| \leqq T(x)$ for every x in J_j by (3). Combining with (4) gives

$$(5)\quad \begin{aligned} s_{m_j+r}(x; \phi_n) = i \exp [i(r + m_j + 1/2)x] \frac{T(x)}{2} \\ - i \exp [-i(r + m_j + 1/2)x] \frac{T_1(x)}{2} \\ + G(x) + H(x) + I(x). \end{aligned}$$

Let x_0 be a particular value of x in J_j. From the definition of J_j

$$|T(x)-T(x_0)| \leqq \frac{\pi^2}{4}\frac{|x-x_0|n^2(\log n)^{2/5}}{(n+1)}$$
$$+\frac{|x-x_0|}{4(n+1)}\sum_{k=j+2}^{n}\frac{1}{[\sin(A_k-x)/2][\sin(A_k-x_0)/2]} \leqq C(\log n)^{2/5}.$$

For the same reasons, $|T_1(x)-T_1(x_0)| \leqq C(\log n)^{2/5}$. We set $T = T(x_0)$, $T_1 = T_1(x_0)$, and $t_j = r+m_j+1/2$ and obtain from (5)

$$s_{m_j+r}(x;\phi_n) = \frac{i}{2}e^{it_jx}T - \frac{i}{2}e^{-it_jx}T_1 + L(x) \tag{6}$$

where $|L(x)| \leqq C_1(\log n)^{2/5}$. Now $T \geqq C_2(\log n)$ if $1 \leqq j < n-n^{1/2}$, [1; p. 71]. If $|T_1| \leqq (1-(\log n)^{-1/2})T$, then

$$|s_{m_j+r}(x;\phi_n)| \geqq C_3(\log n)^{1/2}. \tag{7}$$

If not, put $T_1 = e^{i\alpha}|T_1|$, where α depends on j but not on x. The first two terms of the right side of (6) may be written

$$\frac{i}{2}T\left\{e^{it_jx} - e^{-i(t_jx-\alpha)}\right\} + i\left\{\frac{T-|T_1|}{2}e^{-i(t_jx-\alpha)}\right\}.$$

Since $0 \leqq T - |T_1| \leqq (\log n)^{-1/2}T$, the second term in curly brackets does not exceed in absolute value $2^{-1}(\log n)^{-1/2}T$. The first term above is

$$-Te^{i\alpha/2}\sin(t_jx-\alpha/2).$$

Thus, if x belongs to J_j, $1 \leqq j < n-n^{1/2}$, and if

$$|\sin(t_jx-\alpha/2)| > (\log n)^{-1/2} \tag{8}$$

it follows from (6), (7), and (8) that

$$|s_{m_j+r}(x;\phi_n)| \geqq C_4(\log n)^{-1/2}T \geqq C_5(\log n)^{1/2}. \tag{9}$$

Let E be the set of x which belongs to some J_j for which (8) is satisfied. Here (9) holds. The acceptable J_j have total measure not less than

$$(n-n^{1/2}-2)\left(\frac{4\pi}{2n+1}-\frac{2}{n(\log n)^{1/5}}\right)$$

which equals 2π minus a term which is $O[(\log n)^{-1/5}]$. The part of them for which (8) is not satisfied is $O(\log n)^{-1/2}$.

3. The proof can be completed by familiar means. Choose a sequence $\{n_\mu\}$ going to ∞ sufficiently rapidly and write

$$f(x) = \sum_{\mu=1}^{\infty} (\log n_\mu)^{-1/4} \phi_{n_\mu}(x).$$

The details of what is meant by "sufficiently rapidly" are given in [2, p. 175]. From what has been shown in §2, it follows that for almost every x and for arbitrarily large n, p_ν can be found from our original sequence such that $|s_{p_\nu}(x; f)| \geqq C_6(\log n)^{1/4}$. Appropriate modifications can be made in the construction so as to make f real.

References

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A PROBLEM OF RICHARD BELLMAN[1]

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In the January 1957 number of the Bulletin of the American Mathematical Society, Richard Bellman proposes the following problem:

"Consider the Sturm-Liouville problem

$$u'' + \lambda[f(x) + \epsilon g(x)]u = 0, \tag{1}$$

$$u(0) = u(1) = 0 \tag{2}$$

where $f(x)$ and $g(x)$ are continuous functions over $[0, 1]$ with positive minima.

Let us regard λ_1, the smallest charactistic value as a function of ϵ. Is it true that λ_1, is an analytic function of ϵ for $R(\epsilon) \geqq 0$? In general, where is the singularity nearest the origin ($\epsilon = 0$)?"

In the present note the method of differences is used.

We shall consider $f(x)$ and $g(x)$ defined as continuous functions $-\eta \leqq x \leqq 1+\eta$ where $\eta > 0$ is arbitrarily small. Any definition outside the original interval of definition will do. We shall assume $|f(x)| < N$, $|g(x)| < N$ where N is constant.

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