

## ON THE DIVERGENCE OF REARRANGED TRIGONOMETRIC SERIES

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K. Tandori studied the rearrangement of trigonometric series in [6], and that of Walsh series in [7]. The result of [6] was sharpened by F. Móricz ([1], [2]), then further by the author [3]. That is

**THEOREM A.** *If  $\{\rho(n)\}$  is a sequence of positive numbers with  $\rho(n) = o(\sqrt[4]{\log n})$ , then there exists a sequence of real numbers  $\{a_1, b_1, \dots, a_n, b_n, \dots\}$  for which*

$$\sum_{n=1}^{\infty} (a_n^2 + b_n^2) \rho^2(n) < \infty$$

and such that the trigonometric series

$$(1) \quad \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

can be rearranged into an everywhere divergent series.

The result of [7] was also sharpened in [4] and [5]. Writing  $L_1(n) = \log n$  and  $L_s(n) = L_1(L_{s-1}(n))$  ( $s = 2, 3, \dots$ ), [5] reads as follows.

**THEOREM B.** *For any natural number  $s$ , there exists a sequence of real numbers  $\{a_1, \dots, a_n, \dots\}$  for which*

$$\sum_{n=N+1}^{\infty} a_n^2 \sqrt{L_1(n)} L_2(n) L_3(n) \cdots L_s(n) < \infty^{1)}$$

and such that the Walsh series

$$\sum_{n=1}^{\infty} a_n w_n(x)$$

can be rearranged into an almost everywhere divergent series.

In the present paper, we are going to prove the following theorem by using the methods of [3] and [5].

**THEOREM.** *For any natural number  $s$ , there exists a sequence of real numbers  $\{a_1, b_1, \dots, a_n, b_n, \dots\}$  for which*

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<sup>1)</sup>  $N$  is a natural number depending  $s$  such that  $L_s(N) > 0$ .

$$\sum_{n=N+1}^{\infty} (a_n^2 + b_n^2) \sqrt{L_1(n)} L_2(n) L_3(n) \cdots L_s(n) < \infty$$

and such that the trigonometric series (1) can be rearranged into an almost everywhere divergent series.

**COROLLARY.** For any natural number  $s$ , there exists a sequence of real numbers  $\{a_1, b_1, \dots, a_n, b_n, \dots\} \in l_2$  such that the trigonometric series (1) can be rearranged to satisfy

$$\limsup_{N \rightarrow \infty} \frac{\left| \sum_{j=1}^N (a_{n(j)} \cos n(j)x + b_{n(j)} \sin n(j)x) \right|}{\{L_1(N)\}^{1/4} \cdot \{L_2(N) \cdots L_s(N)\}^{1/2}} > 0$$

almost everywhere.

**1. Lemmas.** Let us modify the original lemmas in [3] or [1].

**LEMMA 1.** Let  $E = \bigcup_i J_i \subset [-\pi/8, \pi/8]$  be a generalized simple set,  $0 < \varepsilon < \max_i |J_i|/2$  and  $0 < \eta \leq 1$  real numbers, and  $n$  a natural number such that  $n \geq (5/4)\pi/\varepsilon\eta - 1$ . Then for any natural number  $N (\geq 4n + 3)$  with  $N \equiv 3 \pmod{4}$ , there exist trigonometric polynomials  $Q_l(x)$  ( $l = 1, 2, 3$ ) with the following properties:

(i) denote by  $\nu_l$  the frequencies occurring in  $Q_l(x)$  ( $l = 1, 2, 3$ ), then

$$\begin{aligned} \nu_1 &\equiv 3 \pmod{4}, & N - 4n &\leq \nu_1 \leq N + 4n, \\ \nu_2 &\equiv 1 \pmod{4}, & N - 4n - 2 &\leq \nu_2 \leq N + 4n + 2, \\ \nu_3 &\equiv 2 \pmod{4}, & 2N - 4n &\leq \nu_3 \leq 2N + 4n; \end{aligned}$$

(ii)  $\left| \sum_{l=1}^3 Q_l(x) \right| < C_1 \eta$  if  $x \in [-\pi/8, \pi/8] - E$  ( $C_1 = 4 + 6\sqrt{2}$ );

(iii)  $\int_{-\pi}^{\pi} \left| \sum_{l=1}^3 Q_l(x) \right|^2 dx \leq C_2 |E|$  ( $C_2 = 9(25 + 12\sqrt{2})\pi^4$ );

(iv) there exists a decomposition  $E^{(\varepsilon)} = E_1 + E_2 + E_3$  such that

$$\sum_{k=1}^l Q_k(x) \geq 1/2 \quad \text{if } x \in E_l \quad (l = 1, 2, 3).$$

**PROOF.** Setting

$$E^{(\varepsilon)} = \bigcup_{i=1}^m [\alpha_i + \varepsilon, \beta_i - \varepsilon],$$

$a = \pi/4(n + 1)$  and  $b_k = 2ak$  ( $k = 0, \pm 1, \pm 2, \dots$ ), we determine the integers  $\rho_i$  and  $\sigma_i$  ( $i = 1, \dots, m$ ) such that

$$b_{\rho_i} - a \leq \alpha_i + \varepsilon < b_{\rho_i} \quad \text{or} \quad b_{\rho_i} \leq \alpha_i + \varepsilon < b_{\rho_i} + a$$

and

$b_{\sigma_i} - a < \beta_i - \varepsilon \leq b_{\sigma_i}$  or  $b_{\sigma_i} < \beta_i - \varepsilon \leq b_{\sigma_i} + a$  respectively. Then the trigonometric polynomial

$$P(x) = 2\pi a \sum_{i=1}^m \sum_{r=\rho_i}^{a_i} K_n(4(x - b_r)^2) = \sum_{\nu=0}^n (a_{4\nu} \cos 4\nu x + b_{4\nu} \sin 4\nu x)$$

has the following properties:

$$P(x) \geq 1 \quad \text{if } x \in E^{(\varepsilon)} ;$$

$$P(x) < \eta \quad \text{if } x \in [-\pi/8, \pi/8] - E ;$$

$$\int_{-\pi}^{\pi} P^2(x) dx \leq (9/2)\pi^4 |E| .$$

Now set

$$Q_1(x) = (\cos Nx)P(x) ,$$

$$Q_2(x) = -2\sqrt{2}(\cos 2x)(\cos Nx)P(x) ,$$

$$Q_3(x) = -(3 + 4\sqrt{2})(\cos 2Nx)P(x)$$

and

$$E_1 = E^{(\varepsilon)} \cap \bigcup_{k=-\infty}^{\infty} \left[ \frac{1}{N} \left( 2k\pi - \frac{1}{3}\pi \right), \frac{1}{N} \left( 2k\pi + \frac{1}{3}\pi \right) \right] ,$$

$$E_2 = E^{(\varepsilon)} \cap \bigcup_{k=-\infty}^{\infty} \left[ \frac{1}{N} \left( 2k\pi + \frac{2}{3}\pi \right), \frac{1}{N} \left( 2k\pi + \frac{4}{3}\pi \right) \right] ,$$

$$E_3 = E^{(\varepsilon)} \cap \bigcup_{k=-\infty}^{\infty} \left( \frac{1}{N} \left( k\pi + \frac{1}{3}\pi \right), \frac{1}{N} \left( k\pi + \frac{2}{3}\pi \right) \right) .$$

Then we can see that (i)-(iv) hold (cf. [3; Lemma 2]).

LEMMA 2. For every natural number  $m$ , there exist generalized simple sets  $E(m, i, k) \subset [-\pi/8, \pi/8]$  and mutually disjoint trigonometric polynomials  $R(m, i, k; x)$  ( $k = 1, \dots, 3^i; i = 0, 1, \dots$ ) with the following properties:

- (i)  $E(m, i, k) \cap E(m, i, k') = \emptyset$  ( $1 \leq k < k' \leq 3^i$ );
- (ii)  $E(m, i, k) \supset \bigcup_{l=1}^3 E(m, i + 1, 3(k - 1) + l)$ ;
- (iii) set

$$E(m, i, k) = \bigcup_{j=1}^{g(m, i, k)} J_j \quad \text{and} \quad v_i(m) = \sum_{k=1}^{3^i} g(m, i, k) ,$$

then

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<sup>2)</sup>  $K_n(x) = \frac{1}{2(n+1)} \left( \frac{\sin(n+1)x/2}{\sin x/2} \right)^2$ .

$$v_0(m) = 1 \quad \text{and} \quad v_i(m) \leq 5^{-1} f_i(m) \quad (i \geq 1)$$

where

$$f_i(m) = m^i (3\sqrt[3]{3})^{i(i+C_3)} \quad (C_3 = 4);$$

(iv) set

$$F(m, i) = \left[ -\frac{\pi}{8}, \frac{\pi}{8} \right] - \bigcup_{k=1}^{3^i} E(m, i, k)$$

and

$$F_m = \lim_{i \rightarrow \infty} F(m, i),$$

then

$$|F_m| \leq \frac{C_4}{m} \quad \left( C_4 = \frac{3\pi}{16} \right);$$

(v) denote by  $\nu$  the frequencies occurring in  $R(m, i, k; x)$ , then

$$\nu \not\equiv 0 \pmod{4} \quad \text{and} \quad \nu \leq f_i(m);$$

$$(vi) \quad \int_{-\pi}^{\pi} \left\{ \sum_{k=1}^{3^i} R(m, i, k; x) \right\}^2 dx \leq C_5 \quad \left( C_5 = C_2 \frac{\pi}{4} \right);$$

$$(vii) \quad \sum_{j=1}^k R(m, i, j; x) \geq \frac{1}{6} \quad \text{if } x \in E(m, i, k).$$

PROOF. We set

$$\varepsilon_i = \varepsilon_i(m) = \begin{cases} \pi/16m & (i = 0), \\ 5\pi/16m3^i f_i(m) & (i \geq 1), \end{cases}$$

$$\eta_i = \begin{cases} 1 & (i = 0), \\ 1/C_1 3^{i+1} & (i \geq 1), \end{cases}$$

$$n_i = n_i(m) = \left[ \frac{(5/4)\pi}{\varepsilon_i \eta_i} \right] = \begin{cases} 20m & (i = 0), \\ [4C_1 m 3^{2i+1} f_i(m)] & (i \geq 1) \end{cases}$$

and

$$N(m, i, k) = 8(n_i + 1)k - 1 \quad (k = 1, \dots, 3^i; i = 0, 1, \dots).$$

Set  $E(m, 0, 1) = [-\pi/8, \pi/8]$  and  $R(m, 0, 1; x) = \cos x$ . Supposing  $E(m, i, k)$  and  $R(m, i, k; x)$  ( $k = 1, \dots, 3^i$ ) defined, we apply Lemma 1 to  $(E(m, i, k), \varepsilon_i(m), \eta_i, n_i(m))$  and  $N(m, i, k)$  and get the trigonometric polynomials  $Q_l(x)$  ( $l = 1, 2, 3$ ) and the decomposition  $E(m, i, k)^{(e_i)} = E_1 + E_2 + E_3$ . Then set  $R(m, i+1, 3(k-1)+l; x) = Q_l(x)$  and  $E(m, i+1, 3(k-1)+l) = E_l$  ( $l = 1, 2, 3; k = 1, \dots, 3^i$ ). And we can see that (i)-(vii) hold (cf. [3; Lemma 3]).

2. **Proof of the theorem.** Setting  $D_n = \{nL_1(n)L_2(n) \cdots L_s(n)\}^{-1}$ , we define a sequence of integers  $\{p(n)\}$  such that

$$\sum_{i=n}^{p(n)} D_i \geq 1 \quad (n = N + 1, N + 2, \dots),$$

and  $\{t_m\}$  such that

$$L_s(t_m - 1) \geq m^2 \quad (m = 1, 2, \dots).$$

Arrange the system of trigonometric polynomials

$$\{D_i R(m, i, k; x)\} \quad (k = 1, \dots, 3^i; i = t_m, \dots, p(t_m))$$

into

$$\{U_j(m; x)\} \quad \left( j = 1, \dots, h(m); h(m) = \sum_{i=t_m}^{p(t_m)} 3^i \right)$$

so that

$$U_{j(i,k)}(m; x) = D_i R(m, i, k; x),$$

$$j(i, k) < j(i + 1, 3k - 2) < j(i + 1, 3k - 1) < j(i + 1, 3k) < j(i, k + 1)$$

hold. Then we see by Lemma 2 that

$$(2) \quad \sum_{j=1}^{\mu_m(x)} U_j(m; x) \geq \frac{1}{6}, \quad 1 \leq \mu_m(x) \leq h(m)$$

for  $x \in [-\pi/8, \pi/8] - F_m$ . Since  $|F_m| \rightarrow 0$  ( $m \rightarrow \infty$ ), (2) holds almost everywhere in  $[-\pi/8, \pi/8]$  for infinitely many  $m$ .

Now let us define the rearranged trigonometric series

$$\sum_{j=1}^{\infty} (a_{n(j)} \cos n(j)x + b_{n(j)} \sin n(j)x)$$

by considering the series

$$\sum_{m=1}^{\infty} \sum_{j=1}^{h(m)} U_j(m; 8^{m-1}x).$$

It is obvious that these series diverge almost everywhere in  $[-\pi, \pi]$  from the second Borel-Cantelli lemma.

Setting  $\lambda(n) = \sqrt{L_1(n)L_2(n) \cdots L_s(n)}$ , we get

$$\begin{aligned} & \sum_{n=N+1}^{\infty} (a_n^2 + b_n^2)\lambda(n) \\ & \leq \pi^{-1} \sum_{m=1}^{\infty} \sum_{i=t_m}^{p(t_m)} D_i^2 \lambda(8^{m-1} f_i(m)) \int_{-\pi}^{\pi} \left\{ \sum_{k=1}^{3^i} R^2(m, i, k; 8^{m-1}x) \right\} dx \\ & \leq \pi^{-1} C_5 \sum_{m=1}^{\infty} \sum_{i=t_m}^{\infty} D_i^2 \lambda(e^{4i^2}) \end{aligned}$$

$$\begin{aligned} &\leq C_6 \sum_{m=1}^{\infty} \sum_{i=t_m}^{\infty} \{iL_1(i) \cdots L_{s-1}(i)\}^{-1} \{L_s(i)\}^{-2} \quad (C_6 = 4\pi^{-1}C_5) \\ &\leq C_6 \sum_{m=1}^{\infty} \{L_s(t_m - 1)\}^{-1} \leq C_6 \sum_{m=1}^{\infty} m^{-2} < \infty . \end{aligned}$$

This completes the proof of the theorem.

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